Local Plastic Deflection of Long Beam under Blast Load

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Dynamic response of a long beam to a pulse load is analyzed on the basis of classical rigid-plastic approach. It is assumed that the load applies transversely with time-decaying intensity and is distributed uniformly over some range of the beam axis of infinite length. Small deformation beam theory yields an explicit closed-form solution of dynamic deflection as a function of time and space. Motion does not terminate in finite time, but locally deflected configuration is determined, as can be regarded as transient response of a finite beam. Two patterns of motion are possible, depending upon the product of the load magnitude and width of the loaded range relative to the full plastic moment. Clarification of the motion pattern makes a simple straight-forward formulation feasible. A purely impulsive load and a rectangular pulse load are considered as examples of application. A few remarks are made with reference to the analytical validity for other loading types.

1. INTRODUCTION

A moderate load can cause a large deflection locally in a beam, when loading is sudden and concentrated in a narrow region; the local deflection can be significant, even when overall deformation is limited. If a load pulse of short duration applies over a certain range in a long beam, inertial resistance prevents parts of the beam distant from the load to deflect until the loading disturbance is transmitted to these parts. During this process, the deflection near the loaded range may attain such an amount that involves plastic deformation. Configuration or distribution of such local plastic deflection is closely related to eventual rupture, and is technically important. Let the beam be infinitely long in comparison with the width of the loaded range. On the basis of rigid-plastic or elastic-perfectly plastic behavior without geometry changes any load of finite magnitude causes plastic bending on the infinite beam. By considering that elastic component fades away eventually due to damping or viscosity, attention is focused on plastic deflection in this paper.

Elastic deflection of an infinite beam due to an impact has been obtained in a closed-form, including Bohnenblust's solution as a special case [1]. Plastic response of an infinite or semi-infinite beam has also been analyzed theoretically for such an impact loading that a beam section is subjected to specified velocity or a concentrated load [2-7]. Rigid-plastic solution was also obtained for built-in beams under distributed dynamic loading [8], in which load-time relation was assumed either monotonic nonincreasing or nondecreasing. A nonincreasing or blast-type load is assumed here to act over a finite region of an infinite beam. By following the classical rigid-plastic approach, straight-forward derivation is presented for the beam deflection as a function of time and space. This formulation is much simpler than in previous papers though along similar lines, utilizing basic principles of elementary small deflection beam theory. It is believed to serve as a first order approximation for the local response of a long ductile beam to a pulse load, when the elastic component of deflection can be neglected in comparison with the plastic. Some remarks will be made on the nature of the resulted solution.

2. ASSUMPTIONS

It is supposed that an infinite beam of uniform cross section is subjected to a load distributed over a width $2a$, as shown in Fig.1. The total load $P(t) \geq 0$ decreases monotonically with time $t \geq 0$. A rectangular pulse and a purely impulsive load belong to this category as special cases. Elastic deformation is neglected based on the assumption that the total plastic work is sufficiently large in comparison with the maximum strain energy that can be stored in the beam [9]. Perfect plasticity is assumed with limit moment M_P; viscosity as well as work-hardening is neglected. Mass per unit length is m. Plane motion is assumed with deflection $y(x,t)$, where x designates a position along the original beam axis with the origin at the center of the loading width. Deflection is assumed to be so small that the square of the slope in the deflection curve is negligible in comparison with unity. Ample ductility and stability are assumed throughout in the beam behavior. Effects of shearing, rotatory inertia and stress waves propagation are neglected.

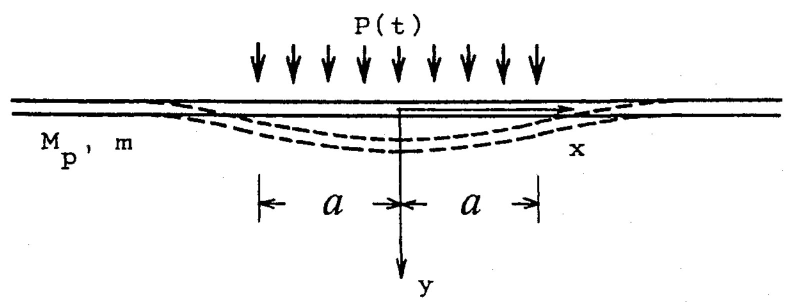

Fig. 1 Infinite Beam under Pulse Loading.

3. ANALYSIS

It turns out that motion starts with either of two deformation patterns of Fig.2, in which plastic and rigid regions are distinguished for a half beam. With symmetry taken into account, consideration is given only of the right half in the discussions to follow. These are the only possible motion patterns that satisfy dynamic equilibrium, velocity continuity and plasticity condition. The plasticity condition does not allow for bending moment to overcome M_P in magnitude, and stipulates for change in curvature to be compatible with the direction of the acting bending moment. The figure also shows the distribution of velocity $\partial y/\partial t$, effective load (external load plus inertia) per unit length q, shearing force Q and bending

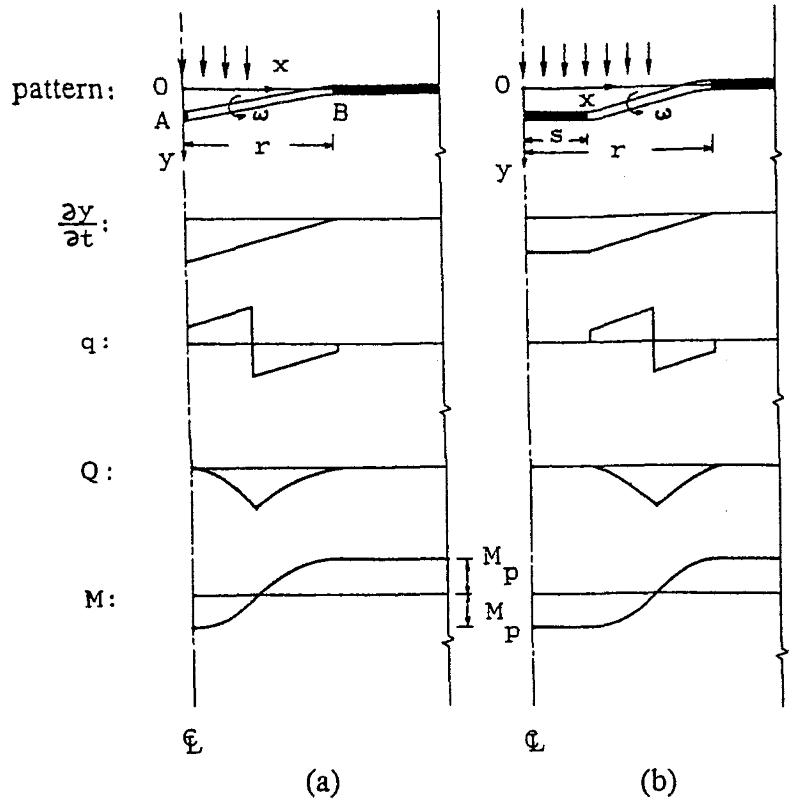

Fig. 2 Patterns of Motion.

moment M. Pattern (a) takes place when P and $''a''$ are relatively small; motion starts with pattern (b) otherwise (The threshold hinges on their product, as determined later in Eqs.(22) and (23).). Since the external load is time-decaying, pattern (b) turns later into pattern (a). This figure affords the key to the simple solution for the rigid-plastic dynamic problem. Its validity shall become clear a posteriori.

In the pattern (a) motion, a rigid segment AB of length $r(t)$ rotates about B with angular velocity $\omega(t)$, while a plastic hinge is formed at A and plastic region (yield zone) is formed beyond B to infinity, both satisfying the yield condition $|M|=M_P$. Conservation of linear momentum

$$I(t)\equiv\int_0^t P dt = m\omega r^2 \tag{1}$$

and conservation of moment of momentum with respect to the origin

$$2M_p t+\frac{aI}{4}=\frac{m\omega r^3}{6} \tag{2}$$

are adequate to determine the two unknowns

$$\omega=\frac{4I^3}{m(24M_p t+3aI)^2} \tag{3}$$

$$r=\frac{3a}{2}+\frac{12M_pt}{I} \tag{4}$$

It follows from Eq.(4) that

$$\frac{dr}{dt}=\frac{12M_p(I-Pt)}{I^2} \tag{5}$$

The definition of blast-type loading is that $I-Pt \geq 0$, which leads to $dr/dt \geq 0$. This means that the rigid-plastic interface B moves toward infinity, leaving the curvature

$$\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=r}=\frac{\omega dt}{dr}=\frac{I^3}{12mM_pr^2(I-Pt)} \tag{6}$$

Deflection $y(x,t)$ can be determined in terms of the specified impulse $I(t)$ from these equations together with

$$\frac{\partial^2 y}{\partial x \partial t}=\begin{cases}-\omega & (0<x<r)\\ 0 & (r<x)\end{cases} \tag{7}$$

$$\frac{\partial y}{\partial t}=\begin{cases}\omega(r-x) & (0\leq x\leq r)\\ 0 & (r\leq x)\end{cases} \tag{8}$$

It follows from Eq.(3) that the time for ω=0 is reached only by the limit $t\rightarrow\infty$. This indicates that the motion does not terminate within a finite time; note that Q=0 for sufficiently large x. Combination of Eqs.(3), (4) and (8) is indicative of the fact that $\lim_{t\rightarrow\infty} \partial y/\partial t=0$, but the final deflection $\lim_{t\rightarrow\infty} y$ is not finite. External work must be finite, however, and is equal to the energy absorbed by plastic bending. In fact, it is seen from Eq.(3) that $\lim_{t\rightarrow\infty}\int_0^t \omega\, dt$ exists. The slope angle at the center, $\lim_{t\rightarrow\infty} \partial y/\partial x\Big|_{x=0+} = -\int_0^\infty \omega\, dt$, which is combined with the curvature of Eq.(6) to give the shape of the deflection curve near the loading range. The total plastic work W is given by

$$W=4M_p\int_0^\infty \omega dt \tag{9}$$

Motion pattern (b) comprises a plastic region at the center as well as the outer plastic region, in addition to a rigid segment in between, as shown in Fig.2(b). The length of the central plastic region $s(t)$ adds to the unknowns $r(t)$ and $\omega(t)$ to be solved. By noting the shearing force distribution, conservation of linear momentum is combined with the velocity continuity across the rigid-plastic interfaces to give

$$\frac{\partial y}{\partial t}=\frac{I}{2ma} \qquad (0\leq x\leq s) \tag{10}$$

$$\omega=\frac{I}{2ma(r-s)} \tag{11}$$

$$r+s=2a \tag{12}$$

Conservation of moment of momentum with respect to the origin leads to

$$2M_pt+\frac{aI}{4}=\frac{Is^2}{4a}+\frac{I(r-s)(r+2s)}{12a} \tag{13}$$

The system of simultaneous Eqs.(11), (12) and (13) provides

$$\omega=\frac{1}{8am}\left(\frac{I^3}{6aM_pt}\right)^{\frac{1}{2}} \tag{14}$$

$$r=a+\left(\frac{24aM_pt}{I}\right)^{\frac{1}{2}} \tag{15}$$

$$s=a-\left(\frac{24aM_pt}{I}\right)^{\frac{1}{2}} \tag{16}$$

It is seen from Eqs.(15) and (16) that $dr/dt \geq 0$ and $ds/dt \leq 0$ for the blast-type loading, and hence that the plastic regions become monotonically narrower with time. Curvature changes at the moving interfaces are found from the relation $\partial^2 y/\partial x^2\Big|_{x=r} = \omega\, dt/dr$ and $\partial^2 y/\partial x^2\Big|_{x=s} = \omega\, dt/ds$, to be

$$\frac{\partial^2 y}{\partial x^2}\Bigg|_{x=r}=-\frac{\partial^2 y}{\partial x^2}\Bigg|_{x=s}=\frac{I^3}{48a^2mM_p(I-Pt)} \tag{17}$$

Deflection can be determined from Eq.(17) and the relations

$$\frac{\partial^2 y}{\partial x\partial t}=\begin{cases}0 & (0<x<s)\\ -\omega & (s<x<r)\\ 0 & (r<x)\end{cases} \tag{18}$$

$$\frac{\partial y}{\partial t}=\begin{cases}\dfrac{I}{2ma} & (0\leq x\leq s)\\ \omega(r-x) & (s\leq x\leq r)\\ 0 & (r\leq x)\end{cases} \tag{19}$$

The motion pattern (b) terminates when $s = 0$ at $t = t_h$, which is found from Eq.(16) or from

$$\frac{24M_pt_h}{a}=I(t_h)\equiv\int_0^{t_h}P\,dt \tag{20}$$

Motion turns into pattern (a) for $t \geq t_h$, and Eqs.(1) to (8) hold. Deflection is the sum of the components acquired in $t \leq t_h$ and $t_h < t < \infty$. Eq.(9) remains valid for the total plastic work.

The condition for pattern (b) to be initiated is determined conveniently from $s > 0$ in Eq.(16) to be

$$aI>24M_pt \qquad (t<t_h) \tag{21}$$

By taking the limit $t \rightarrow 0$ in Eq.(21), this condition reads

$$\frac{P_0 a}{24M_p} > 1 \tag{22}$$

where the subscript 0 denotes the peak value at $t = 0$. The condition otherwise

$$\frac{P_0 a}{24M_p} \leq 1 \tag{23}$$

requires motion to take place in pattern (a). That the afore-derived equations are adequate to determine the response analytically is exemplified below for particular load-time relationship.

4. EXAMPLES AND DISCUSSIONS

With $\dot{I}$ denoting the total impulse $\int_0^\infty Pdt$, purely impulsive loading is considered in terms of Dirac delta function $\delta(t)$ and Heaviside unit step function $u(t)$ defined as

$$P = \hat{I}\,\delta(t - 0_+)\ ; \quad I = \hat{I}\,u(t) \tag{24}$$

Since inequality (22) is satisfied, motion starts with pattern (b), and Eqs.(14) to (17) and Eq.(24) are valid for $0 < t < t_h$ to give

$$\omega = \frac{1}{8am}\left(\frac{\hat{I}^3}{6aM_p t}\right)^{\frac{1}{2}} \tag{25}$$

$$r = a + \left(\frac{24aM_p t}{\hat{I}}\right)^{\frac{1}{2}} \tag{26}$$

$$s = a - \left(\frac{24aM_p t}{\hat{I}}\right)^{\frac{1}{2}} \tag{27}$$

$$\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=r} = -\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=s} = \frac{\hat{I}^2}{48a^2 m M_p} \tag{28}$$

It is seen from Eqs.(26) and (27) that $s_0 = r_0 = a$ and hence that the whole beam becomes plastic at $t = 0$. Deflection for $t < t_h$ is determined by

$$y=\begin{cases}\displaystyle\int_0^t \frac{I}{2ma}dt=\frac{\dot{I}t}{2ma} & (0\leq x\leq s)\\ \displaystyle\int_0^t \frac{I}{2ma}dt+\int_s^x \frac{\partial^2 y}{\partial x^2}\Bigg|_{x=s}(x-s)ds=\frac{\dot{I}t}{2ma}-\frac{\dot{I}^2(x-s)^2}{96a^2mM_p} & (s\leq x\leq a)\\ \displaystyle\int_x^r \frac{\partial^2 y}{\partial x^2}\Bigg|_{x=r}(r-x)dr=\frac{\dot{I}^2(r-x)^2}{96a^2mM_p} & (a\leq x\leq r)\\ 0 & (r\leq x)\end{cases} \tag{29}$$

Eq.(20) gives

$$t_h=\frac{a\dot{I}}{24M_p} \tag{30}$$

Second phase of motion is of pattern (a) with $t \geq t_h$, and it follows from Eqs.(3), (4), (6) and (24) that

$$\omega=\frac{4\dot{I}^3}{m(24M_pt+3a\dot{I})^2} \tag{31}$$

$$r=\frac{3a}{2}+\frac{12M_pt}{\dot{I}} \tag{32}$$

$$\frac{\partial^2 y}{\partial x^2}\Bigg|_{x=r}=\frac{\dot{I}^2}{12mM_pr^2} \tag{33}$$

Deflection is determined in a manner similar to Eq.(29) to be

$$y=\begin{cases}\displaystyle\frac{\dot{I}^2}{12mM_p}\left(\frac{1}{4}-\frac{x}{2a}-\frac{x^2}{8a^2}+\frac{x}{r}+\ln\frac{r}{2a}\right) & (0\leq x\leq a)\\ \displaystyle\frac{\dot{I}^2}{96a^2mM_p}(2a-x)^2+\frac{\dot{I}^2}{12mM_p}\left(-\frac{x}{2a}+\frac{x}{r}+\ln\frac{r}{2a}\right) & (a\leq x\leq 2a)\\ \displaystyle\frac{\dot{I}^2}{12mM_p}\left(-1+\frac{x}{r}+\ln\frac{r}{x}\right) & (2a\leq x\leq r)\\ 0 & (r\leq x)\end{cases} \tag{34}$$

Fig.3 shows how deflection develops as time elapses in a dimensionless form. The impulse is transmitted to the beam portion $0 \leq x/a \leq 1$ as an initial velocity. It is seen that deflection propagates outward, while its increment decreases for a constant time increment. A stationary yield hinge is formed at the beam center for the dimensionless time $24M_pt/(\dot{I}a) \geq 1$, leaving a slope discontinuity $2\theta_m$, where θ_m is the central slope angle given by

$$\theta_m\equiv -\lim_{t\to\infty}\frac{\partial y}{\partial x}\Bigg|_{x=0_+}=\int_{t_h}^{\infty}\omega dt=\frac{\dot{I}^2}{24amM_p} \tag{35}$$

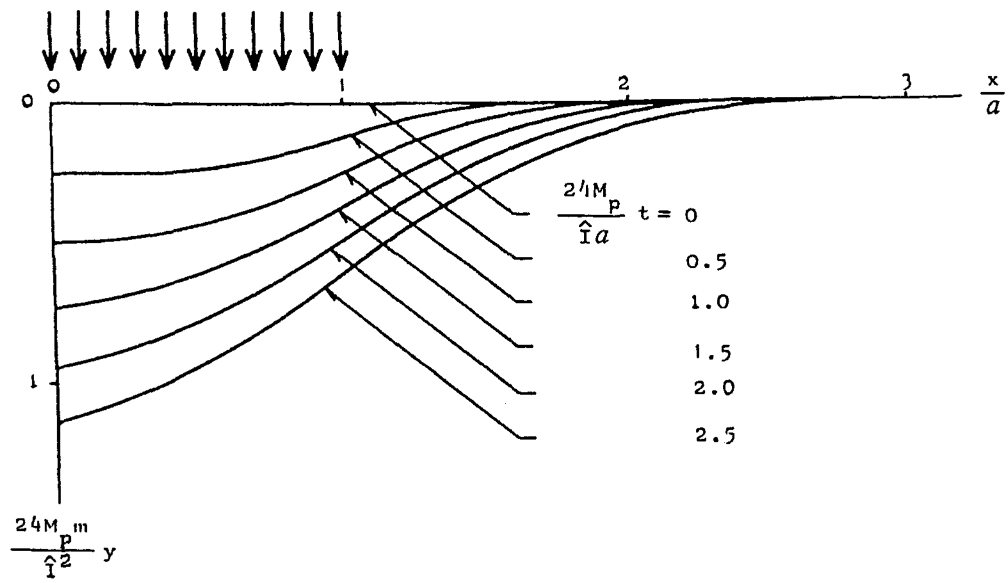

Fig. 3 Deflection Curves due to Pure Impulse.

While the final deflection is not finite, the local configuration of the final deflection relative to that of the center is expressed by $\delta(x)=\lim_{t\to\infty}[y(0,t)-y(x,t)]$. It is found by substituting Eqs.(28), (33) and (35) in

$$\delta=\begin{cases}\theta_m x-\int_0^x \left.\dfrac{\partial^2 y}{\partial x^2}\right|_{x=s}(x-s)ds & (0\leq x\leq a)\\ \theta_m x-\int_0^a \left.\dfrac{\partial^2 y}{\partial x^2}\right|_{x=s}(x-s)ds-\int_a^x \left.\dfrac{\partial^2 y}{\partial x^2}\right|_{x=r}(x-r)dr & (a\leq x)\end{cases} \tag{36}$$

that

$$\delta=\begin{cases}\dfrac{\hat{I}^2}{24mM_p}\left[\dfrac{x}{a}+\left(\dfrac{x}{2a}\right)^2\right] & (0\leq x\leq a)\\ \dfrac{\hat{I}^2}{24mM_p}\left[-\dfrac{1}{2}+\dfrac{2x}{a}-\left(\dfrac{x}{2a}\right)^2\right] & (a\leq x\leq 2a)\\ \dfrac{\hat{I}^2}{24mM_p}\left[\dfrac{5}{2}+2\ln\dfrac{x}{2a}\right] & (2a\leq x)\end{cases} \tag{37}$$

It is checked that the imparted kinematic energy

$$E=\frac{\hat{I}^2}{4r_0 m}=\frac{\hat{I}^2}{4am} \tag{38}$$

is equal to the internal work

$$W=4M_p\left(\int_0^{t_h}\omega\,dt+\int_{t_h}^{\infty}\omega\,dt\right)=\frac{\hat{I}^2}{4am} \tag{39}$$

which has been determined with recourse to Eqs.(9), (25), (30) and (31).

A rectangular pulse load with magnitude P_0 and duration $\hat{I} / P_0$ is expressed as

$$P = P_0[u(t) - u(t - \hat{I}/P_0)] \tag{40}$$

Induced deflection can be determined similarly from the solution derived in the preceding section. The resulting deformation parameters coincide with those for purely impulsive loading in the limit $P_0 \to \infty$. Fig.4 is drawn for the local shape of the final deflection with the ratio $P_0 a$ / $(24M_p)$ as a dimensionless parameter. Unity of this parameter is the threshold that separates the patterns of the initial motion [see Eqs.(22) and (23)]. The feature that the deflection gets more localized as the load magnitude becomes greater (or, as the loading duration becomes shorter) for the same total impulse is governed by the value of this parameter; the loading may be idealized as purely impulsive, unless this parameter takes a magnitude of the order of unity or less. This parameter is taken for the abscissa in Fig.5 to show the dependence of the central slope angle θ_m in the final deflection curve on the loading width "a" for a same load pulse. It is seen that θ_m increases (deformation gets more localized) as the width decreases, and the dimensionless ordinate reaches 6 for a concentrated load. The dependence is not significant when this parameter is greater than the threshold of unity, at which the pattern alteration causes a discontinuity. A conclusion is made of the importance of this parameter on the local deflection characteristics: When the parameter $P_0 a$ / $(24M_p)$ is of magnitude of the order greater than unity, the loading may be regarded as purely impulsive with the same total impulse; otherwise due account should be taken of the shape of load pulse.

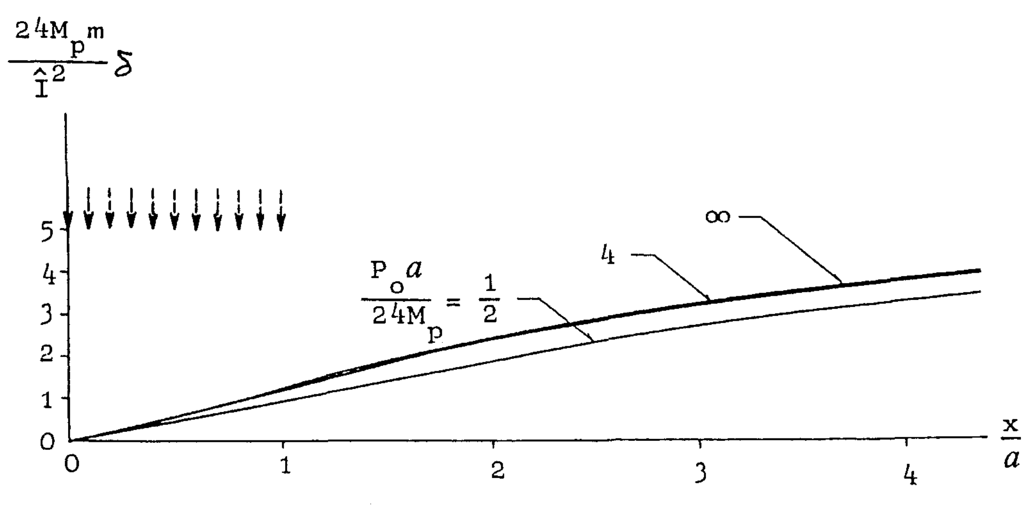

Fig. 4 Dimensionless Permanent Deformation Curves.

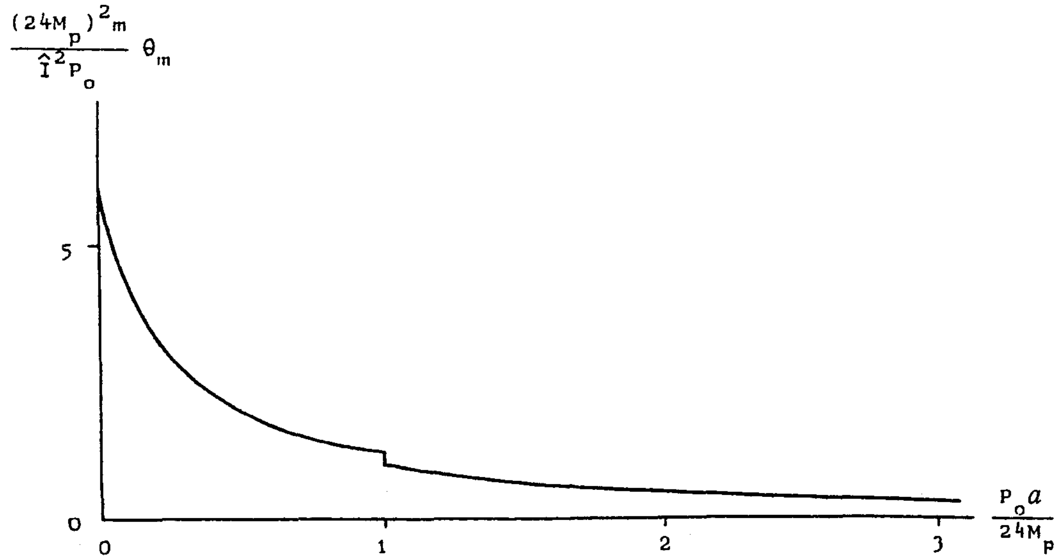

Fig. 5 Effects of Load-Distribution Range on Mid-Point Slope.

5. FURTHER REMARKS

Formulation presented in section 3 is valid as long as motion takes place with one of two types in Fig.2. Discussion has been restricted to the blast-type loading of the case $I-Pt \geq 0$, in which $dr/dt \geq 0$ and $ds/dt \leq 0$. Simplicity of analysis has been due to the fact that the rigid-plastic interfaces travel in definite directions. It is noted that the validity remains as long as these directions do not change in the course of motion, and hence remains valid when these directions are opposite, i.e., $dr/dt \leq 0$ and $ds/dt \geq 0$, which occur in the case of monotone nondecreasing loading, $I-Pt \leq 0$. Further restriction is that $r \geq a$ and $s \leq a$. This means that analysis equations have to be modified when rigid-plastic interfaces either enter into or move out of the loading range. Equations in section 3 are therefore valid in early phases of motion for the monotone nondecreasing loading. Since boundary conditions are irrelevant in the case of the infinite beam, the present analysis can be applied, as representing transient response of finite beams; it is valid, e.g., for a built-in beam, until the outer rigid-plastic interface reaches the beam end.

6. Summary and Conclusions

A simple straight-forward formulation has been presented for the dynamic response of an infinite beam to a blast-type load distributed over some range. Assumption of rigid-perfectly plastic moment-curvature relationship is combined with the assumption that the square of deflection slopes is small in comparison with unity to derive an explicit closed-form dynamic deflection as a function of time and space. The determination of the motion pattern and resulting distributions of equivalent load, shear force and bending moment in d'Alembert sense play a key role in the solution.

Motion takes place with symmetric pattern with two rigid segments, each rotating with respect to an outgoing interface, beyond which a plastic region expands to infinity. There is a central plastic region between the two rigid segments, unless the product of the load magnitude and load width is less than certain multiple of the full plastic moment. The central plastic region decreases in length, and shrinks to an isolated plastic hinge. This pattern of a yield hinge and moving outer interfaces holds from the inception of motion in the counter case ($P_0a \leq 24M_p$), and continues indefinitely. The motion does not terminate in a finite time; the solution determines the local deflection and serves as representing transient response for finite beams. A purely impulsive load and rectangular pulse load are considered as examples of the application of the solution. It is shown that the deflection gets more concentrated near the loading range as the load duration gets shorter and as the loading range gets narrower. Deformation characteristics depends greatly on the ratio $P_0a / (24M_p)$. If this ratio is of magnitude less than unity, the deflection greatly depends on the load magnitude, but for a larger value, the deflection does not depend on the load magnitude significantly, and the loading can be approximated as purely impulsive with the same total impulse for engineering application.

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