Pergamon

Int. J. Impact Engng Vol. 18, Nos 7–8, pp. 889–898, 1996

PII:S0734–743(96)00036–X
0734–743X/96 $15.00 + 0.00

LONGITUDINAL ELASTIC WAVES IN COLUMNS DUE TO EARTHQUAKE MOTION

TAIJIRO NONAKA*, RODNEY J. CLIFTON† and TAICHIRO OKAZAKI‡

*Disaster Prevention Research Institute, Kyoto University, Gokasho, Uji 611, Japan, †Engineering Department, Brown University, Providence, R.I. 02912, U.S.A. and ‡ Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-01, Japan

(*Received* 18 *June* 1996)

Summary—An analysis is presented of longitudinal waves in a thin elastic column. Velocity is specified at one end, and the boundary condition at the other end is expressed in terms of a range of effective impedances of an attached structure. Propagation, reflection and interference of the waves are followed by the method of characteristics. Integration of differential equations along characteristics yields the wave-induced stress, which is then applied to problems of earthquake excitation. Numerical examples are given for recorded updown ground motion of the Kobe Earthquake. Copyright © 1996 Elsevier Science Ltd.

Keywords: elastic wave, stress wave, logitudinal wave, earthquake, seismic disturbance.

INTRODUCTION

Among structural damage due to the Great Hanshin-Awaji Earthquake (the 1995 Hyogoken-Nanbu Earthquake, Kobe Earthquake), there were some uncommon examples of failure mechanisms which had not been observed in previous earthquakes. In a reinforced-concrete column supporting a highway bridge, concrete was ripped off at mid-height, leaving reinforcing bars exposed to the air. Another failure mechanism was observed on a similar column, but of a steel pipe. The coating was ripped off along a buckle taking place along the circumference at some distance from either end. Another striking phenomenon, observed in steel columns of box-type cross section, was their separation into two pieces in a brittle manner across a 20 mm horizontal crack in one case [1]. The authors wish to examine the possibility that stress waves could have caused such destructive effects [2]. It is only after earthquake-generated stress waves propagate, reflect, refract, and/or interfere with one another that a structure begins to vibrate as a whole; the wave propagation phenomenon is a local rather than an overall response. In view of the fact that actual structures and earthquake excitation are too complex to render sufficient information for an exact analysis, simple modeling is proposed in this paper. This is intended to be a preliminary study toward investigation of the effects of wave propagation on structural failure due to earthquake-like excitation. A linear analysis is presented for the simplest case of longitudinal elastic waves to extract useful formulae for application to structural members subjected to imposed velocity at one end. Some examples are then considered for specified velocity variation, including a recorded up-down motion of the Earthquake.

FUNDAMENTALS

Suppose a thin rod of length L is subjected to an imposed longitudinal velocity input at the end $x = 0$. The time history of the velocity variation is specified. The boundary condition at the other end corresponds to one of four cases:

i) free;
ii) non-reflective (L = infinity);
iii) fixed; and
iv) connected to another rod of mis-matched impedance.

The cross section of the rod is assumed to be uniform and of linear elastic material with Young's modulus E such that the axial stress σ (positive for tension) is proportional to the axial strain ε. With density ρ and the speed of wave propagation $c = \sqrt{E/\rho}$, the mechanical or characteristic impedance is ρc. A solution is sought which provides the maximum or minimum value of the stress generated by an imposed velocity.

Solution of equations of motion is equivalent to the solution of $\rho c \mathrm{d}v \pm \mathrm{d}\sigma = 0$ along characteristics $\mathrm{d}x \pm c\,\mathrm{d}t = 0$, respectively, where v is particle velocity, t is time, and x designates position along the longitudinal axis [3]. Constancy of ρ and c provides the relations that the functions $\rho c v \pm \sigma$ take on constant values along characteristics $x \pm c\,t =$ constant, respectively. This situation is represented in the x–t plane in Fig. 1. A longitudinal wave with generic state n emanates from $x = 0$, at which the velocity $v(0, t)$ is prescribed. It propagates along a characteristic line $x - c\,t =$ constant, and attains state $n + 1$ at $x = L$. After reflection it travels along a characteristic line $x + c\,t =$ constant, until attaining a state $n + 2$ at $x = 0$. The wave reflects at $n + 2$ and similar processes are repeated for a number of reflections. With subscripts referring to the corresponding states at the ends of the rod, the equations along characteristics give the following basic relations:

$$\sigma_{n+1} - \rho c v_{n+1} = \sigma_n - \rho c v_n \tag{1}$$

$$\sigma_{n+1} + \rho c v_{n+1} = \sigma_{n+2} + \rho c v_{n+2}. \tag{2}$$

(i) *Free end*

If the end $x = L$ is free from traction, the condition

$$\sigma_{n+1} = 0 \tag{3}$$

is substituted into Eqns (1) and (2), which are then combined to eliminate v_{n+1}, to give:

$$\sigma_{n+2} + \sigma_n = -\rho c(v_{n+2} - v_n). \tag{4}$$

The arithmetic mean

$$\tilde{\sigma} = \tfrac{1}{2}(\sigma_{n+2} + \sigma_n) \tag{5}$$

satisfies the relation

$$\tilde{\sigma} = -\tfrac{1}{2}\rho c(v_{n+2} - v_n). \tag{6}$$

The quantity $v_{n+2} - v_n$ appearing in the right hand side is the increment in velocity acquired at $x = 0$ between the two stages n and $n + 2$. This increment can be approximated by the acceleration, $\mathrm{d}\tilde{v}/\mathrm{d}t$, multiplied by the time difference $2T = 2L/c$ between these two stages. Such a replacement of a finite difference by a derivative may be justified, when the end velocity $v(0, t)$ varies slowly on the time scale T. Thus,

$$v_{n+2} - v_n \cong 2T\frac{\mathrm{d}\tilde{v}}{\mathrm{d}t} \tag{7}$$

where $\tilde{v}$ is to be understood as the mean velocity at $x = 0$. It follows from Eqns (6) and (7) that

$$\tilde{\sigma} = -\rho L\frac{\mathrm{d}\tilde{v}}{\mathrm{d}t}. \tag{8}$$

It is worth noting that the left hand side of this equation is the mean force per unit cross-sectional area at $x = 0$ and the right hand side is the mass ρL of the rod per unit cross-sectional area multiplied by the average acceleration $\mathrm{d}\tilde{v}/\mathrm{d}t$. The minus sign indicates that the axial force is compressive, when the velocity increment is positive. Therefore, Eqn (8) conforms to Newton's equation of motion in the averaged sense. Since $\tilde{\sigma}$ and $\mathrm{d}\tilde{v}/\mathrm{d}t$ are averaged between the two stages, they may represent the stress and acceleration at the time corresponding to the state $n + 1$. If the velocity prescribed at $x = 0$ does not change abruptly, i.e. the velocity variation is smooth, then the right hand side takes on a small magnitude, so that the generated stress may be small in magnitude, as compared with the cases of other boundary conditions.

(ii) *Non-reflective end*

Use is made of the type of Eqn (2), in which both σ_{n+1} and v_{n+1} vanish, because the state $n+1$ is attained at an infinite distance $L = \infty$, so that

$$\sigma_{n+2} + \rho c v_{n+2} = 0. \tag{9}$$

This relation holds for any value of n, and hence

$$\sigma = -\rho c v. \tag{10}$$

This is the well-known relationship between stress and particle velocity, for wave propagation into a semi-infinite rod.

(iii) *Fixed end*

If the end $x = L$ is fully fixed, then $v_{n+1} = 0$, and the elimination of σ_{n+1} between Eqns (1) and (2) leads to:

$$\sigma_{n+2} - \sigma_n = -\rho c(v_{n+2} + v_n). \tag{11}$$

It follows that

$$T\frac{d\tilde{\sigma}}{dt} \cong -\rho c \tilde{v}. \tag{12}$$

If the initial condition is such that $\tilde{\sigma}(t) = 0$, $\tilde{v}(t) = 0$ for $t \leqslant 0$, then integration with respect to time results in:

$$\tilde{\sigma}(t) = -\frac{\rho c}{T}\int_0^t \tilde{v}(\tau)d\tau. \tag{13}$$

The velocity integration appearing in the right hand side of Eqn (13) gives rise to the displacement $\delta(t)$ at $x = 0$. By writing the mean strain $\varepsilon = -\delta/L$, it follows that:

$$\tilde{\sigma} = -\frac{\rho c^2}{L}\delta = \rho c^2 \varepsilon \tag{14}$$

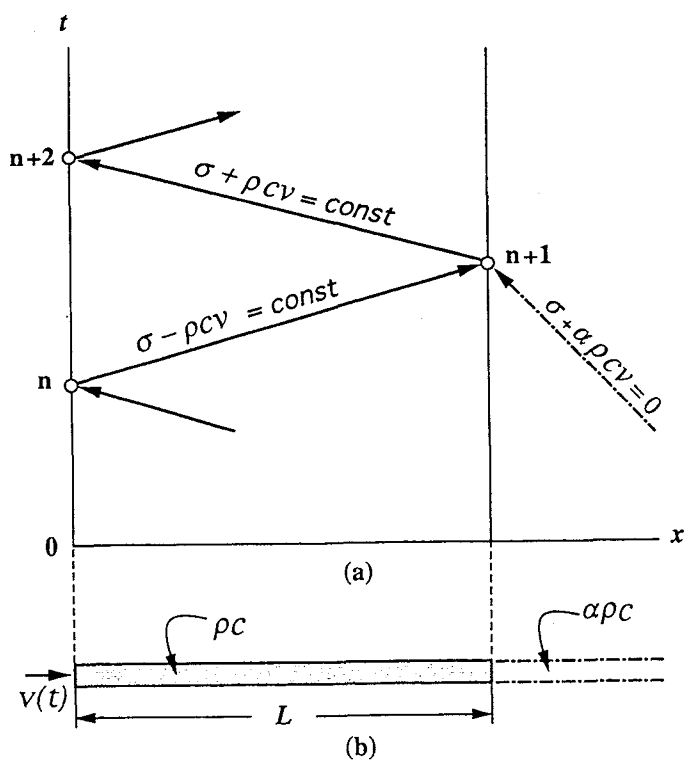

Fig. 1. Space–time diagram (a) and rod (b).

Since $c^2 = E/\rho$, it is seen that:

$$\tilde{\sigma} = E\,\varepsilon. \tag{15}$$

This relation is analogous to Hooke's law, and represents a quasi-static condition.

(iv) *End connected to rod of mis-matched impedance*

In structural applications the top end of the column is connected to structural members that support the rod of the building or the deck of a bridge. These end conditions are not well modelled by any of the three conditions above. A better approximation is to note that stress and velocity at the top of the column are related by an effective impedance, analogous to that of a rod of mis-matched impedance. Thus, suppose that the thin rod is connected at $x = L$ to another thin rod of identical cross section, of infinite length, and of mechanical impedance $\alpha\rho c$ with $\alpha > 0$. In the region of $x > L$, an equation of the form

$$\sigma + \alpha\rho c v = 0 \tag{16}$$

holds along a characteristic line such as the one drawn by dashed-and-dotted line in Fig. 1(a), since no disturbance has reached the infinite region within a finite time. The boundary condition that stress and velocity are continuous at the connection $x = L$ provides the state $n + 1$ with the relation:

$$\sigma_{n+1} + \alpha\rho c v_{n+1} = 0. \tag{17}$$

Substitution of Eqn (17) into Eqns (1) and (2) and elimination of σ_{n+1} and v_{n+1} gives:

$$(\sigma_{n+2} + \sigma_n) + \alpha(\sigma_{n+2} - \sigma_n) = -\rho c\{(v_{n+2} - v_n) + \alpha(v_{n+2} + v_n)\}. \tag{18}$$

This equation relates the values of stress and velocity between the two states n and $n + 2$. For both the stress and the velocity, let half the sum of successive end values be replaced by the mean value, and let the difference of successive end values be replaced by the derivative times the time increment $2T$, as before, to give:

$$\tilde{\sigma} + \alpha T\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}t} = -\rho c\left(\alpha\tilde{v} + T\frac{\mathrm{d}\tilde{v}}{\mathrm{d}t}\right). \tag{19}$$

This differential equation is solved below by excluding the case $\alpha = 0$, which has been treated as case (i). Multiplication by $(1/\alpha T)e^{t/\alpha T}$ for the both sides gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{t/\alpha T}\tilde{\sigma}\right) = -\frac{\rho c}{\alpha T}\left(\alpha\tilde{v} + T\frac{\mathrm{d}\tilde{v}}{\mathrm{d}t}\right)e^{t/\alpha T}. \tag{20}$$

The right hand side of this equation is specified as a function of time; integration by parts provides the relation:

$$\tilde{\sigma}(t) = -\frac{\rho c}{\alpha T}e^{-t/\alpha T}\left\{T\left[\tilde{v}(\tau)e^{\tau/\alpha T}\right]_{\tau=0}^{\tau=t} + \left(\alpha - \frac{1}{\alpha}\right)\int_0^t \tilde{v}(\tau)\,e^{\tau/\alpha T}\mathrm{d}\tau\right\}. \tag{21}$$

It follows that

$$\tilde{\sigma}(t) = -\frac{\rho c}{\alpha}\tilde{v} + \frac{\rho c}{T}\left(\frac{1}{\alpha^2} - 1\right)e^{-t/\alpha T}\int_0^t \tilde{v}(\tau)e^{\tau/\alpha T}\mathrm{d}\tau. \tag{22}$$

This result is checked to coincide with the results obtained earlier in sections (ii) and (iii) for particular cases of $\alpha = 1$, and $\alpha = \infty$, respectively. It should be pointed out that the above results are obtained on the basis of the prescribed velocity, which may be regarded as the outcome of the interaction between the external input at $x = 0$ and the waves reflected in the rod. For earthquake applications, the impedance of the column may be so small in comparison with that of the supporting system of ground and foundation that the motions of the base of a column may be replaced with sufficient accuracy by recorded earthquake motions at nearby locations.

ASYMPTOTIC RELATIONS

The wave-induced stress is governed by impedance ratio α. Cases (i), (ii) and (iii) are special cases of case (iv) corresponding to $\alpha = 0, 1, \infty$, respectively. To make the dependence on α more explicit, it is helpful to obtain approximate evaluations of the integral in Eqn (22). In order to accomplish this, time integration is carried out by change of variables from τ to $\eta \equiv t - \tau$ in Eqn (22), to give:

$$\tilde{\sigma}(t) = -\frac{\rho c}{\alpha}\tilde{v}(t) + \frac{\rho c}{T}\left(\frac{1}{\alpha^2} - 1\right)\int_0^t \tilde{v}(t-\eta)\, e^{-\eta/\alpha T}\, \mathrm{d}\eta. \tag{23}$$

It is noted in executing the integration that the more recent velocity history (i.e. for τ near to t, or η near zero) has the greater effect on the stress. Thus, it means that in the Maclaulin's or Taylor's series expansion

$$\tilde{v}(t-\eta) = \tilde{v}(t) - \tilde{v}'(t)\eta + \frac{1}{2}\tilde{v}''(t)\eta^2 - \frac{1}{6}\tilde{v}'''(t)\eta^3 + \cdots$$

at $\eta = 0$, higher order terms can be neglected, where the primes indicate differentiation with respect to the arguments. The integral in Eqn (23) is approximated to the first order in η as:

$$\int_0^t \tilde{v}(t-\eta)\, e^{-\eta/\alpha T}\, \mathrm{d}\eta \cong \tilde{v}(t)\int_0^t e^{-\eta/\alpha T}\mathrm{d}\eta - \tilde{v}'(t)\int_0^t \eta e^{-\eta/\alpha T}\mathrm{d}\eta$$

$$= \tilde{v}(t)(\alpha T)\left(1 - e^{-t/\alpha T}\right) + \tilde{v}'(t)(\alpha T)\left[t e^{-t/\alpha T} - (\alpha T)\left(1 - e^{-t/\alpha T}\right)\right]. \tag{24}$$

Use has been made of integration by parts in deriving the last equality in the above relationship. Substitution of this integral into Eqn (23) gives:

$$\tilde{\sigma}(t) \cong -\rho c\left[\alpha\left\{1 - \left(1 - \frac{1}{\alpha^2}\right)e^{-t/\alpha T}\right\}\tilde{v}(t) + (1-\alpha^2)\left\{1 - \left(1 + \frac{t}{\alpha T}\right)e^{-t/\alpha T}\right\}T\tilde{v}'(t)\right]. \tag{25}$$

As $\alpha \to 0$, Eqn (25) gives:

$$\tilde{\sigma}(t) \to -\rho c T \tilde{v}'(t), \tag{26}$$

which agrees with Eqn (8). Another useful expression for $t/(\alpha T) \to \infty$ is derived from Eqn (25) to be:

$$\tilde{\sigma}(t) \to -\rho c\left[\alpha\tilde{v}(t) + (1-\alpha^2)\, T\tilde{v}'(t)\right]. \tag{27}$$

To further approximate this relation, comparison is made of the order of magnitude for the two components in terms of their ratio

$$\frac{(1-\alpha^2)\, T\tilde{v}'}{\alpha\tilde{v}} = -\left(1 - \frac{1}{\alpha^2}\right)\frac{\tilde{v}'}{\tilde{v}} = -\left(1 - \frac{1}{\alpha^2}\right)\frac{\mathrm{d}(ln\tilde{v})}{\mathrm{d}(t/\alpha T)} \tag{28}$$

which may be small in comparison with unity, unless the velocity changes at an extremely high rate. Thus, unless α is extremely small or large, the approximation

$$\tilde{\sigma}(t) \cong -\rho c\alpha\tilde{v}(t) \tag{29}$$

may be practical, except for high acceleration and small $t/(\alpha T)$. It may be worth noting that Eqn (29) is analogous to Eqn (16), which holds at end $x = L$.

EXAMPLES

Structural steel has representative values $E = 206\,\mathrm{GPa}$, $\rho = 7850\,\mathrm{kg/m^3}$, and hence $c = 5120\,\mathrm{m/s}$. Concrete has varying moduli and density, but c is of the order of 3000 m/s.

Frequency for the longitudinal wave to travel back and forth along a bar of length L is $1/(2T) = c/(2L)$. For $L = 10$ m, this frequency is approximately 256 Hz for steel, and 150 Hz for concrete. The wave propagation makes at least several thousand round trips along the structural member during an earthquake motion. Thus, the previous replacement of differences by derivatives is justified, although the finite difference equation can be solved just as easily as for the corresponding differential equation. It is also noted that wave interference during a large number of round trips smooths out the stress distribution along the bar axis. It may be taken, therefore, that the stress evaluated for the end $x = 0$ prevails except for regions near the other end and for cases of small α.

Example 1. A trapezoidal velocity pulse is assumed with velocity in the negative direction as shown in Fig. 2(a). The stress variation is determined from Eqn (22) and the results are

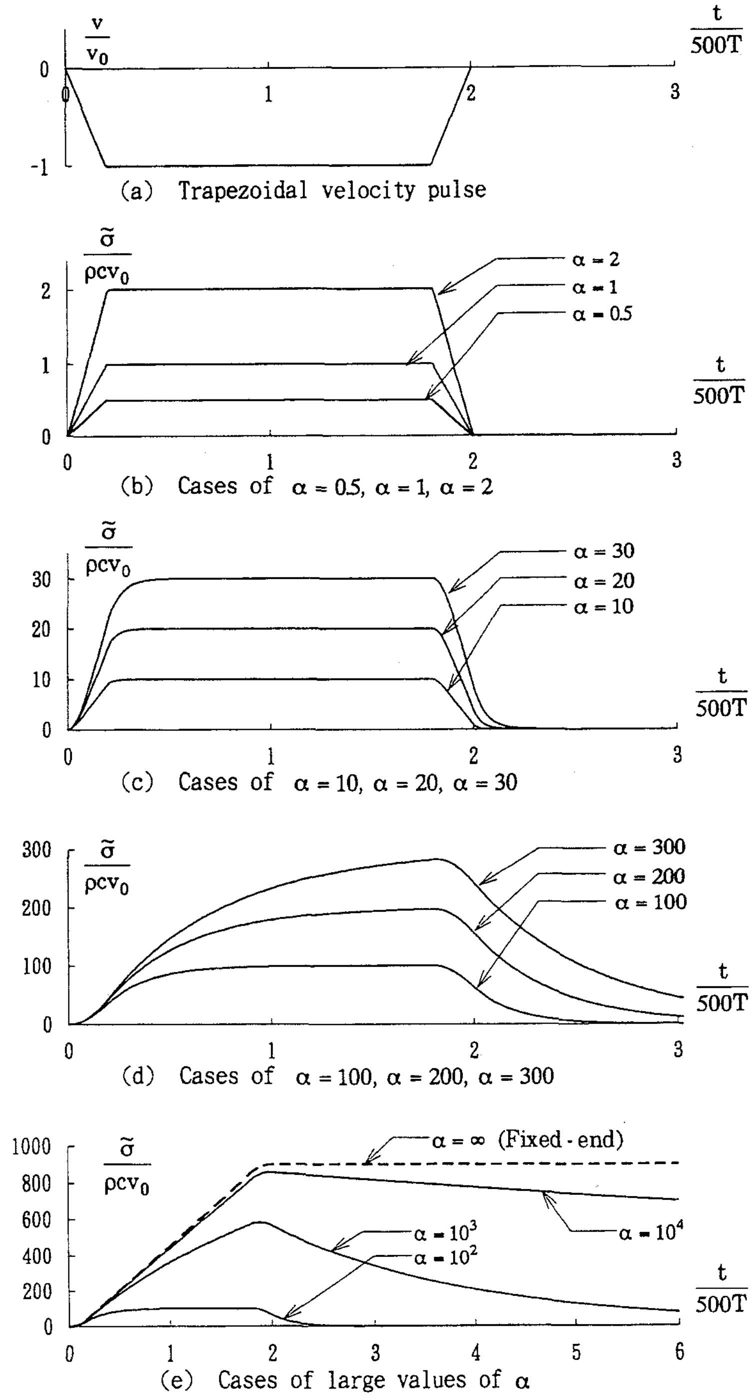

Fig. 2. Trapezoidal velocity pulse and time history of dimensionless stress.

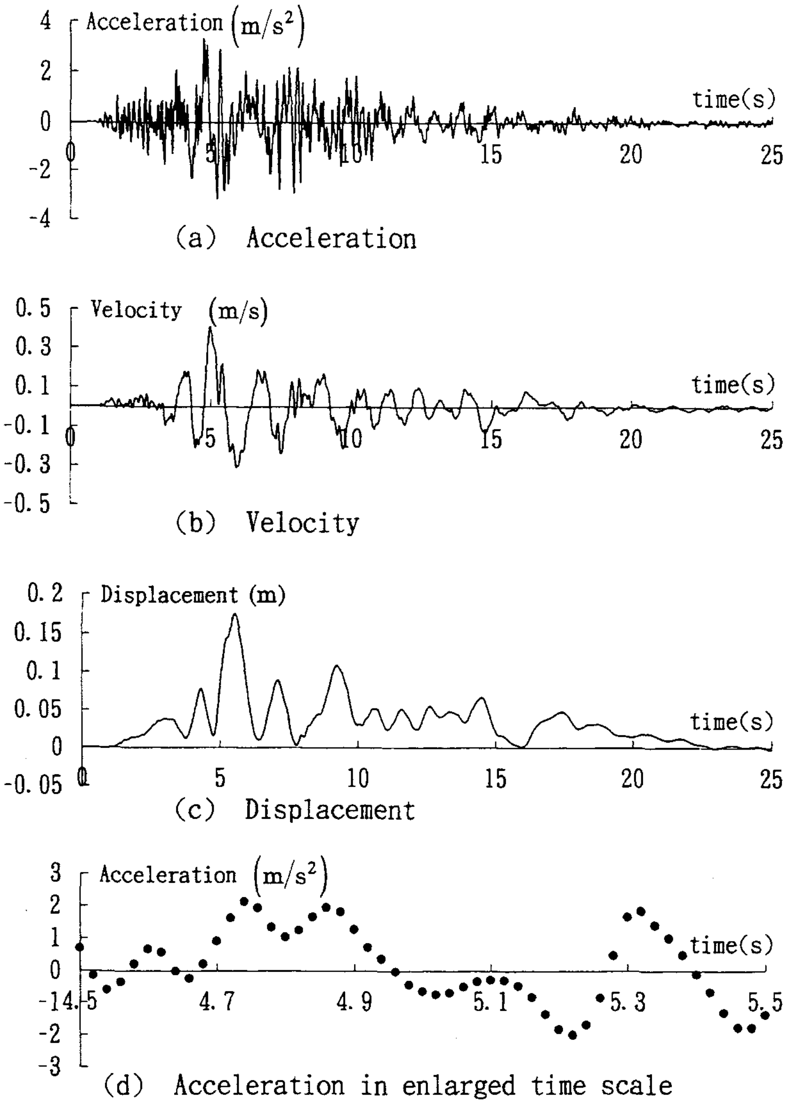

Fig. 3. Up–down ground motion recorded at JMA, Kobe.

shown in dimensionless form for different values of the parameter α in Figs 2(b)–(e); $v_0$is an arbitrary constant velocity. A difference of 2 (e.g. in the abscissa) indicates the time duration for the wave to make 500 round trips. Numerals in the abscissa may, therefore, be roughly regarded as seconds in considering stress waves in a structural member. It is seen, as approximately predicted by Eqn (29), that the stress varies in time in a manner similar to velocity for α of the order of unity, except for narrow regions of sharp velocity change. The stress approaches a peak value proportional to α, except for large values of α, such as $\alpha > 200$. The speed of this approach becomes smaller as α becomes larger. In this example, the magnitude of the prescribed velocity drops after a certain duration, so that for cases of large α, the stress begins to decrease before reaching the peak value that would be attained if the magnitude of the velocity did not drop.

Example 2. The wave-induced stress depends greatly on boundary conditions, which are not clear in most cases of actual structures. In the most severe case of a fixed end, the strain corresponding to that stress is given from Eqn (15) by Hooke's law of linear elasticity. The yield strain of mild steel is at most 0.2%; the crushing strain of concrete may be of the order of 0.3%, unless confined densely by stirrups, hoops or spirals. If an earthquake generates an upward displacement $\delta = 30\,\text{mm}$, e.g. at the foot of a column of length $L = 10\,\text{m}$ then the average strain $\varepsilon = -\delta/L = -0.003$. A reinforced concrete column may crush, and a steel-pipe column may buckle under such a large magnitude of induced strain. A downward motion induces tension, and can cause even more serious damage on a concrete column.

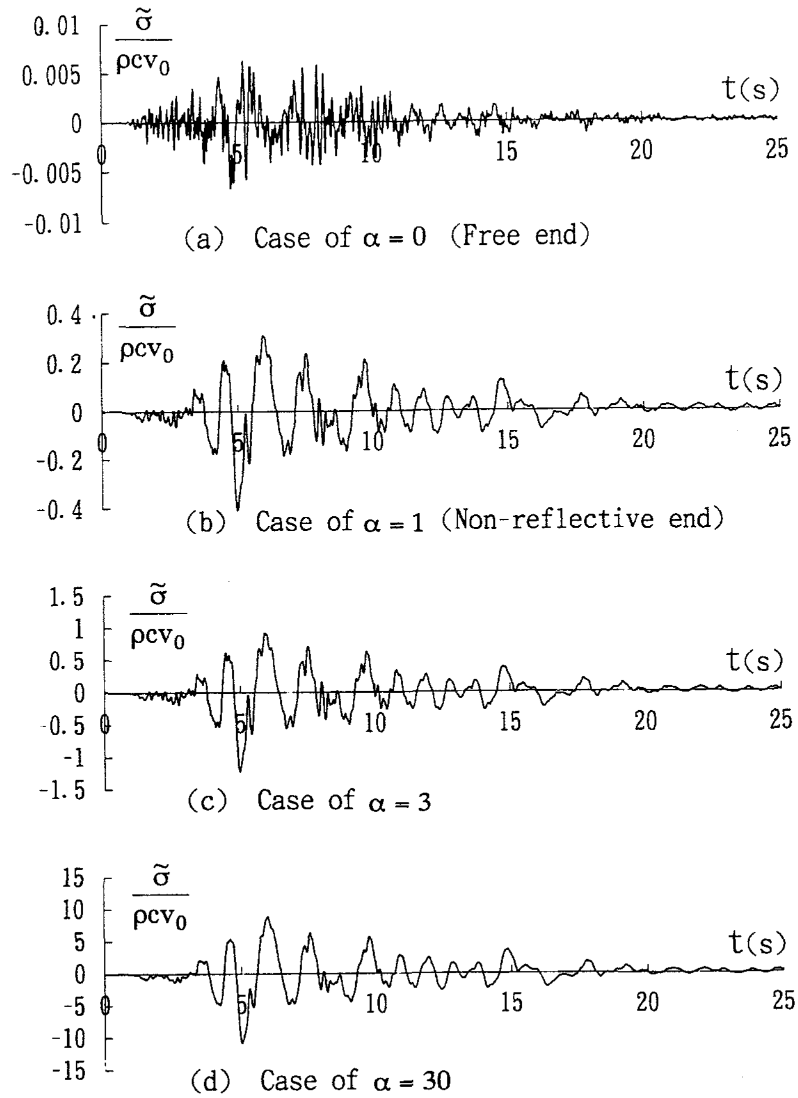

Fig. 4. Time history of dimensionless stress.

Example 3. A realistic situation is considered with reference to the Great Hanshin-Awaji Earthquake. An example of up–down ground motion is shown in Fig. 3; the curve in Fig. 3(a) shows the time history of acceleration for 25s, recorded at the Japan Meteorological Agency, Kobe, just above the source area. The data were obtained with a time interval of 0.02 s. (Plots in Fig. 3(d) show actual data for a second in an enlarged time scale.) Once-integrated velocity variation is shown in Fig. 3(b), with components of frequency less than 0.1 Hz being eliminated through a low-cut-filter. Although less reliable physically, twice integrated displacement variation is shown for reference in Fig. 3(c).

It is supposed that a column is subjected to this up–down motion at its foot. The boundary condition at the top is expressed in terms of α. Numerical analysis is carried out with recourse to Eqn (22) for $T = 0.002$ s. The time history of dimensionless stress is shown in Fig. 4 for various values of α. It is noted that the stress in a reflected pulse is $(\alpha - 1)/(\alpha + 1)$ times that of the incident pulse. This factor of reflection equals 0.5 when $\alpha = 3$. It is seen that as α increases, the stress intensity increases and the curve becomes smoother, or the stress varies with a lesser rate of change. The specific value of $T = 0.002$ s corresponds to $L = 10.24$ m, if $c = 5120$ m/s as for steel, and to $L = 6$ m, if $c = 3000$ m/s as for concrete. The value of stress is determined by multiplying the ordinate by the factor $\rho c v_0$. With $v_0 = 1.0$ m/s, pertinent to the unit of the imposed velocity, this factor is 40.2 MPa for steel. With $\rho = 2400\,\mathrm{kg/m^3}$ for concrete, this factor is 7.2 MPa. It may not be appropriate to make quantitative discussions of the stress level without knowledge of α. Daringly, however, the result that the dimensionless stress

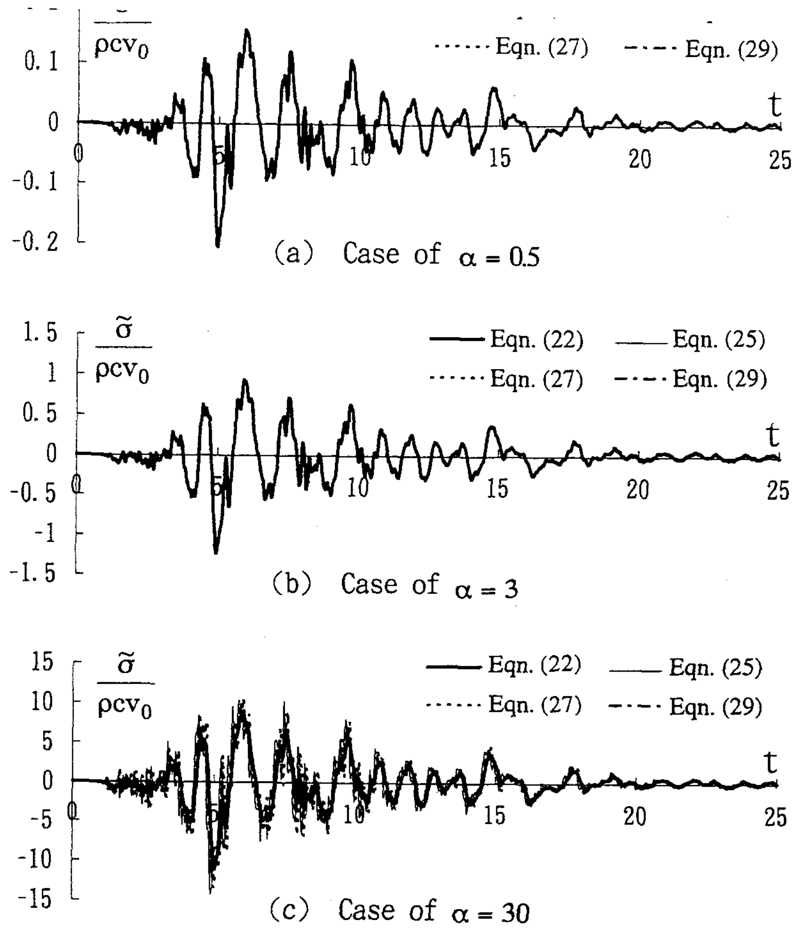

Fig. 5. Comparison between exact and approximate solutions.

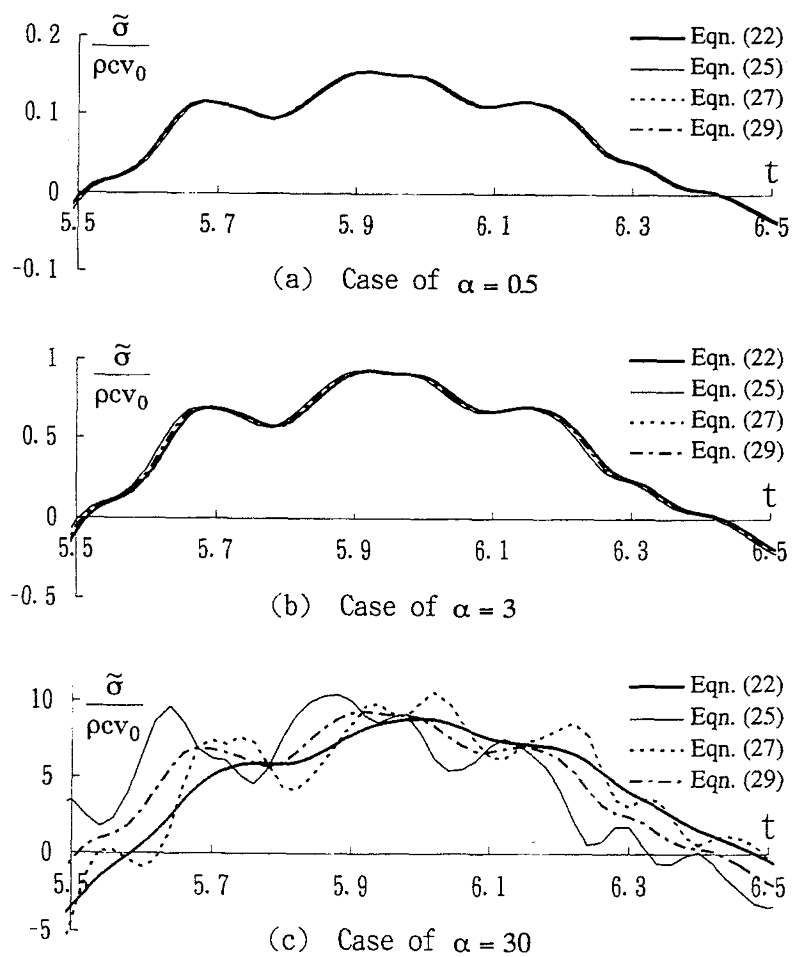

Fig. 6. Comparison in an enlarged scale.

reaches unity for $\alpha = 3$ suggests that a 6 m concerete column cannot help but rupture in tension, disregarding the effect of gravity; the dimensionless stress reaching ± 10 for $\alpha = 30$ suggests that a 10 m steel column can fail both in tension and compression. If the displacement history of Fig. 3(c) were reliable, a fixed-ended column would have undergone an unbearable amount of compressive strain.

In order to examine the accuracy of the afore-derived asymptotic relations, curves are drawn from Eqns (25), (27) and (29) in Fig. 5, and in Fig. 6 for partial enlargement. The solid curves in these figures are drawn from the the basic Eqn (22) for comparison. It is seen that any of these relations are good approximation for α of the order of unity; furthermore these approximations become exact for $\alpha = 1$.

CONCLUSIONS

Longitudinal waves are analyzed for a thin rod on the basis of linear elasticity. Axial velocity is specified at one end as a function of time, as in the case of a column subjected to the up–down motion of an earthquake. The wave-induced stress depends on the boundary condition at the other end, which is assumed to be connected to an infinite rod of effective mechanical impedance α times that of the rod under consideration. The method of characteristics yields Eqn (22) as the basic formula for the stress variation near the loaded end.

It may be expedient to make various degrees of approximation for practical application, considering that actual boundary conditions are often not clear, and the accurate measurement of earthquake motion is difficult to achieve. Unless acceleration $\tilde{v}'(t)$ is large in magnitude, and unless $t/(\alpha T)$ is small, the first order approximation

$$\tilde{\sigma}(t) \cong -\rho c \alpha \tilde{v}(t)$$

may be applied for α of magnitude of the order of unity. For infinitesimal α, the product $\alpha\tilde{v}$ is replaced by $T\mathrm{d}\tilde{v}/\mathrm{d}t$, leading to Eqn (8) for a free end; for infinite α, this product is replaced by $(1/T)\int\tilde{v}\mathrm{d}t$, leading to Eqn (15) for a fixed end.

Although no definite conclusion can be drawn without knowledge of α, there seems to be a fair chance of failure in a steel or concrete column due to longitudinal waves induced by such up–down motion as in the Great Hanshin-Awaji Earthquake.

Acknowledgements — Stimulating discussions and interest by Prof. Shinji Tanimura at the University of Osaka Prefecture are gratefully acknowledged. Much appreciation extends to Mr. Takashi Akazawa of Geo-Research Institute, Osaka, who contributed in providing the Earthquake records.

REFERENCES

1. Preliminary Reconnaissance Report of the 1995 Hyogoken-Nanbu Earthquake. Architectural Institute of Japan (1995).
2. Nonaka, T., Mimura, K., and Tanimura, S., A note on structural failure due to the Great Hanshin Earthquake. *Materials Science Research International*, **1(2)**, 132–133 1995.
3. Cristescu, N., *Dynamic Plasticity*, North-Holland, Amsterdam, Chapter II, 8–100, 1967.