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Analytic solutions for stresses in conical sand heaps piled up with perfect memory

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This paper aims to explain that the closure of fixed principal axes engaged in the physics of sand heaps is too simplistic to characterize stress distributions along the supporting base. New analytic stress solutions based on this closure with Levy's hypothesis on hoop stress in conical sand heaps were rigorously derived. It was found that the analytic stress solutions are in fair agreement with the published experimental data. Moreover, this study indicates that the equations of the original work contain an error and, therefore, the corresponding numeric solutions are not correct. The present study disproves the outcome of the original work and may turn this stress closure from a realistic to an ideal hypothesis. This theoretical finding leads to the conclusion that the memory of sand heaps is no longer perfect.

Key Words: *granular materials, stress distribution, analytic solution*

1. Introduction

Solving mechanics problems using analytic solutions enables us to obtain all stresses at the highest accuracy and to ease the process of verification and validation. Analytic solutions can be explicitly expressed in symbolic expressions, not necessarily in closed-form solutions which are expressed in terms of a bounded number of certain functions and operators. Analytic solutions can be derived for some problems with few unknowns and simple boundary conditions; accordingly, we will find the analytic stress solutions in conical sand heaps under the certain assumptions.

Since the 1900s, numerous researches focused on the counter-intuitive observation of a pressure dip at the base underneath the apex of granular materials. The significance of this puzzling phenomenon was remarked on in a *New Scientist* article by Watson (1991)[1]. He posed an intriguing question: "Which apple is being crushed the most in a pile of apples?" (as depicted in Fig.1). This scientific article was based on the unexpected pattern of stress distribution revealed by Smid & Novosad (1981)[2], Czech scientists who used pressure sensors to measure the stresses acting along the rigid base of granular heaps and reported that it is not the center of the base that feels the greatest downward pressure, but instead a ring of particles a certain distance away from the center. The remarkable appearance of a pressure dip has attracted much curiosity from physicists and mathematicians. Many works devoted to the arching theory using continuum mechanics, as well as computer models using the discrete/distinct element method, were introduced in an attempt to clarify the mechanism of self-weight transmission (e.g. Edwards & Oakeshott (1989)[3], Liffman et al. (1992, 1994)[4, 5], Bouchaud et al. (1995)[6], Hemmingsson (1996)[7]). However they found that this seemingly simple problem is not so easy to tackle. This problem is rooted in a facet of stress indeterminacy and stress history.

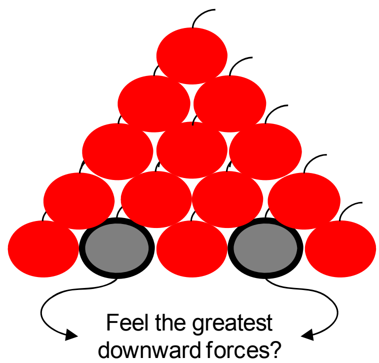

Fig.1 Perplexing pile of apples in a supermarket presented in *New Scientist* (Watson, 1991)

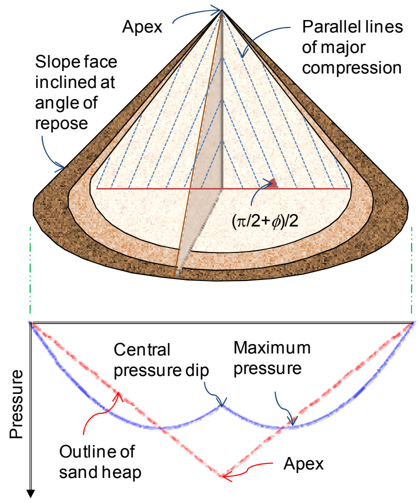

Fig.2 The sand pile puzzle where the greatest pressure is not at the center of the base appeared in *Science* (Watson, 1996)

Among them, the explanation of the pressure dip by Wittmer et al. (1996)[8] emerges as the most outstanding closure of stress relation in a sand heap. This group of physicists from the U.K. and France published their hypothesis on fixed principal axes (FPA) in *Nature* and was given renown in an article in *Science* (Watson, 1996)[9]. They proposed that the major compressive stresses in a pile of sand lie along fixed parallel straight lines buried during deposition, angled in the middle between the angle of repose and the gravitating direction. These stress lines transfer the heap's own weight away from the center, causing a central pressure dip as outlined in Fig.2.

This simple approach can be cast within the existing framework of classical continuum mechanics to suitably reproduce the experimental data reported 15 years before by Smid & Novosad (1981). A detailed explanation of their approach and weight transmission mechanism is given in Wittmer et al. (1997)[10]. The closure of FPA substantially impacts on a wide range of theories in physics and mechanics, looking at the number of citations. Despite acceptance or rejection of the success of their respectable works on sand heaps with perfect memory in the direction of the principal stress, both papers have been later referred to in several papers, up to 200 times, in ISI Journals. Stress distribution in sand heaps became a trendy research topic in the late 1900s and early 2000s (e.g. Bouchaud et al. (1997)[11], Brockbank et al. (1997)[12], Cantelaube & Goddard (1997)[13], Savage (1997, 1998)[14, 15], Cates et al. (1998, 1999)[16, 17], Narayan & Nagel (1999)[18], Vanel et al. (1999)[19], Didwania et al. (2000)[20], Hill & Cox (2000)[21] and Wiesner (2000)[22]).

The authors tackled this problem by linking it with the derivation of coefficient of earth pressure at-rest (Pipatpongsa et al. (2009, 2010)[23, 24]). We found that no article has yet examined the analytic solutions using the FPA closure in conical heap except for numerical solutions appearing in the original work; this work reflects our effort on this derivation.

2. Theoretical Review

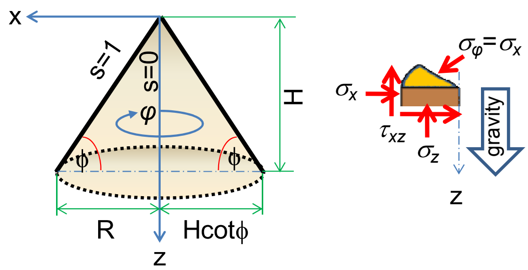

Fig.3 Geometry of a sand heap and the reference cylindrical coordinate system with state of stresses in the positive plane

The basic equations to the problem were briefly reviewed based on the original work of Wittmer et al. (1996, 1997)[8, 10] which were also restated in Bouchaud et al. (1997)[11] and reviewed in Michalowski (2005)[25] and Pipatpongsa et al. (2009)[23]. A particle is considered to be a non-deformable object under their hypothesis. It is beyond the scope of this study to discuss the assumptions used in their analyses but some critical comments against the FPA closure can be found in Savage (1997)[14] and Narayan & Nagel (1999)[18] with corresponding explanations addressed in Cates et al. (1999)[17].

Gravitating granular heaps, poured by noncohesive particles, are formed on the rigid base at their angle of repose. Two common geometries of the heap are planar and conical shapes. Let us concentrate on conical heaps with the geometry shown in Fig.3. A conical sand heap has a rotational symmetry about the central axis; therefore, a cylindrical coordinate system (x,φ,z) is convenient to employ, where x is the radial distance measured from the center line, φ is the angular position measured on the horizontal plane around the central axis and z is the axial position measured from its apex. The state of stresses acting on the positive plane of a sectional element is also shown in Fig.3. Four stress components are horizontal stress σ_x, vertical stress σ_z, shear stress τ_{xz} and hoop stress σ_φ. Considering a conical heap of infinite depth under self-weight loading with constant unit weight γ, all stress components must satisfy the following equilibrium conditions:

$$\partial_x \sigma_x + \partial_z \tau_{xz} + \left(\sigma_x - \sigma_\varphi\right)/x = 0 \qquad (1)$$

$$\partial_x \tau_{xz} + \partial_z \sigma_z + \tau_{xz}/x = \gamma \qquad (2)$$

The original work defined a scaling distance s in Eq.(3) to normalize a half-width base of the heap. Therefore, s=1 refers to the slope face while s=0 refers to the central axis as marked in Fig.3. The spatial derivatives of s are obtained in Eqs.(4)-(5).

$$s = x\tan\phi/z \;,\;\; \partial_x s = \tan\phi/z \;,\;\; \partial_z s = -s/z \qquad (3),\ (4),\ (5)$$

Scaling stresses with respect to a quantity like geostatic pressure γz were also introduced in the original work, equivalent to the radial stress field employed by Sokolovskii (1965)[26] in the problem of planar sand heaps.

$$\sigma_x(x,z) = \gamma z \chi_x(s)\,,\ \sigma_z(x,z) = \gamma z \chi_z(s) \qquad (6),\ (7)$$

$$\tau_{xz}(x,z) = \gamma z \chi_{xz}(s)\,,\ \sigma_\varphi(x,z) = \gamma z \chi_\varphi(s) \qquad (8),\ (9)$$

Therefore, stress fields are assumed to be independently characterized by the depth of interest z and relative location s. Correspondingly, scaling stress variables χ_x, χ_z, χ_{xz} and χ_φ for each stress component are introduced in Eqs.(6)-(9) as a function of s only. Hence, the spatial derivatives of stress required in Eqs.(1)-(2) can be taken:

$$\partial_x\sigma_x = \gamma z \partial_x \chi_x\,,\ \partial_z\sigma_z = \gamma\chi_z + \gamma z \partial_z\chi_z \qquad (10),\ (11)$$

$$\partial_x\tau_{xz} = \gamma z \partial_x \chi_{xz}\,,\ \partial_z\tau_{xz} = \gamma\chi_{xz} + \gamma z \partial_z\chi_{xz} \qquad (12),\ (13)$$

Upon substitution of the above equations into Eqs.(1)-(2), the following system of differential equations is formulated:

$$\begin{Bmatrix} \partial_x\chi_x + \partial_z\chi_{xz} + (\chi_x - \chi_\varphi)\tan\phi/sz \\ \partial_x\chi_{xz} + \partial_z\chi_z + \chi_{xz}\tan\phi/sz \end{Bmatrix} = \frac{1}{z}\begin{Bmatrix} -\chi_{xz} \\ 1-\chi_z \end{Bmatrix} \qquad (14)$$

Partial derivatives of stresses with respect to the rectangular coordinate x and z via the chain rule differentiation can be given, where primes denote derivatives with respect to s:

$$\partial_x\chi_x = \chi'_x\,\partial_x s\,,\ \partial_z\chi_z = \chi'_z\,\partial_z s \qquad (15),\ (16)$$

$$\partial_x\chi_{xz} = \chi'_{xz}\,\partial_x s\,,\ \partial_z\chi_{xz} = \chi'_{xz}\,\partial_z s \qquad (17),\ (18)$$

The substitution of the above equations back into Eq.(14) with minor manipulation yields two equilibrium conditions under scaling stresses.

$$\chi'_x\tan\phi + \chi_{xz} - s\chi'_{xz} + \tan\phi(\chi_x - \chi_\varphi)/s = 0 \qquad (19)$$

$$\chi'_{xz}\tan\phi + \chi_z - s\chi'_z + \chi_{xz}\tan\phi/s = 1 \qquad (20)$$

The above two equations have four unknowns therefore two additional equations are required. One equation is fulfilled by the hypothesis of hoop stress based on Levy (referred to in Sokolovskii (1965)[26], Bouchaud et al. (1995) [6], Wittmer et al.(1996)[8] and Nedderman (2005)[27]).

$$\sigma_\varphi = \sigma_x \ \text{ hence } \ \chi_\varphi = \chi_x \qquad (21)$$

Actually, the contact force depends on how each contact is formed which obviously varies on the shape of the grain; therefore it is noted that, Eq.(21) is valid for uniform materials, but not valid for heterogeneous materials.

Another equation is provided by the closure of FPA given below where the fixed angle of major principal stress (see Fig.2) is $\Psi=(\pi/2+\phi)/2$, simply obtained from Mohr's circle.

$$\tan 2\Psi = \frac{2\tau_{xz}}{\sigma_x - \sigma_z} \ \text{ hence } \ \chi_x = \chi_z - 2\chi_{xz}\tan\phi \qquad (22)$$

The original work also investigated the Haar-von Karman hypothesis of hoop stress, but we will ignore it in this study.

3. Analytical Methods

Since the present analysis is primarily based on continuum mechanics, the analytical stresses would differ to the actual stresses because the effects of grain size distribution, angularity, the properties of individual grains and particle crushing along the base are not considered. These effects to the analysis could be partially resolved by using the appropriate bulk density, $\gamma=\gamma(z/H)$ or $\gamma=\gamma(s)$, as a function of depth or location. However, the present study aims to derive the analytic solution from the same fundamental equations used in the original studies, and the difficulty lies in working out the mathematics.

A set of three dimensionless functions $\chi_x(s)$, $\chi_z(s)$ and $\chi_{xz}(s)$, each determined on a common interval of s, is a solution of the system of first-order linear equations if this set and its derivatives satisfy Eqs.(19)-(22) identically, and satisfy the boundary conditions given at a particular value of s.

3.1 Primary and Auxiliary Boundary Conditions

At the slope face where s=1, all stresses vanish to zero; therefore, all scaling stresses are also zero at this boundary. We regard the following set of equations as the primary boundary conditions for the system of differential equations.

$$\chi_x\big|_{s=1} = 0\,,\ \chi_z\big|_{s=1} = 0\,,\ \chi_{xz}\big|_{s=1} = 0 \qquad (23),\ (24),\ (25)$$

One may notice that the above equations correctly satisfy Eq.(22), so one of them can be ignored. Substituting $\chi_\varphi=\chi_x$ due to Levy's hypothesis, we can rearrange Eqs.(19)-(22) in terms of the first derivative of each scaling stress variable.

$$\chi'_x = (s\chi'_{xz} - \chi_{xz})/\tan\phi \qquad (26)$$

$$\chi'_z = \left(\chi'_{xz}\tan\phi + \chi_z + (\chi_{xz}/s)\tan\phi - 1\right)/s \qquad (27)$$

$$\chi'_{xz} = (\chi'_z - \chi'_x)/(2\tan\phi) = \frac{\chi_{xz}(s^2+\tan^2\phi) + (\chi_z - 1)s\tan\phi}{s\left(s^2 - (1-2s)\tan^2\phi\right)} \qquad (28)$$

Eq.(28) is expanded by substituting χ'_x obtained from Eq.(26) and χ'_z obtained from Eq.(27). Concerning Eqs.(24) and (25), substituting s=1, $\chi_z|_{s=1}$=0 and $\chi_{xz}|_{s=1}$=0 into Eq.(28), the auxiliary boundary condition $\chi'_{xz}|_{s=1}$ results as follows:

$$\chi'_{xz}\big|_{s=1} = -\sin\phi\cos\phi \qquad (29)$$

Substiting of Eq.(29) back into Eqs.(26)-(27) with s=1, using Eqs.(24)-(25) gives other auxiliary boundary conditions.

$$\chi'_x\big|_{s=1} = -\cos^2\phi\,,\ \chi'_z\big|_{s=1} = \cos^2\phi - 2 \qquad (30),\ (31)$$

3.2 Fundamental Form of the Differential Equations

The scaling stress variable χ_{xz} and its derivatives χ'_{xz} appear in Eq.(19). Let us start by solving χ_{xz} from Eq.(19).:

$$\chi_{xz} = s\chi'_{xz} - \chi'_x \tan\phi\,,\ \ \chi''_{xz} = \chi''_x \tan\phi / s \qquad (32),\ (33)$$

Substituting χ_{xz} in Eq.(32) into Eq.(19) will remove χ_{xz} from Eq.(19) with an increasing order of differentiation. Again, χ_z and its derivatives χ'_z appear in Eq.(20). In order to remove χ_z from the original equation, let us solve for χ_z from Eq.(20).

$$\chi_z = 1 - \chi'_{xz} \tan\phi + s\chi'_z - \chi_{xz} \tan\phi / s \qquad (34)$$

Substituting χ_z in Eq.(34) into Eq.(20) can remove χ_z and χ'_z but obtain χ''_z in a similar manner to Eq.(33).

$$\chi'_{xz} = \chi_{xz}/s - s\chi''_{xz} + s^2\chi''_z / \tan\phi \qquad (35)$$

One might observe that the resulting euqtions, Eqs.(33)-(35), can be identically obtained by taking the differentiation according to s, of Eqs.(19)-(20) respectively. Thus, we can regard a further differentiation as a variable removal method which will be generalized later to a systematic technique. We shall remove the term χ''_{xz} from the system of differential equations in Eq.(35) by substituting Eq.(33) into Eq.(35):

$$\chi'_{xz} = \chi_{xz}/s - \chi''_x \tan\phi + s^2\chi''_z / \tan\phi \qquad (36)$$

We find that substitution of χ'_{xz} in Eq.(36) back to Eq.(32) will remove both χ_{xz} and χ'_{xz} from the equations.

$$\chi'_x \tan\phi + s\chi''_x \tan\phi - s^3\chi''_z / \tan\phi = 0 \qquad (37)$$

So far, Eq.(22) has not yet been involved in the system; therefore, we can rearrange Eq.(22) in terms of χ_z as follows:

$$\chi_z = \chi_x + 2\chi_{xz} \tan\phi \qquad (38)$$

Substituting of χ_z from Eq.(38) into Eq.(37) yields the following differential equation:

$$\chi'_x \tan\phi + \left(\tan\phi - s^2/\tan\phi\right) s\chi''_x - 2s^3\chi''_{xz} = 0 \qquad (39)$$

Finally, χ''_{xz} can be removed from Eq.(39) by substituting it with Eq.(33), then the differential equation only in terms of χ_x can be rearranged into Eq.(40):

$$\chi''_x - \chi'_x \sin^2\phi \Big/ \left(s\left(s^2 - (1-s)^2 \sin^2\phi\right)\right) = 0 \qquad (40)$$

Therefore, the variable χ_x is completely separated from the coupled differential equations shown in Eqs.(19)-(22) by suitable steps of substitution and rearrangement. The authors realize that a lengthy manipulation from Eqs.(32)-(40) to seek the substantial form of the differential equations may be tedious and time consuming. As a consequence, the systematic technique of variable separation for a linear system of coupled differential equations is introduced in Appendix A. With this technique, two second-order differential equations for χ_z and χ_{xz}, unconstrained from other variables, can be systematically manipulated as explained in Appendix A.

$$\chi''_{xz} + \frac{(\chi_{xz} - s\chi'_{xz})\sin^2\phi}{s^2\left(s^2 - (1-s)^2\sin^2\phi\right)} = 0 \qquad (41)$$

$$\chi''_z + \frac{\sin^2\phi\left(1 - (1-s)\chi'_z - \chi_z\right)}{s\left(\dfrac{s + (1-s)\sin^2\phi}{s + (2-s)\sin^2\phi}\right)\left(s^2 - (1-s)^2\sin^2\phi\right)} = 0 \qquad (42)$$

Now the ordinary linear differential equations shown in Eqs.(40)-(42) subject to the primary and auxiliary boundary conditions can be solved numerically using either the Runge-Kutta method or Newton's method. However, more advanced handling of these equations can derive analytic solutions which will be presented in the next section.

3.3 Particular Solutions of the Differential Equations

We can see that Eq. (40) and Eq.(41) are homogeneous differential equations while Eq.(42) is a nonhomogeneous equation. Fortunately, Eq.(40) has only χ'_x and χ''_x; χ_x does not appear in the expression. Due to its less complicated terms, we shall start by considering Eq.(40) with the expression rearranged to Eq.(43), so that χ'_x can be simply integrated through the order-reduction method.

$$\frac{d\chi'_x}{ds} = \chi'_x \sin^2\phi \Big/ \left(s\left(s^2 - (1-s)^2\sin^2\phi\right)\right) \qquad (43)$$

It is obvious that Eq.(43) has singular points at s=sinϕ/(1+sinϕ) and s=0; therefore, the function of χ'_x is discontinuous in a domain of s and a singular solution is required in addition to a general solution. We can figure out that χ'_x equaled to a constant value is a trivial solution for Eq.(43). Let us disregard it at this moment and consider a nontrivial solution as a general solution. As long as $\chi'_x \neq 0$, χ'_x can be solved by integrating the following partial fractions, where c_x is a constant of integration and $|C_x|$=exp(c_x) is taken as an equivalent constant:

$$\int \frac{d\chi'_x}{\chi'_x} = \int \left(\begin{array}{l} \dfrac{1}{2}\dfrac{1+\sin\phi}{\left(s - \left(\sin\phi/(1+\sin\phi)\right)\right)} \\ +\dfrac{1}{2}\dfrac{1-\sin\phi}{\left(s + \left(\sin\phi/(1-\sin\phi)\right)\right)} - \dfrac{1}{s} \end{array} \right) ds + c_x$$

$$\ln|\chi'_x| = \left(\begin{array}{l} \ln\left|s - \dfrac{\sin\phi}{1+\sin\phi}\right|^{\frac{1+\sin\phi}{2}} \\ + \ln\left|s + \dfrac{\sin\phi}{1-\sin\phi}\right|^{\frac{1-\sin\phi}{2}} - \ln|s| + \ln|C_x| \end{array} \right) \qquad (44)$$

$$= \ln\left| \frac{C_x\left(s - \frac{\sin\phi}{1+\sin\phi}\right)^{\frac{1+\sin\phi}{2}}}{s\left(s + \frac{\sin\phi}{1-\sin\phi}\right)^{-\frac{1-\sin\phi}{2}}} \right|$$

Owing to the multiplicativeness of the absolute function, χ'_x can be solved from Eq.(44), provided $\chi'_x \neq 0$, s>sinϕ/(1+sinϕ) and $s \neq 0$.

$$\chi'_x = \frac{C_x}{s}\sqrt{\left(s - \frac{\sin\phi}{1+\sin\phi}\right)^{\sin\phi+1} \Big/ \left(s + \frac{\sin\phi}{1-\sin\phi}\right)^{\sin\phi-1}} \qquad (45)$$

The arbitrary constant C_x is determined from the auxiliary boundary condition $\chi'_x|_{s=1}=-\cos^2\phi$, as shown in Eq.(30).

$$C_x = -\left((1+\sin\phi)/\cos\phi\right)^{\sin\phi}\cos^3\phi \tag{46}$$

Consequently, a particular solution of χ_x can be obtained by integrating an integrand χ'_x with the primary boundary condition $\chi_x|_{s=1}=0$, as shown in Eq.(23).

$$\begin{aligned}\chi_x - \chi_x|_{s=1} &= \int_1^s \chi'_x dt \\ \chi_x &= C_x\int_1^s \frac{1}{t}\sqrt{\frac{\left(t-\frac{\sin\phi}{1+\sin\phi}\right)^{\sin\phi+1}}{\left(t+\frac{\sin\phi}{1-\sin\phi}\right)^{\sin\phi-1}}}dt\end{aligned} \tag{47}$$

Eq.(47) cannot be integrated to the closed form or explicit terms of elementary functions due to a term involving a hypergeometric function so we leave it as a symbolic expression which can be solved by means of series methods and numerical methods. In addition, we notice that Eq.(47) gives a real range of χ_x for a domain of $\sin\phi/(1+\sin\phi)\le s\le 1$ due to the appearance of a square root appeared in Eq.(45). This reflects the limited range of a nontrivial solution. For the sake of completeness of the solution in an applied domain of $0\le s\le 1$, a trivial solution in which $\chi'_x=c'_x$=constant is considered to satisfy the other domain of $0\le s\le\sin\phi/(1+\sin\phi)$.

$$\chi'_x|_{0\le s\le\bar{s}} = c'_x \quad \text{where} \quad \bar{s}=\sin\phi/(1+\sin\phi) \tag{48}$$

Despite the singularity arising in the nontrivial solution in Eq.(45) at the point $s=\sin\phi/(1+\sin\phi)$, the one-sided limit for $s\to\sin\phi/(1+\sin\phi)$ from the positive direction and the one-sided limit from the negative direction can be taken:

$$\lim_{s\to\bar{s}^+}\chi'_x = 0\,,\quad \lim_{s\to\bar{s}^-}\chi'_x = c'_x \tag{49), (50}$$

The singularity is removable if the limits of both sides as expressed above, are identical; therefore, we shall continuously link the χ'_x of both trivial and nontrivial solutions by equating Eq.(49) and Eq.(50) so that $c'_x=0$. Since $\chi'_x|_{s=\sin\phi/(1+\sin\phi)}=0$, substituting $c'_x=0$ in Eq.(48) gives Eq.(51).

$$\chi'_x|_{0\le s\le\bar{s}} = \chi'_x|_{s=\bar{s}} = 0 \tag{51}$$

Integrating Eq.(51) according to s results in a constant χ_x throughout a domain of the trivial solution because $\chi'_x=0$. We shall merge χ_x of both trivial and nontrivial solutions to satisfy the condition of stress continuity. In other words, χ_x in Eq.(47) at $s=\sin\phi/(1+\sin\phi)$ provides a new boundary condition for Eq.(51). We can then obtain the trivial solution of χ_x as follows:

$$\chi_x|_{0\le s\le\bar{s}} = \chi_x|_{s=\bar{s}} = C_x\int_1^{\bar{s}}\frac{1}{t}\sqrt{\frac{\left(t-\frac{\sin\phi}{1+\sin\phi}\right)^{\sin\phi+1}}{\left(t+\frac{\sin\phi}{1-\sin\phi}\right)^{\sin\phi-1}}}dt \tag{52}$$

One might notice that the present solution of χ_x satisfies two conditions of tangency applied to Eqs.(50) and (52), where a general (nontrivial) solution is seamlessly connected with a singular (trivial) solution; hence, χ_x is smooth throughout the applied domain of s. As mentioned earlier, Eq.(43) also has the singular point under the apex of sand heaps where $s=0$. This undefined solution is now removed by replacing it with the trivial solution which suggests that χ_x is constant for a domain of $0\le s\le\sin\phi/(1+\sin\phi)$.

Because Eq.(41) is also a homogeneous differential equation, we can follow an identical procedure in solving for χ_{xz}. Rather than manipulating Eq.(41) by repeating long sequences similar to Eq.(44)-(52), we shall seek a solution of χ_{xz} from Eq.(19) based on the derived expression of χ_x for a shorter derivation. We will check the obtained solution with Eq.(41) later because both methods should give the same result. Multiplying Eq.(19) throughout by the integration factor $1/s^2$, provided $s\neq 0$, will arrange the differential equation into a solvable form.

$$\begin{aligned}\chi'_x\tan\phi/s^2 + \chi_{xz}/s^2 - \chi'_{xz}/s &= 0 \\ d\left(\chi_{xz}/s\right)/ds &= \chi'_x\tan\phi/s^2\end{aligned} \tag{53}$$

Eq.(53) can be integrated using the derived form of χ'_x and the primary boundary condition $\chi_{xz}|_{s=1}=0$, as shown in Eq.(25). A particular solution of χ_{xz} cannot be obtained in closed form because an integrand is associated with χ'_x.

$$\begin{aligned}\int_1^s d\left(\chi_{xz}/t\right) &= \tan\phi\int_1^s \chi'_x/t^2\,dt \\ \left(\chi_{xz}/s\right)-\left(\chi_{xz}/s\right)|_{s=1} &= \tan\phi\int_1^s \chi'_x/t^2\,dt \\ \chi_{xz} &= s\tan\phi\int_1^s \chi'_x/t^2\,dt\end{aligned} \tag{54}$$

A particular solution of χ_{xz} also suffers at a singular point because a term involves χ'_x as obtained in Eq.(45). This identically leads Eq.(54) to a nontrivial solution in a domain of $\sin\phi/(1+\sin\phi)\le s\le 1$.

$$\chi_{xz} = C_x s\tan\phi\int_1^s\frac{1}{t^3}\sqrt{\frac{\left(t-\frac{\sin\phi}{1+\sin\phi}\right)^{\sin\phi+1}}{\left(t+\frac{\sin\phi}{1-\sin\phi}\right)^{\sin\phi-1}}}dt \tag{55}$$

We can find a trivial solution of χ_{xz} in a domain of $0\le s\le\sin\phi/(1+\sin\phi)$ by considering Eq.(19) again with $\chi'_x=0$ according to Eq.(51). We can determine χ_{xz} as follows and find that χ_{xz} is linearly dependent with s where c_{xz} is a constant of integration and $|C_{xz}|=\exp(c_{xz})$ is taken as an equivalent constant.

$$\begin{aligned}\chi_{xz} - s\chi'_{xz} &= 0 \\ d\chi_{xz}/\chi_{xz} &= ds/s \\ \int 1/\chi_{xz}\,d\chi_{xz} &= \int 1/s\,ds + c_{xz} \\ \ln|\chi_{xz}| &= \ln|s| + \ln|C_{xz}| \\ \chi_{xz} &= C_{xz}s\end{aligned} \tag{56}$$

Consequently, the following conditions are required to merge nontrivial and trivial solutions at $s=\sin\phi/(1+\sin\phi)$, so that the profile of χ_{xz} along s is smooth.

$$\lim_{s\to\bar{s}^-} \chi_{xz}\big|_{0\le s\le \bar{s}} = \lim_{s\to\bar{s}^+} \chi_{xz}\big|_{\bar{s}<s\le 1} \tag{57}$$

$$\lim_{s\to\bar{s}^-} \chi'_{xz}\big|_{0\le s\le \bar{s}} = \lim_{s\to\bar{s}^+} \chi'_{xz}\big|_{\bar{s}<s\le 1} \tag{58}$$

Using Eqs.(54) and (56), C_{xz} can be solved from Eq.(57):

$$\begin{aligned} C_{xz} &= \tan\phi \int_1^{\bar{s}} \chi'_x / t^2\, dt \\ &= C_x \tan\phi \int_1^{\bar{s}} \frac{1}{t^3} \sqrt{\frac{\left(t - \frac{\sin\phi}{1+\sin\phi}\right)^{\sin\phi+1}}{\left(t + \frac{\sin\phi}{1-\sin\phi}\right)^{\sin\phi-1}}}\, dt \end{aligned} \tag{59}$$

Therefore, Eq.(56) can be expressed by the following equation.

$$\chi_{xz}\big|_{0\le s\le \bar{s}} = s\tan\phi \int_1^{\bar{s}} \chi'_x / t^2\, dt = \left(s/\bar{s}\right)\left(\chi_{xz}\big|_{s=\bar{s}}\right) \tag{60}$$

Let us find χ'_{xz} in a nontrivial range by differentiating Eq.(54):

$$\chi'_{xz} = \tan\phi\left(\chi'_x / s + \int_1^s \chi'_x / t^2\, dt\right) \tag{61}$$

From the above equation, $\chi'_{xz}|_{s=\sin\phi/(1+\sin\phi)} = C_{xz}$ because $\chi'_x|_{s=\sin\phi/(1+\sin\phi)} = 0$ due to Eq.(51). Comparison with χ'_{xz} in a trivial range obtained by differentiating Eq.(56) satisfies Eq.(58); hence, two conditions of tangency are satisfied, confirming χ_{xz} is smooth throughout the applied domain of s.

Let us check the nontrivial solution of χ_{xz} with Eq.(41). The further differentiation of Eq.(61) can obtain χ''_{xz}, using Eq.(40).

$$\chi''_{xz} = \tan\phi \frac{\chi''_x}{s} = \frac{\tan\phi \sin^2\phi}{s^2\left(s^2 - (1-s)^2 \sin^2\phi\right)} \chi'_x \tag{62}$$

By substituting Eq.(62), Eq.(61) and Eq.(54) for χ''_{xz}, χ'_{xz} and χ_{xz}, respectively, we find that Eq.(41) is reduced to zero, satisfying the system of equation. Similarly, Eq.(41), with the substitution of the trivial solution of χ_{xz} based on Eq.(56), also satisfies the system of equations. Finally, the particular solution of χ_z can be simply rearranged from Eq.(22), using Eq.(47) and Eq.(54) for the nontrivial and trivial solutions using Eq.(52) and Eq.(56). Both solutions of χ_z certainly satisfy two conditions of tangency, so χ_z is smooth throughout the applied domain of s. Also, we can check that both non-trivial and trivial solutions of χ_z automatically satisfy Eq.(42).

Thus far, the solutions of the differential equations rooted in the system of equation are analytically derived in symbolic terms of the definite integral. For convenience, the order of integral range will be changed because the upper limit s is less than the lower limit. Defining $f(t)$ in an interval $s\le t\le 1$, we can reorder the ranges of limit in the following ways:

$$\int_s^1 f(t)dt = -\int_1^s f(t)dt \quad \text{hence} \quad \frac{d}{ds}\int_s^1 f(t)dt = -f(s) \tag{63}$$

$$\int_s^1 f(t)dt = \int_s^{\bar{s}} f_{0\le s\le \bar{s}}(t)dt + \int_{\bar{s}}^1 f_{\bar{s}\le s\le 1}(t)dt \tag{64}$$

Furthermore, we will encapsulate terms of symbolic integration to particular functions in the next section.

4. Numerical Results

Consistent with Eqs.(45)-(46), we can write $\chi'_x = -I(s)s^2\cos^2\phi$ by introducing a particular function $I(s)$ with the expression:

$$\begin{aligned} I(s) &= \frac{\left(\frac{1+\sin\phi}{\cos\phi}\right)^{\sin\phi}\cos\phi}{s^3} \sqrt{\frac{\left(s - \frac{\sin\phi}{1+\sin\phi}\right)^{\sin\phi+1}}{\left(s + \frac{\sin\phi}{1-\sin\phi}\right)^{\sin\phi-1}}} \\ &= \frac{1}{s^3}\sqrt{\frac{\left(s-(1-s)\sin\phi\right)^{\sin\phi+1}}{\left(s+(1-s)\sin\phi\right)^{\sin\phi-1}}} \end{aligned} \tag{65}$$

It is clear that $\chi'_x|_{s=1} = -\cos^2\phi$ satisfies the auxiliary boundary condition since $I(1)=1$. Next, we can define functions $A(s)$ and $J(s)$ as the area integral and second moment of area of the function $I(s)$ by the following equations:

$$A(s) = \int_s^1 I(t)dt\,, \quad J(s) = \int_s^1 t^2 I(t)\, dt \tag{66), (67}$$

Both functions $A(s)$ and $J(s)$ are defined in symbolic expressions due to the difficulty of finding the closed-form integration for the given integrands. Nonetheless, we can simply approximate the definite integrals by numerical integration using quadrature rules. For illustration purposes, using $\phi=30^{\circ}$, the curve of $I(s)$ is shown in Fig.4 where the real range of $I(s)$ can be plotted for $s\ge \sin\phi/(1+\sin\phi)$. The area integral $A(s)$ and the second moment of area $J(s)$ under the curve $I(s)$ delimited by $s=1$ are shown in Fig.5 and Fig.6, respectively, in the applied domain of s between $\sin\phi/(1+\sin\phi)$ and 1. The dependence of $J(s)$ on $I(s)$ and $A(s)$ is further considered in Appendix B. Typically, $I(s)$, $A(s)$ and $J(s)$ are positive in a domain of $\sin\phi/(1+\sin\phi)\le s\le 1$, but these functions are undefined in a domain of $0\le s<\sin\phi/(1+\sin\phi)$.

According to Eq.(66), we can combine Eq.(54) and Eq.(60) into a conditional function χ_{xz} in the applied domain of $s=x\tan\phi/z$. It is clear that $\chi_{xz}|_{s=1}=0$ satisfies the primary boundary condition since $A(1)=0$. And in addition, $\chi_{xz}|_{s=0}=0$ satisfies the condition of zero shear stress along the center line.

$$\chi_{xz} = \frac{\tau_{xz}}{\gamma z} = \begin{cases} sA(\bar{s})\sin\phi\cos\phi & \text{if } 0\le s\le \bar{s} \\ sA(s)\sin\phi\cos\phi & \text{if } \bar{s}\le s\le 1 \end{cases} \tag{68}$$

In the same manner, we can combine Eq.(47) and Eq.(52) according to Eq.(67), into a conditional function χ_x in the applied domain of s. It is clear that $\chi_x|_{s=1}=0$ satisfies the primary boundary condition since $J(1)=0$.

$$\chi_x = \frac{\sigma_x}{\gamma z} = \begin{cases} J(\bar{s})\cos^2\phi & \text{if } 0\le s\le \bar{s} \\ J(s)\cos^2\phi & \text{if } \bar{s}\le s\le 1 \end{cases} \tag{69}$$

Finally, we can formulate a conditional function χ_z according to Eq.(22), using Eqs.(68)-(69):

$$\chi_z = \frac{\sigma_z}{\gamma z} = \begin{cases} J(\bar{s})\cos^2\phi + 2sA(\bar{s})\sin^2\phi & \text{if } 0\le s\le \bar{s} \\ J(s)\cos^2\phi + 2sA(s)\sin^2\phi & \text{if } \bar{s}\le s\le 1 \end{cases} \tag{70}$$

It is noted that a friction angle ϕ is assumed to be uniform in this study, however, ϕ depends on the position and orientation of particles which is related to construction history. Minimal contact between particles results in less friction; hence, sands along the slope boundary have a smaller friction angle comparison to the other parts. The appropriate $\phi=\phi(s)$, which is considered as a function of position (Michalowski, 2004)[28], can be selected to enhance the present analytical method for investigating a variety of construction sequences of a sand heap.

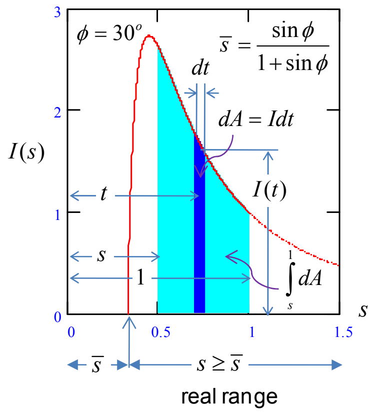

Fig.4 Real range of a function $I(s)$ using ϕ=30° for $s \ge \sin\phi/(1+\sin\phi)$, where s refers to a scaling distance limited in the interval from 0 to 1

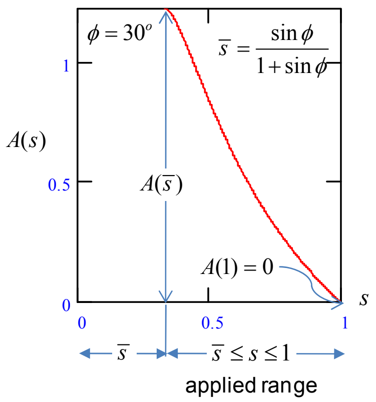

Fig.5 Area integral $A(s)$ taken from s to 1 under the curve $I(s)$ in the domain of $\sin\phi/(1+\sin\phi) \le s \le 1$, where ϕ=30°

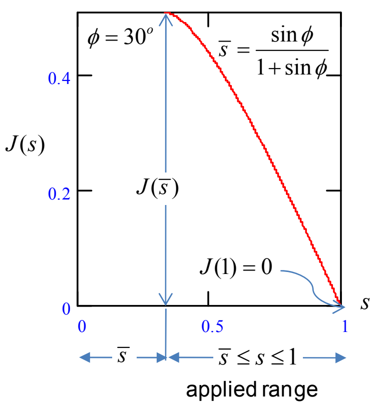

Fig.6 Second moment of area $J(s)$ taken from s to 1 under the curve $I(s)$ in the domain of $\sin\phi/(1+\sin\phi) \le s \le 1$, where ϕ=30°

5. Verification and Validation

Analytic solutions for scaling stress variables χ_x, χ_z and χ_{xz} derived in Eqs.(68)-(70) basically satisfy the equilibrium conditions, as proven in Appendix B. Moreover, the analytic solutions are verified with numerical results using Newton's method and validated with published experimental data.

5.1 Verification by Newton's method

Following the numerical approach suggested in Pipatpongsa et al. (2008)[29], the solutions for scaling stress variables are determined by the mid-point scheme. The scaling distance s, ranging from 0 to 1, is discretized into 20 equal intervals. Appropriate initial guessed values are trialed until the solutions are converged. The numerical solutions for χ_x, χ_z and χ_{xz} are compared with those obtained from analytic solutions. As shown in Fig.7, stress distributions in a half-width section of a conical heap with ϕ=33° have been correctly verified.

5.2 Validation with published experimental data

Smid & Novosad (1981)[2] measured the vertical and shear stresses underneath the base of conical heaps during the pouring of quartz sand and granulated NPK-1 fertilizer from a bunker at a constant rate onto the 2×2 m^2 rigid steel plates. The platform was equipped with 13 pressure cells which can simultaneously and independently measure the normal and shear components of the acting stresses. Their experiments showed that under the apex of the heap, the vertical pressure attains a significant depression and the shear stress at the center is almost zero due to the balance of the frictional forces. The final heights and the averaged radii were measured after the formation. The final shape of the heaps gave the angles of repose: 32.6° for sand and the fertilizer was 33.7° for. The average bulk weight of the sand was 1567 kg/m^3 and that of fertilizer was 1054 kg/m^3. Measurements were recorded for six stages of deposition, when the apex reached a certain heights of H, at 0.2, 0.3, 0.4, 0.5, 0.55 and 0.6 meters.

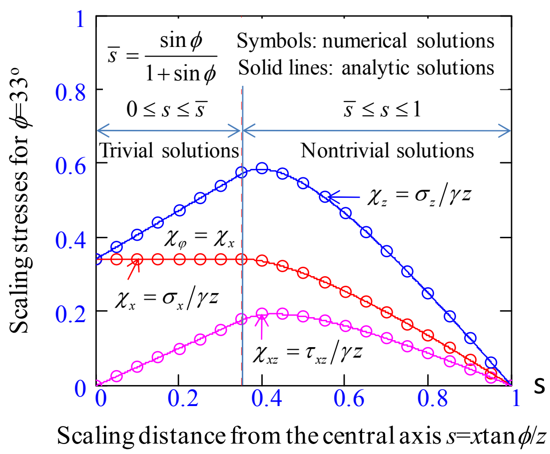

Fig.7 Verification of analytic to numerical solutions under the FPA closure with Levy's hypothesis on a conical heap

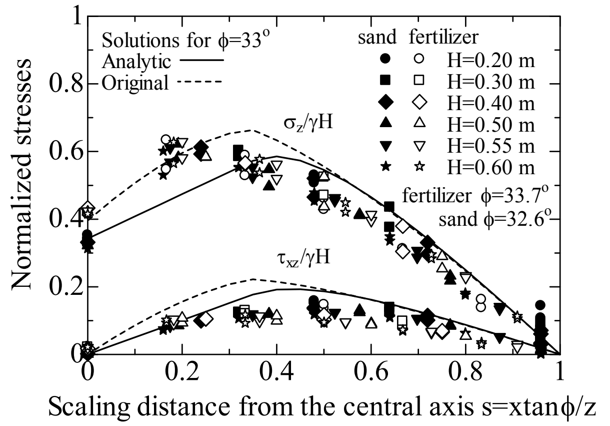

Fig. 8 Comparisons between the experimental data and the theory based on the closure of FPA with Levy's assumption

Wittmer et al. (1996)[8] presented these data featuring normalized stresses $\sigma_z/\gamma H$ and $\tau_{xz}/\gamma H$ versus normalized half-width distance $x\tan\phi/H$, as shown in Fig. 8. One can observe that the shapes of normalized stress distributions do not tend to change with the height of the heap, for both sand and fertilizer, even though the heights of the heap reached differences of up to to three times; therefore, the results of their investigation appear to justify the scaling stress hypothesis employed in the analysis. There are no experimental results on horizontal stress; therefore, normalized stresses $\sigma_z/\gamma H$ and $\tau_{xz}/\gamma H$ obtained from the measurement are compared with χ_z and χ_{zx} obtained from the analytic solutions under the closure of FPA with Levy's assumption, using ϕ=33° as approximation for sand and fertilizer. Despite the original work reporting an acceptable result with experimental data, we found significant differences near the apex of the heap. Under Levy's hypothesis, the numerical results reported by Wittmer et al.[8] are superimposed to our analytic solutions. We observed that our solutions and Wittmer et al.'s solutions are obviously different near the apex but almost the same near the toe of the heaps. Even though our analytic solutions passed verification by both algebraic and numerical methods, this surprising difference leads us to verify our solutions again by the weight-balanced condition. The volume, weight and radius of the conical heaps, as well as the vertical reaction, were obtained as follow.

$$V = \pi R^2 H/3\,,\ W = \gamma V\,,\ R = H\cot\phi \qquad (71),\ (72),\ (73)$$

$$F_z = \int_0^R \sigma_z \frac{d\left(\pi x^2\right)}{dx}dx = 2\pi\gamma H^3 \cot^2\phi \int_0^1 \chi_z\left(s\right)sds \qquad (74)$$

where $x = sH\cot\phi$ according to Eq.(3) using $z=H$

The error determined by weight balance $(F_z-W)/W$ is used to verify the results. We found there is no error in our solution but the numerical results reported in Wittmer et al.[8, 10] contain an error of about 5.1%. Thus, the original work's result is not accurate as it gives the force to be larger than the weight; $F_z>W$.

6. Discussion of Error in the Original Work

The error found in the original work clearly points out that its numerical solutions do not strictly satisfy the equilibrium conditions. Though the original work contains a slight error of about 5.1%, this error obviously appeared near the center line because the area of the inner ring about the center is comparatively smaller than that of the outer ring. We found that the predicted pressure distribution of the original work is somewhat overestimated. Moreover, the vertical pressure profile bowed towards the center should not be a curve but a straight line due to the trivial solution described in Eq.(70).

The authors found that the error came from the equilibrium equations, in terms of scaling variables, because Wittmer et al. (1997)[10] wrongly derived these equations through Eq.(75) instead of the correctly derived Eqs.(19)-(20).

$$\begin{Bmatrix} \chi'_x \tan\phi + \chi_{xz} - s\chi'_{xz} + \chi_x - \chi_\varphi \\ \chi'_{xz}\tan\phi + \chi_z - s\chi'_z + \chi_{xz} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \qquad (75)$$

We believe that such an error is due neither to a kind of typo nor to numerical inaccuracy because the original work really employed Eq.(75) in their analyses (Wittmer et al.[8, 10]). We solved the stress distributions with the Newton's method by employing Eq.(75) and can reproduce the same results reported in the original work as shown in Fig. 8. This outcome reflects the inferiority of numerical solutions to analytic solutions in the absence of verification. The analytic solutions limited to planar heaps were confirmed by different groups of researchers using different approaches (i.e. Cantelaube & Goddard (1997)[13] and Didwania et al. (2000)[20]) because the geometry of a wedge is simpler than that of a cone; therefore, this study does not disprove the closure of FPA but points out the erroneous equations for a conical shape. Despite wrong solutions in the original work, the closure of FPA is still influential in describing the phenomenon of pressure dips. Many systematic experiments were encouraged (e.g. Brockbank et al. (1997)[12], Vanel et al. (1999)[19]) to examine the phenomenon. Current researches into the simulation of sand heaps still address the closure of FPA (e.g. Tejchman & Wu (2008)[30]).

7. Conclusion

New analytic stress distributions in conical sand heaps under the closure of FPA with Levy's hypothesis on hoop stress were derived by introducing the particular symbolic functions. The study covered the technique to separate variables from a differential equation system. The obtained solutions were algebraically and numerically proved. In addition, an error in the original work was identified. The inconsistency with experimental data leads to the conclusion that the memory of sand heaps is no longer perfect upon having principal axes fixed. One might turn the FPA closure into an ideal hypothesis.

Appendices

Appendix A: Separation of variables in a linear system of differential equations

The usual method (e.g. the standard textbook[31)]) for solving a linear system of differential equations is to eliminate variables until we obtain the single linear equation, similar to the way we reduce variables in linear algebra. Let us define a differential operator $D^n=d^n/ds^n$ where n denotes the order of differentiation. Hence, Eqs.(19)-(22) can be rewritten into the following system of equations which are equaled to zero.

$$E_1(s)=\tan\phi D\chi_x+(-sD+1)\chi_{xz} \tag{76}$$

$$E_2(s)=(-sD+1)\chi_z+\tan\phi(D+1/s)\chi_{xz}-1 \tag{77}$$

$$E_3(s)=\chi_x-\chi_z+2\chi_{xz}\tan\phi \tag{78}$$

The coefficients of χ_x, χ_z and χ_{xz} are polynomial operators in the above system of three linear differential equations. We can see that these coefficients are not constant due to the appearance of s; therefore, the method of reducing the equation using an equivalent triangular system would be unfavorable because mathematical skill with trial and error is required. We will show that a linear system of differential equations with variable coefficient can be reduced with less effort by means of the variable separation technique.

This systematic process begins by determining the determinant of the system. We can find from Eq.(79) that the determinant is of the second order in this problem. Hence, the general solution of the system can be achieved and must contain only two arbitrary constants.

$$\begin{vmatrix}\tan\phi D & 0 & -sD+1\\ 0 & -sD+1 & \tan\phi(D+1/s)\\ 1 & -1 & 2\tan\phi\end{vmatrix} =\begin{pmatrix}\left((1-2s)\tan^2\phi-s^2\right)D^2\\ +\left((2+1/s)\tan^2\phi+2s\right)D-1\end{pmatrix}\neq 0 \tag{79}$$

Therefore, the order of the differential equations E_1, E_2 and E_3 is raised to the second order by further differentiation. Let us add these higher-order equations, which are also equaled to zero, to the system of differential equations.

$$E_4(s)=dE_1/ds=\tan\phi D^2\chi_x-sD^2\chi_{xz} \tag{80}$$

$$\begin{aligned}E_5(s)&=dE_2/ds\\&=-sD^2\chi_z+\tan\phi\left(D^2+s^{-1}D-s^{-2}\right)\chi_{xz}\end{aligned} \tag{81}$$

$$E_6(s)=dE_3/ds=D\chi_x-D\chi_z+2\tan\phi D\chi_{xz} \tag{82}$$

$$E_7(s)=d^2E_1/ds^2=\tan\phi D^3\chi_x-\left(sD^3+D^2\right)\chi_{xz} \tag{83}$$

$$\begin{aligned}E_8(s)&=d^2E_2/ds^2\\&=\begin{pmatrix}-\left(sD^3+D^2\right)\chi_z+\\ \tan\phi\left(D^3+s^{-1}D^2-2s^{-2}D+2s^{-3}\right)\chi_{xz}\end{pmatrix}\end{aligned} \tag{84}$$

$$E_9(s)=d^2E_3/ds^2=D^2\chi_x-D^2\chi_z+2\tan\phi D^2\chi_{xz} \tag{85}$$

Eqs.(83) and (84) can be ignored because the highest order of differential equations is three, which is greater than the required order. However, we will leave these equations to show that they do not need to be considered. Now, a linear combination $\Xi(s)$ is defined as a summation of $E_i(s)$, each multiplied by an arbitrary coefficient a_i where i=1 to 9. Since all of E_i=0, $\Xi(s)$ is also zero. Let us rearrange $\Xi(s)$ in terms of polynomial operators.

$$\begin{aligned}\Xi(s)&=\sum_{i=1}^{9}a_iE_i(s)\\&=-a_2+A(\chi_x)+B(\chi_z)+C(\chi_{xz})\end{aligned} \tag{86}$$

The polynomial operators A, B and C contain differential terms involving χ_x, χ_z and χ_{xz} as described respectively below.

$$A(\chi_x)=\begin{pmatrix}(a_7\tan\phi)D^3+(a_4\tan\phi+a_9)D^2\\ +(a_1\tan\phi+a_6)D+a_3\end{pmatrix}\chi_x \tag{87}$$

$$B(\chi_z)=\begin{pmatrix}-(a_8s)D^3+(-a_5s-a_8-a_9)D^2\\ +(-a_2s-a_6)D+(a_2-a_3)\end{pmatrix}\chi_z \tag{88}$$

$$C(\chi_{xz})=\begin{pmatrix}(-a_7s+a_8\tan\phi)D^3\\ +\begin{pmatrix}-a_4s+a_5\tan\phi-a_7+\\ a_8s^{-1}\tan\phi+2a_9\tan\phi\end{pmatrix}D^2\\ +\begin{pmatrix}-a_1s+a_2\tan\phi+a_5s^{-1}\tan\phi\\ +2a_6\tan\phi-2a_8s^{-2}\tan\phi\end{pmatrix}D\\ +\begin{pmatrix}a_1+a_2s^{-1}\tan\phi+2a_3\tan\phi\\ -a_5s^{-2}\tan\phi+2a_8s^{-3}\tan\phi\end{pmatrix}\end{pmatrix}\chi_{xz} \tag{89}$$

The coefficients a_i are superfluous constants of combination which will be imposed corresponding to the preferred separation of variables. A certain combination in $\Xi(s)$ can be freed of χ_x and χ_z if the polynomial operators A and B are zero. We can satisfy this requirement by imposing the coefficients of each differential operator to zero respectively as given below.

$$\begin{Bmatrix}a_7\tan\phi\\ a_4\tan\phi+a_9\\ a_1\tan\phi+a_6\\ a_3\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix},\quad \begin{Bmatrix}-a_8s\\ -a_5s-a_8-a_9\\ -a_2s-a_6\\ a_2-a_3\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix} \quad (90),\ (91)$$

Simultaneously solving the algebraic systems of the eight equations shown in Eqs.(90)-(91) with the nine unknowns for a_1 to a_9, we can calculate the required constants in proportion to a_4, provided that $a_4\neq0$.

$$\begin{cases} a_1 = a_2 = a_3 = a_6 = a_7 = a_8 = 0 \\ a_5 = a_4 \tan\phi/s \\ a_9 = -a_4 \tan\phi \end{cases} \tag{92}$$

Substituting the coefficients obtained from Eq.(92) into Eq.(86), we obtain a certain combination of $\Xi(s)$ relating to χ_{xz}.

$$\Xi(s) = \frac{-a_4}{s^3}\begin{pmatrix} s^2\left(s^2-(1-2s)\tan^2\phi\right)\chi''_{xz} \\ -\left(s\tan^2\phi\right)\chi'_{xz}+\left(\tan^2\phi\right)\chi_{xz} \end{pmatrix} \tag{93}$$

Because $\Xi(s)=0$ and $a_4 \neq 0$, an arbitrary a_4 can be removed from Eq.(93) and rearranged to the standard form of a homogeneous second-order ordinary differential equation (ODE), which is undefined at $s=0$ and $s=\sin\phi/(1+\sin\phi)$.

$$\chi''_{xz} - \frac{\sin^2\phi}{s\left(s^2-(1-s)^2\sin^2\phi\right)}\left(\chi'_{xz}-\frac{\chi_{xz}}{s}\right)=0 \tag{94}$$

Likewise, a certain combination in $\Xi(s)$ can be freed of χ_x and χ_{xz} if the polynomial operators A and C are zero. We can satisfy this requirement by imposing the coefficients of each differential operator to zero as given below for $C(\chi_{xz})=0$ (the required condition for $A(\chi_x)=0$ is already given in Eq.(90)).

$$\begin{Bmatrix} -a_7 s + a_8\tan\phi \\ \begin{pmatrix} -a_4 s + a_5\tan\phi - a_7 \\ +a_8\tan\phi/s + 2a_9\tan\phi \end{pmatrix} \\ \begin{pmatrix} -a_1 s + a_2\tan\phi + a_5\tan\phi/s \\ +2a_6\tan\phi - 2a_8\tan\phi/s^2 \end{pmatrix} \\ \begin{pmatrix} a_1 + a_2\tan\phi/s + 2a_3\tan\phi \\ -a_5\tan\phi/s^2 + 2a_8\tan\phi/s^3 \end{pmatrix} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{95}$$

Simultaneously solving the algebraic systems of the eight equations in Eqs.(90) and (95) with the nine unknowns for a_1 to a_9, we can get the required constants in proportion to $a_4 \neq 0$.

$$\begin{cases} a_3 = a_7 = a_8 = 0 \\ a_1 = a_4\left(2\tan^2\phi+s\right)/\left(s\left(\tan^2\phi+s\right)\right) \\ a_2 = a_4\tan\phi\left(2\tan^2\phi+s\right)/\left(s\left(\tan^2\phi+s\right)\right) \\ a_5 = a_4\left(2\tan^2\phi+s\right)/\tan\phi \\ a_6 = -a_4\tan\phi\left(2\tan^2\phi+s\right)/\left(s\left(\tan^2\phi+s\right)\right) \\ a_9 = -a_4\tan\phi \end{cases} \tag{96}$$

Substituting the coefficients obtained from Eq.(96) into Eq.(86), we obtain a certain combination of $\Xi(s)$ relating to χ_z.

$$\Xi(s) = \frac{-a_4}{\tan\phi}\begin{pmatrix} \left(s^2-(1-2s)\tan^2\phi\right)\chi''_z + \\ \dfrac{\tan^2\phi\left(2\tan^2\phi+s\right)}{s\left(\tan^2\phi+s\right)}\begin{pmatrix} -(1-s)\chi'_z \\ -\chi_z+1 \end{pmatrix} \end{pmatrix} \tag{97}$$

Because $\Xi(s)=0$ and $a_4 \neq 0$, an arbitrary a_4 can be removed from Eq.(93) and rearranged to the standard form of a nonhomogeneous second-order ODE, which is undefined at $s=0$ and $s=\sin\phi/(1+\sin\phi)$ in the applied domain of s.

$$\begin{pmatrix} \chi''_z + \dfrac{-\sin^2\phi\left(s-(s-2)\sin^2\phi\right)(1-s)\chi'_z}{s\left(s+(1-s)\sin^2\phi\right)\left(s^2-(1-s)^2\sin^2\phi\right)} \\ +\dfrac{-\sin^2\phi\left(s-(s-2)\sin^2\phi\right)\chi_z}{s\left(s+(1-s)\sin^2\phi\right)\left(s^2-(1-s)^2\sin^2\phi\right)} \\ +\dfrac{\sin^2\phi\left(s-(s-2)\sin^2\phi\right)}{s\left(s+(1-s)\sin^2\phi\right)\left(s^2-(1-s)^2\sin^2\phi\right)} \end{pmatrix} = 0 \tag{98}$$

A certain combination in $\Xi(s)$ can be freed as well of χ_{xz} and χ_z if the polynomial operators B and C are zero. We can satisfy this requirement by imposing B=0 and C=0 (the required conditions for B and C are already given in Eq.(91) and Eq.(95), respectively). Simultaneously solving algebraic systems of the eight equations in Eqs.(91) and (95) with the nine unknowns a_1 to a_9, we can calculate the required constants in proportion to $a_5 \neq 0$.

$$\begin{cases} a_2 = a_3 = a_6 = a_7 = a_8 = 0 \\ a_1 = a_5\tan\phi/s^2 \\ a_4 = a_5(1-2s)\tan\phi/s \\ a_9 = -a_5 s \end{cases} \tag{99}$$

Substituting the coefficients obtained from Eq.(99) into Eq.(86), we obtain a certain combination of $\Xi(s)$ relating to χ_x.

$$\Xi(s) = \frac{-a_5}{s^2}\begin{pmatrix} s\left(s^2-(1-2s)\tan^2\phi\right)\chi''_x \\ -\left(\tan^2\phi\right)\chi'_x \end{pmatrix} \tag{100}$$

Because $\Xi(s)=0$ and $a_5 \neq 0$, a_5 is removed from Eq.(100). Using $\tan^2\phi=\sin^2\phi/(1-\sin^2\phi)$), the standard form of a second-order homogeneous ODE similar to Eq.(40), which is undefined at $s=0$ and $s=\sin\phi/(1+\sin\phi)$ in the applied domain of s can be rearranged; therefore, the manipulation addressed in Eqs.(97)-(100) are equivalent to Eqs.(32)-(39).

According to Eqs.(92), (96) and (99), it is clear that $a_7=0$ and $a_8=0$ under all conditions because the required order is two, as indicated by Eq.(79). So Eqs.(83) and (84) are not involved in the system and can be ignored as mentioned earlier. Because the separated form of χ_x has the simplest term, we should start with Eq.(40), which requires at least two boundary conditions. According to Eqs.(26)-(28), we can see that only two primary conditions, $\chi_z|_{s=1}=0$ and $\chi_{xz}|_{s=1}=0$, are necessary to determine three auxiliary boundary conditions. Even if we do not know the primary condition $\chi_x|_{s=1}=0$, it can be determined from the stress relation shown in Eq.(22) using the other two primary conditions. Instead of using Eqs.(98) and (94) once a particular solution of χ_x is determined, those of χ_z and χ_{xz} can be subsequently solved using Eqs.(19) and (20). So, the method described here can help point out the simplest form to start with.

Appendix B: Verification with the equilibrium conditions

The equilibrium conditions are the basic requirement that internal stresses in the body must satisfy whether the body behaves elastically or plastically. Because stresses in equilibrium are expressed in partial differentiation with respect to a coordinate system, let us consider the derivatives of the particular functions $I(s)$, $A(s)$ and $J(s)$ with respect to scaling distance s according to Eqs.(65)-(67) as follows:

$$I'(s)=\left(\frac{\left(2s^2-4s+3\right)\sin^2\phi-2s^2}{s\left(s^2-(1-s)^2\sin^2\phi\right)}\right)I(s) \quad (101)$$

$$A'(s)=-I(s),\ J'(s)=-s^2I(s) \quad (102),\ (103)$$

Since the three scaling stress variables are expressed as conditional functions based on Eqs.(68)-(70), their derivatives with respect to s are also expressed in conditional functions.

$$\chi'_{xz}=\begin{cases}A(\bar{s})\sin\phi\cos\phi & \text{if } 0\le s\le\bar{s}\\ \sin\phi\cos\phi\left(A(s)-sI(s)\right) & \text{if } \bar{s}\le s\le 1\end{cases} \quad (104)$$

$$\chi'_{x}=\begin{cases}0 & \text{if } 0\le s\le\bar{s}\\ -s^2I(s)\cos^2\phi & \text{if } \bar{s}\le s\le 1\end{cases} \quad (105)$$

$$\chi'_{z}=\begin{cases}2A(\bar{s})\sin^2\phi & \text{if } 0\le s\le\bar{s}\\ \begin{pmatrix}-s^2I(s)\cos^2\phi\\ +2\sin^2\phi\left(A(s)-sI(s)\right)\end{pmatrix} & \text{if } \bar{s}\le s\le 1\end{cases} \quad (106)$$

Axi-symmetric condition with a hoop stress $\sigma_\varphi=\sigma_x$ based on Levy's hypothesis are employed in this verification. Partial differentiation of stress components appearing in the equilibrium conditions under cylindrical coordinate system is obtained by chain rule via Eqs.(3)-(13) and Eqs.(104)-(106).

$$\begin{aligned}\partial_x\sigma_x&=\gamma\chi'_x\tan\phi\\ &=\gamma\begin{cases}0 & \text{if } 0\le s\le\bar{s}\\ -s^2I(s)\sin\phi\cos\phi & \text{if } \bar{s}\le s\le 1\end{cases}\end{aligned} \quad (107)$$

$$\begin{aligned}\partial_z\tau_{xz}&=\gamma\left(\chi_{xz}-s\chi'_{xz}\right)\\ &=\gamma\begin{cases}0 & \text{if } 0\le s\le\bar{s}\\ s^2I(s)\sin\phi\cos\phi & \text{if } \bar{s}\le s\le 1\end{cases}\end{aligned} \quad (108)$$

$$\begin{aligned}\partial_z\sigma_z&=\gamma\left(\chi_z-s\chi'_z\right)\\ &=\gamma\begin{cases}J(\bar{s})\cos^2\phi & \text{if } 0\le s\le\bar{s}\\ \begin{pmatrix}\left(J(s)+s^3I(s)\right)\cos^2\phi\\ +2s^2I(s)\sin^2\phi\end{pmatrix} & \text{if } \bar{s}\le s\le 1\end{cases}\end{aligned} \quad (109)$$

$$\begin{aligned}\partial_x\tau_{xz}&=\gamma\chi'_{xz}\tan\phi\\ &=\gamma\begin{cases}A(\bar{s})\sin^2\phi & \text{if } 0\le s\le\bar{s}\\ \left(A(s)-sI(s)\right)\sin^2\phi & \text{if } \bar{s}\le s\le 1\end{cases}\end{aligned} \quad (110)$$

$$\tau_{xz}/x=\gamma\begin{cases}A(\bar{s})\sin^2\phi & \text{if } 0\le s\le\bar{s}\\ A(s)\sin^2\phi & \text{if } \bar{s}\le s\le 1\end{cases} \quad (111)$$

Two different solutions in accordance with a domain of s are verified with each equilibrium condition. We find that along the x-direction, Eqs.(107)-(108) satisfies Eq.(1) for the whole domain of s. Along the z direction, Eq.(2) is replaced with Eqs.(109)-(110) for nontrivial solutions as follows.

$$\gamma\begin{pmatrix}2A(s)\sin^2\phi+J(s)\cos^2\phi\\ +s\left(s^2-(1-s)^2\sin^2\phi\right)I(s)\end{pmatrix}=\gamma \quad (112)$$

The term $s(s^2-(1-s)^2\sin^2\phi)I(s)$ appears in the above equation. Because $I(1)=1$, this term of expression can be extended and rearranged by the following expressions according to the method of integration by parts for a domain of $\sin\phi/(1+\sin\phi)\le s\le 1$.

$$\begin{aligned}&s\left(s^2-(1-s)^2\sin^2\phi\right)I(s)\\ &=1-\left(\left.\left(s\left(s^2-(1-s)^2\sin^2\phi\right)I(s)\right)\right|_s^1\right)\\ &=1-\begin{pmatrix}\int_s^1 s\left(s^2-(1-s)^2\sin^2\phi\right)I'(s)\,ds\\ +\int_s^1 I(s)\dfrac{d\left(s\left(s^2-(1-s)^2\sin^2\phi\right)\right)}{ds}ds\end{pmatrix}\\ &=1-\begin{pmatrix}\int_s^1\left(\left(2s^2-4s+3\right)\sin^2\phi-2s^2\right)I(s)\,ds\\ +\int_s^1\left(-\left(3s^2-4s+1\right)\sin^2\phi+3s^2\right)I(s)\,ds\end{pmatrix}\\ &=1-\int_s^1\left(2\sin^2\phi+s^2\cos^2\phi\right)I(s)\,ds\\ &=1-2A(s)\sin^2\phi-J(s)\cos^2\phi\end{aligned} \quad (113)$$

The result of Eq.(113) indicates that we can alternatively determine $J(s)$ from $A(s)$ and $I(s)$, instead of computing the integration as defined in Eq.(67).

$$J(s)=\frac{\begin{pmatrix}1-2A(s)\sin^2\phi\\ -s\left(s^2-(1-s)^2\sin^2\phi\right)I(s)\end{pmatrix}}{\cos^2\phi} \quad (114)$$

By replacing the term of Eq.(113) on the left-hand side of Eq.(112), the equilibrium condition along the z-direction is satisfied for a domain of $\sin\phi/(1+\sin\phi)\le s\le 1$. We have another equations for a domain of $0\le s\le \sin\phi/(1+\sin\phi)$ by a similar way.

$$\gamma\left(2A(\bar{s})\sin^2\phi+J(\bar{s})\cos^2\phi\right)=\gamma \quad (115)$$

$$J(\bar{s})=\left(1-2A(\bar{s})\sin^2\phi\right)/\cos^2\phi \quad (116)$$

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