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Anisotropic constitutive equation for friction with transition from static to kinetic friction and vice versa

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High friction coefficient is first observed when a sliding between bodies commences, which is called the *static friction*. Then, the friction coefficient decreases approaching the lowest stationary value, which is called the *kinetic friction*. Thereafter, if the sliding stops for a while and then it starts again, the friction coefficient recovers and a similar behavior as that in the first sliding is reproduced. In this article the *subloading-friction model*[1] with a smooth elastic-plastic sliding transition (Hashiguchi, 2005) is extended so as to describe the reduction from the static to kinetic friction and the recovery of the static friction. The reduction is formulated as the plastic softening due to the separations of the adhesions of surface asperities induced by the sliding and the recovery is formulated as the viscoplastic hardening due to the reconstructions of the adhesions of surface asperities during the elapse of time under a quite high actual contact pressure between edges of asperities. Further, the anisotropy of friction is described by incorporating the rotation and the orthotropy of sliding-yield surface.

Key Words: *anisotropy; constitutive equation; friction; elastoplasticity; hardening/softening; subloading surface model; viscoplasticity.*

1. Introduction

Description of the friction phenomenon by a constitutive equation has been attained first as a rigid-plasticity[2), 3)]. Further, it has been extended to an elastoplasticity[4)-17)] in which the penalty concept, i.e. the elastic springs between contact surfaces is incorporated and the isotropic hardening is taken into account so as to describe the test results[18)] exhibiting the smooth contact traction vs. sliding displacement curve reaching the static-friction. However, the interior of the sliding-yield surface has been assumed to be an elastic domain and thus the plastic sliding velocity due to the rate of traction inside the sliding-yield surface is not described. Therefore, the accumulation of plastic sliding due to the cyclic loading of contact traction within the sliding-yield surface cannot be described by these models. They could be called the conventional friction model in accordance with the classification of plastic constitutive models by Drucker[19)]. On the other hand, the first author of the present article has proposed the *subloading surface model*[20), 21)] within the framework of unconventional plasticity, which is capable of describing the plastic strain rate by the rate of stress inside the yield surface. Based on the concept of subloading surface, the authors proposed the *subloading-friction model*[1), 22)] which describes the smooth transition from the elastic to plastic sliding state and the accumulation of sliding displacement during a cyclic loading of tangential contact traction. Besides, in this model the reduction of friction coefficient with the increase of normal contact traction observed in experiments[15), 23), 24)] is formulated by incorporating the nonlinear sliding-yield surface, while the decrease has not been taken into account in Coulomb sliding-yield surface, which has been adopted widely in constitutive models for friction so far.

It is widely known that when bodies at rest begin to slide to each other, a high friction coefficient appears first, which is called the *static friction*, and then it decreases approaching a stationary value, called the *kinetic friction*. However, this process has not been formulated pertinently so far, although the increase of friction coefficient up to the peak has been described as the isotropic hardening, i.e. the expansion of sliding-yield surface as described above.

Further, it has been found that if the sliding ceases for a while and then it starts again, the friction coefficient recovers and the similar behavior as that in the initial sliding is reproduce[25)-35)]. The recovery has been formulated by equations including the time elapsed after the stop of sliding[26),28),30),31),33),35)]. However, the inclusion of time itself leads to the loss of objectivity in constitutive equations as known from the fact that the evaluation of elapsed time varies depending on the judgment of time when the sliding stops, which is accompanied with the arbitrariness especially for the state varying sliding velocity in low level. Generally speaking, the variation of material property cannot be described pertinently by the elapse from a particular time but has to be described by state of internal variables without the inclusion of time itself.

The reduction of friction coefficient from the static to kinetic friction and the recovery of friction coefficient mentioned above are to be the fundamental characteristics in friction phenomenon, which have been recognized widely for a long time. Difference of the static and kinetic frictions often reaches up to several ten percents. Therefore, the formulation of the transition from the static to kinetic friction and vice versa are of importance for the development of mechanical design in the field of engineering. However, the rational formulation has not been attained so far.

The difference of friction coefficients is observed in the mutually opposite sliding directions. It could be described by the rotation of sliding-yield surface, whilst the anisotropy of soils has been described by the rotation of yield surface[36)-38)]. Further, the difference of the range of friction coefficients is observed in the different sliding directions. It could be described by concept of orthotropy of sliding-yield surface[14)].

In this article, the *subloading-friction model*[1)] is extended so as to describe the reduction of friction coefficient from the static to kinetic friction as the plastic softening due to the sliding and the recovery of friction coefficient as the viscoplastic hardening due to the creep phenomenon induced with the elapse of time under a high contact pressure between edges of surface asperities. It is further extended so as to describe the anisotropy by incorporating the rotation and the orthotropy of sliding-yield surface.

2. Formulation of the constitutive equation for friction

The subloading-friction mode[1)] is extended below so as to describe the *static-kinetic friction transition*, i.e. the transition from static and kinetic friction, and vice versa.

2.1 Decomposition of sliding velocity

The sliding velocity $\bar{\mathbf{v}}$ is defined as the relative velocity of the counter body and is additively decomposed into the normal part $\bar{\mathbf{v}}_n$ and the tangential part $\bar{\mathbf{v}}_t$ as follows (see Fig. 1):

$$\bar{\mathbf{v}} = \bar{\mathbf{v}}_n + \bar{\mathbf{v}}_t, \tag{1}$$

where

$$\left.\begin{aligned} \bar{\mathbf{v}}_n &= (\bar{\mathbf{v}}\bullet\mathbf{n})\mathbf{n} = (\mathbf{n}\otimes\mathbf{n})\bar{\mathbf{v}} = -\bar{v}_n\mathbf{n}, \\ \bar{\mathbf{v}}_t &= \bar{\mathbf{v}} - \bar{\mathbf{v}}_n = (\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\bar{\mathbf{v}}. \end{aligned}\right\} \tag{2}$$

$\mathbf{n}$ is the unit outward-normal vector at the contact surface, $(\bullet)$ and $\otimes$ denote the scalar and the tensor products, respectively, and $\mathbf{I}$ is the second-order identity tensor having the components of Kronecker's delta $\delta_{ij}=1$ for $i=j$, $\delta_{ij}=0$ for $i\neq j$. $\bar{v}_n$ is the normal component of the sliding velocity, i.e.

$$\bar{v}_n \equiv -\mathbf{n}\bullet\bar{\mathbf{v}}, \tag{3}$$

where the sign of $\bar{v}_n$ is selected to be plus when the counter body approaches to the relevant body.

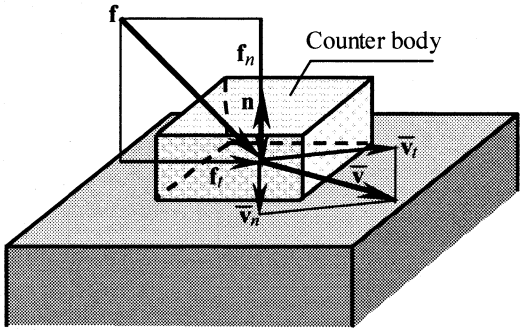

Fig. 1. Contact traction $\mathbf{f}$ and sliding velocity $\bar{\mathbf{v}}$.

Further, it is assumed that $\bar{\mathbf{v}}$ is additively decomposed into the elastic sliding velocity $\bar{\mathbf{v}}^e$ and the plastic sliding velocity $\bar{\mathbf{v}}^p$, i.e.

$$\bar{\mathbf{v}} = \bar{\mathbf{v}}^e + \bar{\mathbf{v}}^p \tag{4}$$

with

$$\left.\begin{aligned} \bar{\mathbf{v}}_n &= \bar{\mathbf{v}}_n^e + \bar{\mathbf{v}}_n^p, \\ \bar{\mathbf{v}}_t &= \bar{\mathbf{v}}_t^e + \bar{\mathbf{v}}_t^p, \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} \bar{\mathbf{v}}_n^e &= (\bar{\mathbf{v}}^e\bullet\mathbf{n})\mathbf{n} = (\mathbf{n}\otimes\mathbf{n})\bar{\mathbf{v}}^e = -\bar{v}_n^e\mathbf{n}, \\ \bar{\mathbf{v}}_t^e &= \bar{\mathbf{v}}^e - \bar{\mathbf{v}}_n^e = (\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\bar{\mathbf{v}}^e, \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} \bar{\mathbf{v}}_n^p &= (\bar{\mathbf{v}}^p\bullet\mathbf{n})\mathbf{n} = (\mathbf{n}\otimes\mathbf{n})\bar{\mathbf{v}}^p = -\bar{v}_n^p\mathbf{n}, \\ \bar{\mathbf{v}}_t^p &= \bar{\mathbf{v}}^p - \bar{\mathbf{v}}_n^p = (\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\bar{\mathbf{v}}^p. \end{aligned}\right\} \tag{7}$$

where $\bar{v}_n^e$ and $\bar{v}_n^p$ are the elastic and the plastic part, respectively, of $\bar{v}_n$.

The contact traction $\mathbf{f}$ acting on the body is decomposed into the normal part, i.e. normal-traction $\mathbf{f}_n$ and the tangential part, i.e. tangential traction $\mathbf{f}_t$ as follows:

$$\mathbf{f} = \mathbf{f}_n + \mathbf{f}_t, \tag{8}$$

$$\left.\begin{aligned} \mathbf{f}_n &\equiv (\mathbf{n}\bullet\mathbf{f})\mathbf{n} = (\mathbf{n}\otimes\mathbf{n})\mathbf{f} = -f_n\mathbf{n}, \\ \mathbf{f}_t &\equiv \mathbf{f} - \mathbf{f}_n = (\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\mathbf{f}, \end{aligned}\right\} \tag{9}$$

whilst $\mathbf{n}$ is identical to the normalized direction vectors of $\mathbf{f}_n$, i.e.

$$\mathbf{n} \equiv \frac{\mathbf{f}_n}{\|\mathbf{f}_n\|} \tag{10}$$

and f_n is the normal part of the contact traction $\mathbf{f}$, i.e.

$$f_n \equiv -\mathbf{n} \bullet \mathbf{f} \tag{11}$$

where the sign of f_n is selected to be plus when the relevant body is compressed by the counter body. Here, note that the directions of the tangential contact traction and the tangential sliding velocity are not necessary identical in general and $\mathbf{I}-\mathbf{n}\otimes\mathbf{n} \neq \mathbf{t}\otimes\mathbf{t}$ $(\mathbf{t} \equiv \mathbf{f}_t/\|\mathbf{f}_t\|)$ in the three-dimensional behavior.

Now, let the elastic sliding velocity be given by the following hypo-elastic relation, whilst the elastic sliding velocity is usually far small compared with the plastic sliding velocity in the friction phenomenon.

$$\overline{\mathbf{v}}_n^e = \frac{1}{\alpha_n}\overset{\circ}{\mathbf{f}}_n, \quad \overline{\mathbf{v}}_t^e = \frac{1}{\alpha_t}\overset{\circ}{\mathbf{f}}_t, \tag{12}$$

where $\overset{\circ}{\mathbf{f}}_n$ and $\overset{\circ}{\mathbf{f}}_t$ are the normal component and tangential component, respectively, of $\overset{\circ}{\mathbf{f}}$, $(^\circ)$ denoting the corotational rate, which are related to the material-time derivative denoted by $(^\bullet)$ as follows:

$$\overset{\circ}{\mathbf{f}} = \overset{\bullet}{\mathbf{f}} - \mathbf{\Omega}\mathbf{f}, \quad \overset{\circ}{\mathbf{f}}_n = \overset{\bullet}{\mathbf{f}}_n - \mathbf{\Omega}\mathbf{f}_n, \quad \overset{\circ}{\mathbf{f}}_t = \overset{\bullet}{\mathbf{f}}_t - \mathbf{\Omega}\mathbf{f}_t \tag{13}$$

which is derived from

$$\begin{aligned}\overset{\circ}{\mathbf{f}} &= \overset{\bullet}{\mathbf{f}} - \mathbf{\Omega}\mathbf{f} = (\mathbf{f}_n + \mathbf{f}_t)^\bullet - \mathbf{\Omega}(\mathbf{f}_n + \mathbf{f}_t) \\ &= \overset{\bullet}{\mathbf{f}}_n - \mathbf{\Omega}\mathbf{f}_n + \overset{\bullet}{\mathbf{f}}_t - \mathbf{\Omega}\mathbf{f}_t = \overset{\circ}{\mathbf{f}}_n + \overset{\circ}{\mathbf{f}}_t\end{aligned} \tag{14}$$

where the skew-symmetric tensor $\mathbf{\Omega}$ is the spin describing the rigid-body rotation of the contact surface. α_n and α_t are the contact elastic moduli in the normal and the tangential directions to the contact surface. On the other hand, the sliding velocity $\overline{\mathbf{v}}$ is not an absolute velocity of a point on the body surface but the relative velocity between two points on the contact surface, and thus it can be adopted to the constitutive relation as it is since it has the objectivity. It follows from Eq. (12) that

$$\overset{\circ}{\mathbf{f}} = \overset{\circ}{\mathbf{f}}_n + \overset{\circ}{\mathbf{f}}_t = \mathbf{C}^e\overline{\mathbf{v}}^e \tag{15}$$

where the second-order tensor $\mathbf{C}^e$ is the fictitious contact elastic modulus tensor given by

$$\left.\begin{aligned}\mathbf{C}^e &= \alpha_n \mathbf{n}\otimes\mathbf{n} + \alpha_t(\mathbf{I}-\mathbf{n}\otimes\mathbf{n}) \\ \mathbf{C}^{e-1} &= \frac{1}{\alpha_n}\mathbf{n}\otimes\mathbf{n} + \frac{1}{\alpha_t}(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\end{aligned}\right\}, \tag{16}$$

2.2 Normal sliding-yield and sliding-subloading surfaces

Assume the following isotropic *sliding-yield surface* with the isotropic hardening/softening, which describes the *sliding-yield condition.*

$$f(\mathbf{f}, \boldsymbol{\beta}) = F, \tag{17}$$

where F is the isotropic hardening/softening function denoting the variation of the size of sliding-yield surface. $\boldsymbol{\beta}$ is the vector describing anisotropy due to the rotation around the null traction point without the normal component of contact surface, while it is assumed that it does not evolve leading to $\overset{\bullet}{\boldsymbol{\beta}} = \mathbf{0}$, and thus it holds that

$$\mathbf{n}\bullet\boldsymbol{\beta} = 0, \tag{18}$$

$$\overset{\bullet}{\boldsymbol{\beta}} = \mathbf{\Omega}\boldsymbol{\beta}. \tag{19}$$

The anisotropy of metals is described by the translation of yield surface but the anisotropy of frictional materials such as soils is described by the rotation[36)-38)]. Therefore, it is assumed that the anisotropy of friction phenomenon is described by the rotation of the sliding-yield surface. Then, it holds that

$$f(s\mathbf{f}, \boldsymbol{\beta}) = sf(\mathbf{f}, \boldsymbol{\beta}) \tag{20}$$

$$\frac{\partial f(\mathbf{f}, \boldsymbol{\beta})}{\partial \mathbf{f}} \bullet \mathbf{f} = f(\mathbf{f}, \boldsymbol{\beta}) \tag{21}$$

where s is an arbitrary positive scalar quantity, whilst Eq. (21) is based on the Euler's homogeneous function of degree-one. Therefore, the sliding-yield surface keeps the similar shape and orientation with respect to the origin of contact traction space, i.e. $\mathbf{f} = \mathbf{0}$ for $\boldsymbol{\beta} = \text{const.}$

In what follows, we assume that the interior of the sliding-yield surface is not a purely elastic domain but that the plastic sliding velocity is induced by the rate of traction inside that surface. Therefore, let the surface described by Eq. (17) be renamed as the *normal sliding-yield surface.*

Then, based on the concept of *subloading surface*[20),21)], we introduce the *sliding-subloading surface*, which always passes through the current contact traction point $\mathbf{f}$ and keeps a similar shape and orientation to the normal sliding-yield surface with respect to the origin of contact traction space, i.e. $\mathbf{f} = \mathbf{0}$ for $\boldsymbol{\beta} = \text{const.}$ Then, the sliding-subloading surface fulfills the following geometrical characteristics.

i) All lines connecting an arbitrary point inside the sliding-subloading surface and its conjugate point inside the normal sliding-yield surface join at a unique point, called the *similarity-center*, which is the origin of the contact traction space in the present model.

ii) All ratios of length of an arbitrary line-element connecting two points inside the sliding-subloading surface to that of an arbitrary conjugate line-element connecting two conjugate points inside the normal sliding-yield surface are identical. The ratio is called the *similarity-ratio,* which coincides with the ratio of the sizes of these surfaces.

Let the ratio of the size of the sliding-subloading surface to that of the normal sliding-yield surface be called the normal sliding-yield *ratio*, denoted by $\overline{R}$ $(0 \le \overline{R} \le 1)$, where $\overline{R} = 0$ corresponds to the null traction state ($f = 0$) as the most elastic state, $0 < \overline{R} < 1$ to the *subsliding state* ($0 < f < F$), and $\overline{R} = 1$ to the *normal sliding-yield state* in which the contact traction lies on the normal sliding-yield surface ($f = F$). Therefore, the normal sliding-yield ratio $\bar{R}$ plays the role of three-dimensional measure of the degree of approach to the normal sliding-yield state. Then, the sliding-subloading surface is described by

$$f(\mathbf{f},\ \boldsymbol{\beta}) = \bar{R}F\,. \tag{22}$$

The material-time derivative of Eq. (22) leads to

$$\bar{\mathbf{N}}\bullet\overset{\circ}{\mathbf{f}} = \bar{R}\overset{\bullet}{F} + \overset{\bullet}{\bar{R}}F\,, \tag{23}$$

where

$$\bar{\mathbf{N}} \equiv \frac{\partial f(\mathbf{f},\ \boldsymbol{\beta})}{\partial \mathbf{f}}\,, \tag{24}$$

Here, note that the direct transformation of the material-time derivative to the corotational derivative is verified by substituting Eq. (13) into Eq.(23), noting $\mathbf{a}\bullet(\boldsymbol{\Omega}\mathbf{a}) = 0$ for an arbitrary vector $\mathbf{a}$. The direct transformation of the material-time derivative to the corotational derivative is verified for the general scalar function[39].

2.3 Evolution rules of the hardening function and the normal sliding-yield ratio

It could be stated from experiments that

1) If the sliding commences, the friction coefficient reaches first the maximal value of static-friction and then it reduces to the minimal stationary value of kinetic-friction. Physically, this phenomenon could be interpreted to be caused by the separations of the adhesions of surface asperities between contact bodies due to the sliding[40]. Then, let it be assumed that the reduction is caused by the contraction of the normal sliding-yield surface, i.e. the plastic softening due to the sliding.

2) If the sliding ceases after the reduction of friction coefficient, the friction coefficient recovers gradually with the elapse of time and the identical behavior as the initial sliding behavior with the static friction is reproduced after an elapse of sufficient time. Physically, this phenomenon could be interpreted to be caused by the reconstructions of the adhesions of surface asperities during the elapsed time under a quite high contact pressure between edges of surface asperities. Then, let it be assumed that the recovery is caused by the viscoplastic hardening due to the creep phenomenon.

Taking account of these facts, let the evolution rule of the isotropic hardening/softening function F be postulated as follows:

$$\overset{\bullet}{F} = -\kappa\left\{\left(\frac{F}{F_k}\right)^m - 1\right\}\|\bar{\mathbf{v}}^p\| + \xi\left\{1 - \left(\frac{F}{F_s}\right)^n\right\}, \tag{25}$$

where F_s and F_k ($F_s \geq F \geq F_k$) are the maximum and minimum values of F for the static and kinetic frictions, respectively. κ and m are the material constants influencing the decreasing rate of F due to the plastic sliding, and ξ and n are the material constants influencing the recovering rate of F due to the elapse of time, while they would be functions of absolute temperature in general. The first and the second terms in Eq. (25) stand for the deteriorations and the formations, respectively, of the adhesions between surface asperities. On the other hand, so far these phenomena have been described by separate formulations for the softening due to the sliding displacement and the hardening due to the time elapsed after the stop of sliding. Here, the inclusion of the time itself in constitutive equations[26),28),30),31),33),35)] is not allowed violating the objectivity since the evaluation of elapsed time from the stop of sliding depends on the subjectivity as known from the state varying sliding velocity in low level.

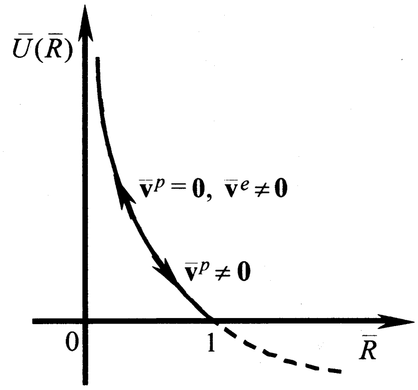

Fig. 2. Function $\bar{U}(\bar{R})$ for the evolution rule of the normal sliding-yield ratio $\bar{R}$.

It is observed in experiments that the tangential traction increases almost elastically with the plastic sliding when it is zero but thereafter it increases gradually approaching the normal sliding-yield surface and it does not increase any more when it reaches the normal sliding-yield surface. Then, we assume the evolution rule of the normal sliding-yield ratio as follows:

$$\overset{\bullet}{\bar{R}} = \bar{U}(\bar{R})\|\bar{\mathbf{v}}^p\| \ \text{ for } \bar{\mathbf{v}}^p \neq \mathbf{0}\,, \tag{26}$$

where $\bar{U}(\bar{R})$ is a monotonically decreasing function of $\bar{R}$ fulfilling the following conditions (Fig 2).

$$\left.\begin{array}{l} \bar{U}(\bar{R}) \to +\infty \ \text{ for } \bar{R} \to 0, \\ \bar{U}(\bar{R}) = 0 \ \text{ for } \bar{R} = 1, \\ (\bar{U}(\bar{R}) < 0 \ \text{ for } \bar{R} > 1). \end{array}\right\} \tag{27}$$

Let the function $\bar{U}$ satisfying Eq. (27) be simply given by

$$\bar{U}(\bar{R}) = \tilde{u}\cot\left(\frac{\pi}{2}\bar{R}\right) \tag{28}$$

where $\tilde{u}$ is the material constant. Eq. (26) with Eq. (28) can lead to the analytical integration of $\bar{R}$ for the accumulated plastic sliding $\bar{u}^p \equiv \int\|\bar{\mathbf{v}}^p\|dt$ under the initial condition $\bar{u}^p - \bar{u}_0^p : \bar{R} = \bar{R}_0$ as follows:

$$\frac{2}{\pi}\frac{1}{\tilde{u}}\log\frac{\cos\left(\frac{\pi}{2}\bar{R}\right)}{\cos\left(\frac{\pi}{2}\bar{R}_0\right)} + (\bar{u}^p - \bar{u}_0^p) = 0 \tag{29}$$

On the other hand, the following function has been used widely so far.

$$\bar{U}(\bar{R}) = -\bar{u}\ln\bar{R} \tag{30}$$

However, an analytical integration cannot be obtained from Eq. (26) with Eq. (30) and thus Eq. (30) is inconvenient to formulate the return-mapping method attracting the contact traction to the subloading surface[41].

2.4 Relationships of contact traction rate and sliding velocity

The substitution of Eqs. (25) and (26) into Eq. (23) gives rise to the *consistency condition* for the sliding-subloading surface:

$$\bar{\mathbf{N}}\bullet\mathring{\mathbf{f}} = \bar{R}\left\{-\kappa\left\{\left(\frac{F}{F_k}\right)^m - 1\right\}\|\bar{\mathbf{v}}^p\| + \xi\left\{1-\left(\frac{F}{F_s}\right)^n\right\}\right] + \bar{U}\|\bar{\mathbf{v}}^p\|F. \tag{31}$$

Assume that the direction of plastic sliding velocity is tangential to the contact plane and outward-normal to the curve generated by the intersection of sliding-yield surface and the constant normal traction plane $\mathbf{f}_n = \text{const.}$, leading to the *tangential associated flow rule*, i.e.

$$\bar{\mathbf{v}}^p = \bar{\lambda}\bar{\mathbf{t}}_n, \tag{32}$$

where $\bar{\lambda}\ (>0)$ is a positive proportionality factor and

$$\bar{\mathbf{t}}_n \equiv (\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\bullet\bar{\mathbf{N}}, \tag{33}$$

Substituting Eq. (32) into Eq. (31), the proportionality factor $\bar{\lambda}$ is derived as follows:

$$\bar{\lambda} = \frac{\bar{\mathbf{N}}\bullet\mathring{\mathbf{f}} - m^c}{m^p} \tag{34}$$

and thus

$$\bar{\mathbf{v}}^p = \frac{\bar{\mathbf{N}}\bullet\mathring{\mathbf{f}} - m^c}{m^p}\bar{\mathbf{t}}_n, \tag{35}$$

where

$$m^p \equiv -\kappa\left\{\left(\frac{F}{F_k}\right)^m - 1\right\}\bar{R} + F\bar{U}, \tag{36}$$

$$m^c \equiv \xi\left\{1-\left(\frac{F}{F_s}\right)^n\right\}\bar{R}\ (\geq 0). \tag{37}$$

Substituting Eqs. (4) and (35) into Eq. (31), the sliding velocity is given by

$$\bar{\mathbf{v}} = \mathbf{C}^{e-1}\mathring{\mathbf{f}} + \frac{\bar{\mathbf{N}}\bullet\mathring{\mathbf{f}} - m^c}{m^p}\bar{\mathbf{t}}_n. \tag{38}$$

The positive proportionality factor in terms of the sliding velocity, denoted by the symbol $\bar{\Lambda}$, is given from Eqs. (38) as

$$\bar{\Lambda} = \frac{\bar{\mathbf{N}}\bullet\mathbf{C}^e\bullet\bar{\mathbf{v}} - m^c}{m^p + \bar{\mathbf{N}}\bullet\mathbf{C}^e\bullet\bar{\mathbf{t}}_n}. \tag{39}$$

The traction rate is derived from Eqs. (4), (15), (32) and (39) as follows:

$$\mathring{\mathbf{f}} = \mathbf{C}^e\left(\bar{\mathbf{v}} - \left\langle\frac{\bar{\mathbf{N}}\bullet\mathbf{C}^e\bullet\bar{\mathbf{v}} - m^c}{m^p + \bar{\mathbf{N}}\bullet\mathbf{C}^e\bullet\bar{\mathbf{t}}_n}\right\rangle\bar{\mathbf{t}}_n\right), \tag{40}$$

where $\langle\ \rangle$ is the McCauley's bracket, i.e. $\langle s\rangle = (s+|s|)/2$ for an arbitrary scalar variable s.

2.5 Loading criterion

The loading criterion for the constitutive equation formulated in the foregoing is given in this section.

First, note the following facts:

1. It is required that

$$\bar{\lambda} = \bar{\Lambda} > 0 \tag{41}$$

in the loading (plastic sliding) process $\bar{\mathbf{v}}^p \neq \mathbf{0}$.

2. It holds that

$$\bar{\mathbf{N}}\bullet\mathring{\mathbf{f}} \leq 0 \tag{42}$$

in the unloading (elastic sliding) process $\bar{\mathbf{v}}^p = \mathbf{0}$. Further, because of $\bar{\mathbf{v}} = \bar{\mathbf{v}}^e$ leading to $\bar{\mathbf{N}}\bullet\mathbf{C}^e\bullet\bar{\mathbf{v}} = \bar{\mathbf{N}}\bullet\mathbf{C}^e\bullet\bar{\mathbf{v}}^e = \bar{\mathbf{N}}\bullet\mathring{\mathbf{f}}$ in this process it holds that

$$\bar{\Lambda} = \frac{\bar{\mathbf{N}}\bullet\mathring{\mathbf{f}} - m^c}{m^p + \bar{\mathbf{N}}\bullet\mathbf{C}^e\bullet\bar{\mathbf{t}}_n} \tag{43}$$

while it should be noted that $m^c \geq 0$ (Eq. (36)).

3. The plastic modulus m^p takes both signs of positive and negative in hardening/softening materials. On the other hand, noting that the contact elastic modulus $\mathbf{C}^e$ is the positive definite tensor and thus it holds that $\bar{\mathbf{N}}\bullet\mathbf{C}^e\bullet\bar{\mathbf{N}} \gg m^p$ in general and postulating that the plastic relaxation does not proceed infinitely, let the following inequality be assumed.

$$m^p + \bar{\mathbf{N}}\bullet\mathbf{C}^e\bullet\bar{\mathbf{t}}_n > 0 \tag{44}$$

Then, in the unloading process $\bar{\mathbf{v}}^p = \mathbf{0}$ the following inequalities hold depending on the sign of the plastic modulus m^p, i.e. the hardening, perfectly-plastic and softening states from Eqs. (34) and (41)-(44).

$$\left.\begin{array}{l} \bar{\lambda} \leq 0 \text{ and } \bar{\Lambda} \leq 0 \text{ when } m^p > 0 \\ \bar{\lambda} \to -\infty \text{ or indeterminate and } \bar{\Lambda} \leq 0 \text{ when } m^p = 0 \\ \bar{\lambda} \geq 0 \text{ and } \bar{\Lambda} \leq 0 \text{ when } m^p < 0 \end{array}\right\} \tag{45}$$

Therefore, the sign of $\bar{\lambda}$ in the unloading process from the state, in which the perfectly-plastic or softening proceeds if the plastic sliding is induced, is not negative. On the other hand, $\bar{\Lambda}$ is negative in the unloading process. Thus, the distinction between a loading and an unloading processes cannot be judged by the sign of $\bar{\lambda}$ but can be done by that of $\bar{\Lambda}$. Therefore, the loading criterion is given as follows:

$$\left.\begin{array}{l} \bar{\mathbf{v}}^p \neq \mathbf{0} : \bar{\Lambda} > 0, \\ \bar{\mathbf{v}}^p = \mathbf{0} : \text{otherwise}. \end{array}\right\} \tag{46}$$

or

$$\left.\begin{array}{l} \bar{\mathbf{v}}^p \neq \mathbf{0} :\ \bar{\mathbf{N}}\bullet\mathbf{C}^e\bullet\bar{\mathbf{v}} - m^c > 0, \\ \bar{\mathbf{v}}^p = \mathbf{0} :\ \text{otherwise}. \end{array}\right\} \tag{47}$$

in lieu of Eq. (44).

3. Specific sliding-yield surfaces

It can be stated from experiments that the friction coefficient decreases with the increase of contact pressure[15),23),24),42)]. Therefore, the normal sliding-yield surface cannot be described appropriately by the Coulomb sliding-yield surface in which the tangential contact traction and the normal contact traction are linearly related to each other using the angle of external friction and the adhesion. In what follows, the sliding-yield surface with the nonlinear relation of tangential contact traction and normal contact traction is assumed below, by which the reduction of friction coefficient with the increase of normal contact traction is described.

The closed normal sliding-yield and the sliding-subloading surfaces can be described by putting

$$f(\mathbf{f},\ \boldsymbol{\beta})= f_n\, g(\hat{\chi}) \tag{48}$$

as follows:

$$f_n\, g(\hat{\chi})= F,\ f_n\, g(\hat{\chi})= \bar{R}F \tag{49}$$

where

$$\hat{\chi} \equiv \frac{\|\hat{\boldsymbol{\eta}}\|}{M},\ \hat{\boldsymbol{\eta}} \equiv \boldsymbol{\eta} - \boldsymbol{\beta},\ \boldsymbol{\eta} \equiv \frac{\mathbf{f}_t}{f_n}. \tag{50}$$

M is the material constant denoting the traction ratio $\eta\ (=f_t/f_n)$ at the maximum point of f_t. The simple examples of the function $g(\chi)$ in the sliding-yield function in Eq. (48) are as follows:

$$g(\hat{\chi}) = \exp(\hat{\chi}),\ g'(\chi) = \exp(\hat{\chi}), \tag{51}$$

$$g(\hat{\chi}) = 1+\hat{\chi}^2,\ g'(\hat{\chi}) = 2\hat{\chi}, \tag{52}$$

$$g(\hat{\chi}) = \exp(\hat{\chi}^2/2),\ g'(\chi) = \hat{\chi}\exp(\hat{\chi}), \tag{53}$$

$$g(\hat{\chi}) - \frac{1}{1-\hat{\chi}/2},\ g'(\hat{\chi}) - \frac{1}{2(1-\hat{\chi}/2)^2}, \tag{54}$$

All the sets of Eqs. (17) and (48) with Eqs. (51)-(54) exhibit the closed surfaces passing through the points $f_n = 0$ and $f_n = F$ at $f_t = 0$. Eq. (51) and (52) are based on the original Cam-clay yield surface[43)] and the modified Cam-clay yield surfaces[44)], respectively, for soils. Eq. (53) exhibits the tear-shaped surface[1),45),46)] which is reversed from the surface of Eq. (51) on the axis of normal contact traction. Eq. (54) exhibits the parabola[1)].

It holds for Eq. (48) that

$$\left.\begin{aligned} \frac{\partial f_n}{\partial \mathbf{f}} &= \frac{\partial(-\mathbf{n}\cdot\mathbf{f})}{\partial \mathbf{f}} = -\mathbf{n}, \\ \frac{\partial \mathbf{f}_t}{\partial \mathbf{f}} &= \frac{\partial\{(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\cdot\mathbf{f}\}}{\partial \mathbf{f}} = \mathbf{I}-\mathbf{n}\otimes\mathbf{n} \end{aligned}\right\} \tag{55}$$

$$\frac{\partial \boldsymbol{\eta}}{\partial \mathbf{f}} = \frac{\partial \hat{\boldsymbol{\eta}}}{\partial \mathbf{f}} = \frac{\partial \mathbf{f}_t/f_n}{\partial \mathbf{f}} = \frac{f_n(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})-\mathbf{f}_t\otimes(-\mathbf{n})}{f_n^2}$$

$$= \frac{\mathbf{I}-\mathbf{n}\otimes\mathbf{n}+\boldsymbol{\eta}\otimes\mathbf{n}}{f_n} \tag{56}$$

$$\frac{\partial\|\hat{\boldsymbol{\eta}}\|}{\partial \mathbf{f}} = \frac{\partial\|\hat{\boldsymbol{\eta}}\|}{\partial \hat{\boldsymbol{\eta}}}\cdot\frac{\partial \hat{\boldsymbol{\eta}}}{\partial \mathbf{f}} = \hat{\boldsymbol{\tau}}\cdot\frac{\mathbf{I}-\mathbf{n}\otimes\mathbf{n}+\boldsymbol{\eta}\otimes\mathbf{n}}{f_n}$$

$$= \frac{1}{f_n}\{(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\mathbf{n}+\hat{\boldsymbol{\tau}}\} \tag{57}$$

where

$$\hat{\boldsymbol{\tau}} \equiv \frac{\hat{\boldsymbol{\eta}}}{\|\hat{\boldsymbol{\eta}}\|}\ \ (\mathbf{n}\cdot\hat{\boldsymbol{\tau}}=0) \tag{58}$$

Further, it holds from Eqs. (16) and (55)-(57) that

$$\frac{\partial \hat{\chi}}{\partial \mathbf{f}} = \frac{\partial\|\hat{\boldsymbol{\eta}}\|/M}{\partial \mathbf{f}} = \frac{1}{Mf_n}\{(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\mathbf{n}+\hat{\boldsymbol{\tau}}\} \tag{59}$$

$$\bar{\mathbf{N}} = -g(\hat{\chi})\mathbf{n} + f_n g'(\hat{\chi})\frac{1}{Mf_n}\{(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\mathbf{n}+\hat{\boldsymbol{\tau}}\}$$

$$= -\left\{g(\hat{\chi}) - \frac{g'(\hat{\chi})}{M}(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\right\}\mathbf{n} + \frac{g'(\hat{\chi})}{M}\hat{\boldsymbol{\tau}} \tag{60}$$

$$(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\cdot\bar{\mathbf{N}} = (\mathbf{I}-\mathbf{n}\otimes\mathbf{n})$$

$$\cdot\left[-\left\{g(\hat{\chi}) - \frac{g'(\hat{\chi})}{M}(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\right\}\mathbf{n} + \frac{g'(\hat{\chi})}{M}\hat{\boldsymbol{\tau}}\right] = \frac{g'(\hat{\chi})}{M}\hat{\boldsymbol{\tau}} \tag{61}$$

$$\hat{\mathbf{t}}_n = \hat{\boldsymbol{\tau}} \tag{62}$$

$$\bar{\mathbf{N}}\cdot\mathbf{C}^e = \left[-\left\{g(\hat{\chi}) - \frac{g'(\hat{\chi})}{M}(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\right\}\mathbf{n} + \frac{g'(\hat{\chi})}{M}\hat{\boldsymbol{\tau}}\right]$$

$$\cdot\{\alpha_n\mathbf{n}\otimes\mathbf{n}+\alpha_t(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\}$$

$$= -\alpha_n\left\{g(\hat{\chi}) - \frac{g'(\hat{\chi})}{M}(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\right\}\mathbf{n} + \alpha_t\frac{g'(\hat{\chi})}{M}\hat{\boldsymbol{\tau}} \tag{63}$$

$$\bar{\mathbf{N}}\cdot\mathbf{C}^e\cdot\hat{\mathbf{t}}_n = \left[-\alpha_n\left\{g(\hat{\chi}) - \frac{g'(\hat{\chi})}{M}(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\right\}\mathbf{n} + \alpha_t\frac{g'(\hat{\chi})}{M}\hat{\boldsymbol{\tau}}\right]\cdot\hat{\boldsymbol{\tau}} = \alpha_t\frac{g'(\hat{\chi})}{M} \tag{64}$$

The substitution of Eqs. (16) and (59)-(64) into Eqs. (38) and (40) leads to the sliding velocity vs. contact traction rate and its inverse relation are given as follows:

$$\bar{\mathbf{v}} = \left\{\frac{1}{\alpha_n}\mathbf{n}\otimes\mathbf{n} + \frac{1}{\alpha_t}(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\right\}\mathring{\mathbf{f}} + \frac{\left[-\left\{g(\hat{\chi}) - \frac{g'(\hat{\chi})}{M}(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\right\}\mathbf{n} + \frac{g'(\hat{\chi})}{M}\hat{\boldsymbol{\tau}}\right]\cdot\mathring{\mathbf{f}} - m^c}{m^p}\hat{\boldsymbol{\tau}}$$

$$= \left(\frac{1}{\alpha_n} - \frac{1}{\alpha_t}\right)(\mathbf{n}\cdot\mathring{\mathbf{f}})\mathbf{n} + \frac{1}{\alpha_t}\mathring{\mathbf{f}} + \frac{-\left\{g(\hat{\chi}) - \frac{g'(\hat{\chi})}{M}(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\right\}(\mathbf{n}\cdot\mathring{\mathbf{f}}) + \frac{g'(\hat{\chi})}{M}(\hat{\boldsymbol{\tau}}\cdot\mathring{\mathbf{f}}) - m^c}{m^p}\hat{\boldsymbol{\tau}} \tag{65}$$

$$\overset{\circ}{\mathbf{f}}=\{\alpha_n\mathbf{n}\otimes\mathbf{n}+\alpha_t(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\}\left(\bar{\mathbf{v}}-\left\langle\frac{\left[-\alpha_n\left\{g(\hat{\chi})-\frac{g'(\hat{\chi})}{M}(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\right\}\mathbf{n}+\alpha_t\frac{g'(\hat{\chi})}{M}\hat{\boldsymbol{\tau}}\right]\cdot\bar{\mathbf{v}}-m^c}{m^p+\alpha_t\frac{g'(\hat{\chi})}{M}}\right\rangle\hat{\boldsymbol{\tau}}\right)$$

$$=(\alpha_n-\alpha_t)(\mathbf{n}\cdot\bar{\mathbf{v}})\mathbf{n}+\alpha_t\bar{\mathbf{v}}-\alpha_t\left\langle\frac{-\alpha_n\left\{g(\hat{\chi})-\frac{g'(\hat{\chi})}{M}(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\right\}(\mathbf{n}\cdot\bar{\mathbf{v}})+\alpha_t\frac{g'(\hat{\chi})}{M}(\hat{\boldsymbol{\tau}}\cdot\bar{\mathbf{v}})-m^c}{m^p+\alpha_t\frac{g'(\hat{\chi})}{M}}\right\rangle\hat{\boldsymbol{\tau}} \quad (66)$$

On the other hand, the normal sliding-yield and the sliding-subloading surfaces for the circular cone of the Coulomb friction condition is given by putting

$$f(\mathbf{f},\ \boldsymbol{\beta})=\|\hat{\boldsymbol{\eta}}\|,\ F=\mu \quad (67)$$

as follows:

$$\|\hat{\boldsymbol{\eta}}\|=\mu,\ \|\hat{\boldsymbol{\eta}}\|=\bar{R}\mu \quad (68)$$

where μ is the friction coefficient and the evolution rule is given in the identical form with Eq. (25) as follows:

$$\dot{\mu}=-\kappa\left\{\left(\frac{\mu}{\mu_k}\right)^m-1\right\}\|\bar{\mathbf{v}}^p\|+\xi\left\{1-\left(\frac{\mu}{\mu_s}\right)^n\right\} \quad (69)$$

μ_s and μ_k are material constants designating the maximum and the minimum frictions, i.e. static and kinetic friction coefficients, respectively. $f(\mathbf{f},\boldsymbol{\beta})$ in Eq. (67) is the homogeneous function of $\mathbf{f}$ in degree-zero, and the normal sliding-yield and sliding-subloading surfaces in Eq. (68) are open surfaces having a conical shape and thus expand/contact with the increase/decrease of μ and $\bar{R}$.

It holds for Eq. (68) that

$$m^p\equiv-\kappa\left\{\left(\frac{\mu}{\mu_k}\right)^m-1\right\}\bar{R}+\mu\bar{U} \quad (70)$$

$$m^c\equiv\xi\left\{1-\left(\frac{\mu}{\mu_s}\right)^n\right\}\bar{R}\ (\geq 0) \quad (71)$$

Further, it holds from Eqs. (16) and (57) that

$$\bar{\mathbf{N}}=\frac{\partial\|\hat{\boldsymbol{\eta}}\|}{\partial\mathbf{f}}=\frac{1}{f_n}\{(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\mathbf{n}+\hat{\boldsymbol{\tau}}\} \quad (72)$$

$$(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\cdot\bar{\mathbf{N}}=(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\cdot\frac{1}{f_n}\{(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\mathbf{n}+\hat{\boldsymbol{\tau}}\}=\frac{1}{f_n}\hat{\boldsymbol{\tau}} \quad (73)$$

$$\hat{\mathbf{t}}_n=\hat{\boldsymbol{\tau}} \quad (74)$$

$$\bar{\mathbf{N}}\cdot\mathbf{C}^e=\frac{1}{f_n}\{(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\mathbf{n}+\hat{\boldsymbol{\tau}}\}\cdot\{\alpha_n\mathbf{n}\otimes\mathbf{n}+\alpha_t(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\}=\frac{1}{f_n}\{\alpha_n(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\mathbf{n}+\alpha_t\hat{\boldsymbol{\tau}}\} \quad (75)$$

$$\bar{\mathbf{N}}\cdot\mathbf{C}^e\cdot\hat{\mathbf{t}}_n=\frac{1}{f_n}\{\alpha_n(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\mathbf{n}+\alpha_t\hat{\boldsymbol{\tau}}\}\cdot\hat{\boldsymbol{\tau}}=\frac{\alpha_t}{f_n} \quad (76)$$

The substitution of Eqs. (16) and (72)-(76) into Eqs. (38) and (40) leads to the sliding velocity vs. contact traction rate and its inverse relation are given as follows:

$$\bar{\mathbf{v}}=\left\{\frac{1}{\alpha_n}\mathbf{n}\otimes\mathbf{n}+\frac{1}{\alpha_t}(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\right\}\overset{\circ}{\mathbf{f}}+\frac{\frac{1}{f_n}\{(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\mathbf{n}+\hat{\boldsymbol{\tau}}\}\cdot\overset{\circ}{\mathbf{f}}-m^c}{m^p}\hat{\boldsymbol{\tau}}$$

$$=\left(\frac{1}{\alpha_n}-\frac{1}{\alpha_t}\right)(\mathbf{n}\cdot\overset{\circ}{\mathbf{f}})\mathbf{n}+\frac{1}{\alpha_t}\overset{\circ}{\mathbf{f}}+\frac{\frac{1}{f_n}\{(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})(\mathbf{n}\cdot\overset{\circ}{\mathbf{f}})+(\hat{\boldsymbol{\tau}}\cdot\overset{\circ}{\mathbf{f}})\}-m^c}{m^p}\hat{\boldsymbol{\tau}} \quad (77)$$

$$\overset{\circ}{\mathbf{f}}=\{\alpha_n\mathbf{n}\otimes\mathbf{n}+\alpha_t(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\}\left(\bar{\mathbf{v}}-\left\langle\frac{\frac{1}{f_n}\{\alpha_n(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})\mathbf{n}+\alpha_t\hat{\boldsymbol{\tau}}\}\cdot\bar{\mathbf{v}}-m^c}{m^p+\frac{\alpha_t}{f_n}}\right\rangle\hat{\boldsymbol{\tau}}\right)$$

$$=(\alpha_n-\alpha_t)(\mathbf{n}\cdot\bar{\mathbf{v}})\mathbf{n}+\alpha_t\bar{\mathbf{v}}-\frac{\alpha_t}{f_n}\left\langle\frac{\alpha_n(\hat{\boldsymbol{\tau}}\cdot\boldsymbol{\eta})(\mathbf{n}\cdot\bar{\mathbf{v}})+\alpha_t(\hat{\boldsymbol{\tau}}\cdot\bar{\mathbf{v}})-m^c}{m^p+\frac{\alpha_t}{f_n}}\right\rangle\hat{\boldsymbol{\tau}} \quad (78)$$

4. Extension to orthotropic anisotropy

The difference of friction coefficients in the mutually opposite sliding directions can be described by the aforementioned rotational anisotropy. However, the difference of the range of friction coefficients in the different sliding directions cannot be described by the rotational anisotropy. In order to extend so as to describe it, let the concept of orthotropy be further incorporated below.

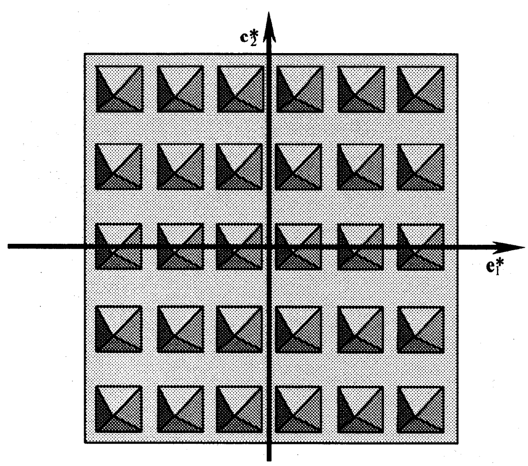

Fig. 3. Surface asperity model suggesting the rotational and the orthotropic anisotropy.

The simple surface asperity model is illustrated in order to obtain an insight into the anisotropy in Fig. 3. Here, the directions in the inclination of surface asperities would lead to the rotational anisotropy, and the anisotropic shapes and intervals of surface asperities to the orthotropic anisotropy. Now, choosing the bases $\mathbf{e}_1^*$ and $\mathbf{e}_2^*$ in the directions of the maximum and the minimum principal directions of anisotropy, respectively, and letting $\mathbf{e}_3^*$ coincide with $\mathbf{n}$ so as to make the right-hand coordinate system $(\mathbf{e}_1^*,\ \mathbf{e}_2^*,\ \mathbf{e}_3^*)$, it can be written as

$$\left.\begin{aligned}\mathbf{f} &= f_1^*\mathbf{e}_1^* + f_2^*\mathbf{e}_2^* + f_3^*\mathbf{e}_3^*\\ \boldsymbol{\beta} &= \beta_1^*\mathbf{e}_1^* + \beta_2^*\mathbf{e}_2^* + \beta_3^*\mathbf{e}_3^*\end{aligned}\right\} \tag{79}$$

while the spin $\boldsymbol{\Omega}$ of the base $(\mathbf{e}_1^*,\ \mathbf{e}_2^*,\ \mathbf{e}_3^*)$ is described as

$$\boldsymbol{\Omega} \equiv \dot{\mathbf{e}}_r^* \otimes \mathbf{e}_r^*,\quad \dot{\mathbf{e}}_i^* = \boldsymbol{\Omega}\mathbf{e}_i^*, \tag{80}$$

Eq. (79) is rewritten by $f_1^* = f_{t1}^*,\ f_2^* = f_{t2}^*,\ f_3^* = -f_n^*,\ \beta_1^* = \beta_{t1}^*,\ \beta_2^* = \beta_{t2}^*,\ \beta_3^* = 0$ as follows:

$$\left.\begin{aligned}\mathbf{f} &= f_{t1}^*\mathbf{e}_1^* + f_{t2}^*\mathbf{e}_2^* - f_n\mathbf{e}_3^*\\ \boldsymbol{\beta} &= \beta_{t1}^*\mathbf{e}_1^* + \beta_{t2}^*\mathbf{e}_2^*\end{aligned}\right\} \tag{81}$$

In Fig. 4 the section of the sliding-yield surface with the rotational and the orthotropic anisotropy is depicted in the coordinate system with the bases $(\mathbf{e}_1^*,\ \mathbf{e}_2^*)$.

Invoking the orthotropic anisotropy proposed by Mroz and Stupkiewicz (1994), let Eq. (48) with Eq. (49) taken account of the rotational anisotropy be extended as follows:

$$f(\mathbf{f},\ \boldsymbol{\beta}) = f_n\, g(\hat{\chi}^*) \tag{82}$$

$$f_n\, g(\hat{\chi}^*) = F,\quad f_n\, g(\hat{\chi}^*) = \bar{R}F, \tag{83}$$

where

$$\hat{\chi}^* \equiv \sqrt{\hat{\chi}_1^{*2} + \hat{\chi}_2^{*2}},\quad \hat{\chi}_1^* \equiv \frac{\hat{\eta}_1^*}{M_1},\quad \hat{\chi}_2^* \equiv \frac{\hat{\eta}_2^*}{M_2} \tag{84}$$

M_1 and M_2 are the material constants standing for the values of M in the maximum and the minimum principal directions of anisotropy, respectively.

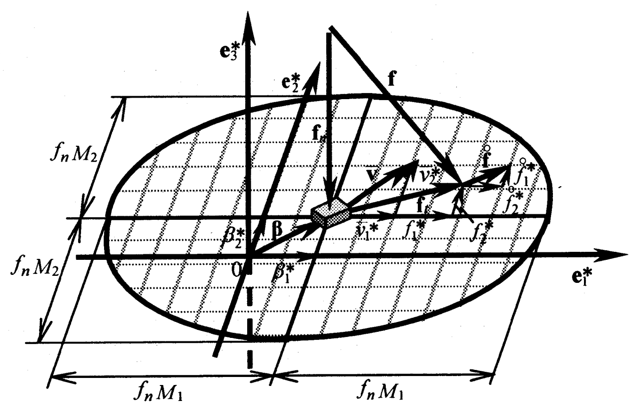

Fig. 4. Sliding-yield surface with the rotational and the orthotropic anisotropy.

The partial derivatives for Eq. (82) are given as

$$\left.\begin{aligned}
\frac{\partial \hat{\chi}_i^*}{\partial f_{ti}^*} &= \frac{\partial (f_{ti}^*/f_n - \beta_i^*)/M_i}{\partial f_{ti}^*} = \frac{1}{f_n M_i}\\
\frac{\partial \hat{\chi}_i^*}{\partial f_n} &= \frac{\partial (f_{ti}^*/f_n - \beta_i^*)/M_i}{\partial f_n} = \frac{-f_{ti}^*}{f_n^2 M_i} = -\frac{\chi_i^*}{f_n}\\
\frac{\partial \hat{\chi}^*}{\partial \hat{\chi}_i^*} &= \frac{1}{2\hat{\chi}^*}2\hat{\chi}_i^* = \hat{\zeta}_i^*\\
\frac{\partial \hat{\chi}^*}{\partial f_{ti}^*} &= \frac{\partial \hat{\chi}^*}{\partial \hat{\chi}_i^*}\frac{\partial \hat{\chi}_i^*}{\partial f_{ti}^*} = \frac{\hat{\chi}_i^*}{\hat{\chi}^*}\frac{1}{f_n M_i} = \frac{1}{f_n}\frac{\hat{\zeta}_i^*}{M_i}\\
\frac{\partial \hat{\chi}^*}{\partial f_n} &= \frac{\partial \hat{\chi}^*}{\partial \hat{\chi}_1^*}\frac{\partial \hat{\chi}_1^*}{\partial f_n} + \frac{\partial \hat{\chi}^*}{\partial \hat{\chi}_2^*}\frac{\partial \hat{\chi}_2^*}{\partial f_n} = -\frac{1}{f_n}(\hat{\zeta}_1^*\chi_1^* + \hat{\zeta}_2^*\chi_2^*)
\end{aligned}\right\} \tag{85}$$

where

$$\hat{\zeta}_i \equiv \frac{\hat{\chi}_i^*}{\hat{\chi}^*} \tag{86}$$

The subscript i takes 1 or 2 and is not summed even when it is repeated.

It holds from Eqs. (16) and (85) that

$$\begin{aligned}\frac{\partial \hat{\chi}^*}{\partial \mathbf{f}} &= \frac{\partial \hat{\chi}^*}{\partial f_{t1}^*}\mathbf{e}_1^* + \frac{\partial \hat{\chi}^*}{\partial f_{t2}^*}\mathbf{e}_2^* - \frac{\partial \hat{\chi}^*}{\partial f_n}\mathbf{n}\\ &= \frac{1}{f_n}\left\{\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^* + \frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^* + (\hat{\zeta}_1^*\chi_1^* + \hat{\zeta}_2^*\chi_2^*)\mathbf{n}\right\}\end{aligned} \tag{87}$$

$$\begin{aligned}\bar{\mathbf{N}} &= \frac{\partial (f_n\, g(\hat{\chi}^*))}{\partial \mathbf{f}} = \frac{\partial f_n}{\partial \mathbf{f}} g(\hat{\chi}^*) + f_n\, g'(\hat{\chi}^*)\frac{\partial \hat{\chi}^*}{\partial \mathbf{f}}\\
&= -g(\hat{\chi}^*)\mathbf{n}\\
&\quad + f_n\, g'(\hat{\chi}^*)\frac{1}{f_n}\left\{\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^* + \frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^* + (\hat{\zeta}_1^*\chi_1^* + \hat{\zeta}_2^*\chi_2^*)\mathbf{n}\right\}\\
&= -g(\hat{\chi}^*)\mathbf{n}\\
&\quad + g'(\hat{\chi}^*)\left\{\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^* + \frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^* + (\hat{\zeta}_1^*\chi_1^* + \hat{\zeta}_2^*\chi_2^*)\mathbf{n}\right\}\\
&= g'(\hat{\chi}^*)\left(\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^* + \frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^*\right)\\
&\quad - \{g(\hat{\chi}^*) - g'(\hat{\chi}^*)(\hat{\zeta}_1^*\chi_1^* + \hat{\zeta}_2^*\chi_2^*)\}\mathbf{n}\end{aligned} \tag{88}$$

$$\begin{aligned}(\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\bullet\bar{\mathbf{N}} &= (\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\bullet\Big[g'(\hat{\chi}^*)\left(\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^* + \frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^*\right)\\ &\qquad - \{g(\hat{\chi}^*) - g'(\hat{\chi}^*)(\hat{\zeta}_1^*\chi_1^* + \hat{\zeta}_2^*\chi_2^*)\}\mathbf{n}\Big]\\ &= g'(\hat{\chi}^*)\left(\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^* + \frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^*\right)\end{aligned} \tag{89}$$

$$\bar{\mathbf{t}}_n = \frac{1}{\sqrt{\left(\frac{\hat{\zeta}_1^*}{M_1}\right)^2 + \left(\frac{\hat{\zeta}_2^*}{M_2}\right)^2}}\left(\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^* + \frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^*\right) \tag{90}$$

$$\bar{\mathbf{N}}\bullet\mathbf{C}^e = \Big[g'(\hat{\chi}^*)\left(\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^* + \frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^*\right)$$

$$-\{g(\hat{\chi}^*)-g'(\hat{\chi}^*)(\hat{\zeta}_1^*\chi_1^*+\hat{\zeta}_2^*\chi_2^*)\}\mathbf{n}\Big]$$
$$\bullet\{\alpha_n\mathbf{e}_3^*\otimes\mathbf{e}_3^*+\alpha_t(\mathbf{e}_1^*\otimes\mathbf{e}_1^*+\mathbf{e}_2^*\otimes\mathbf{e}_2^*)\}$$
$$=\alpha_t g'(\hat{\chi}^*)\Big(\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^*+\frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^*\Big)$$
$$-\alpha_n\{g(\hat{\chi}^*)-g'(\hat{\chi}^*)(\hat{\zeta}_1^*\chi_1^*+\hat{\zeta}_2^*\chi_2^*)\}\mathbf{n} \quad (91)$$

$$\bar{\mathbf{N}}\bullet\mathbf{C}^e\bullet\bar{\mathbf{t}}_n=\Big[\alpha_t g'(\hat{\chi}^*)\Big(\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^*+\frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^*\Big)$$
$$-\alpha_n\{g(\hat{\chi}^*)-g'(\hat{\chi}^*)(\hat{\zeta}_1^*\chi_1^*+\hat{\zeta}_2^*\chi_2^*)\}\mathbf{n}\Big]$$
$$\bullet\frac{1}{\sqrt{\Big(\frac{\hat{\zeta}_1^*}{M_1}\Big)^2+\Big(\frac{\hat{\zeta}_2^*}{M_2}\Big)^2}}\Big(\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^*+\frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^*\Big)$$
$$=\alpha_t g'(\hat{\chi}^*)\sqrt{\Big(\frac{\hat{\zeta}_1^*}{M_1}\Big)^2+\Big(\frac{\hat{\zeta}_2^*}{M_2}\Big)^2} \quad (92)$$

The substitution of Eqs. (16) and (88)-(92) into Eqs. (38) and (40) leads to the sliding velocity vs. contact traction rate and its inverse relation are given as follows:

$$\bar{\mathbf{v}}=\Big\{\frac{1}{\alpha_n}\mathbf{n}\otimes\mathbf{n}+\frac{1}{\alpha_t}(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\Big\}(\mathring{f}_1\mathbf{e}_1^*+\mathring{f}_2\mathbf{e}_2^*-\mathring{f}_n\mathbf{n})$$
$$+\frac{\Big[g'(\hat{\chi}^*)\Big(\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^*+\frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^*\Big)-\{g(\hat{\chi}^*)-g'(\hat{\chi}^*)(\hat{\zeta}_1^*\chi_1^*+\hat{\zeta}_2^*\chi_2^*)\}\mathbf{n}\Big]\bullet(\mathring{f}_1\mathbf{e}_1^*+\mathring{f}_2\mathbf{e}_2^*-\mathring{f}_n\mathbf{n})-m^c}{m^p}$$
$$=\frac{1}{\alpha_t}(\mathring{f}_1\mathbf{e}_1^*+\mathring{f}_2\mathbf{e}_2^*)-\frac{1}{\alpha_n}\mathring{f}_n\mathbf{n}$$
$$+\frac{g'(\hat{\chi}^*)\Big(\frac{\hat{\zeta}_1^*}{M_1}\mathring{f}_1+\frac{\hat{\zeta}_2^*}{M_2}\mathring{f}_2\Big)+\{g(\hat{\chi}^*)-g'(\hat{\chi}^*)(\hat{\zeta}_1^*\chi_1^*+\hat{\zeta}_2^*\chi_2^*)\}\mathring{f}_n-m^c}{m^p}$$
$$\frac{1}{\sqrt{\Big(\frac{\hat{\zeta}_1^*}{M_1}\Big)^2+\Big(\frac{\hat{\zeta}_2^*}{M_2}\Big)^2}}\Big(\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^*+\frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^*\Big) \quad (93)$$

$$\mathring{\mathbf{f}}=\{\alpha_n\mathbf{n}\otimes\mathbf{n}+\alpha_t(\mathbf{e}_1^*\otimes\mathbf{e}_1^*+\mathbf{e}_2^*\otimes\mathbf{e}_2^*)\}\Big(\bar{v}_1\mathbf{e}_1^*+\bar{v}_2\mathbf{e}_2^*-\bar{v}_n\mathbf{n}$$
$$-\Big\langle\frac{\Big\{\alpha_t g'(\hat{\chi}^*)\Big(\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^*+\frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^*\Big)-\alpha_n\{g(\hat{\chi}^*)-g'(\hat{\chi}^*)(\hat{\zeta}_1^*\chi_1^*+\hat{\zeta}_2^*\chi_2^*)\}\mathbf{n}\Big\}\bullet(\bar{v}_1\mathbf{e}_1^*+\bar{v}_2\mathbf{e}_2^*-\bar{v}_n\mathbf{n})-m^c}{m^p+\alpha_t g'(\hat{\chi}^*)\sqrt{\Big(\frac{\hat{\zeta}_1^*}{M_1}\Big)^2+\Big(\frac{\hat{\zeta}_2^*}{M_2}\Big)^2}}\Big\rangle$$
$$\frac{1}{\sqrt{\Big(\frac{\hat{\zeta}_1^*}{M_1}\Big)^2+\Big(\frac{\hat{\zeta}_2^*}{M_2}\Big)^2}}\Big(\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^*+\frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^*\Big)\Big)$$
$$=\alpha_t(\bar{v}_1\mathbf{e}_1^*+\bar{v}_2\mathbf{e}_2^*)-\alpha_n\bar{v}_n\mathbf{n}$$
$$-\Big\langle\frac{\Big\{\alpha_t g'(\hat{\chi}^*)\Big(\frac{\hat{\zeta}_1^*}{M_1}\bar{v}_1+\frac{\hat{\zeta}_2^*}{M_2}\bar{v}_2\Big)+\alpha_n\{g(\hat{\chi}^*)-g'(\hat{\chi}^*)(\hat{\zeta}_1^*\chi_1^*+\hat{\zeta}_2^*\chi_2^*)\}\bar{v}_n-m^c}{m^p+\alpha_t g'(\hat{\chi}^*)\sqrt{\Big(\frac{\hat{\zeta}_1^*}{M_1}\Big)^2+\Big(\frac{\hat{\zeta}_2^*}{M_2}\Big)^2}}\Big\rangle$$
$$\frac{\alpha_t}{\sqrt{\Big(\frac{\hat{\zeta}_1^*}{M_1}\Big)^2+\Big(\frac{\hat{\zeta}_2^*}{M_2}\Big)^2}}\Big(\frac{\hat{\zeta}_1^*}{M_1}\mathbf{e}_1^*+\frac{\hat{\zeta}_2^*}{M_2}\mathbf{e}_2^*\Big) \quad (94)$$

Eq. (67) with Eq. (68) for Coulomb friction condition with the rotational anisotropy is extended to the orthotropic anisotropy as follows:

$$f(\mathbf{f},\ \boldsymbol{\beta})=\hat{\chi}_c^*,\quad F=\mu \quad (95)$$

$$\hat{\chi}_c^*=\mu,\ \hat{\chi}_c^*=\bar{R}\mu \quad (96)$$

where

$$\hat{\chi}_c^*\equiv\sqrt{\hat{\chi}_{c1}^{*2}+\hat{\chi}_{c2}^{*2}},\ \hat{\chi}_{c1}^*\equiv\frac{\hat{\eta}_1^*}{C_1},\ \hat{\chi}_{c2}^*\equiv\frac{\hat{\eta}_2^*}{C_2} \quad (97)$$

C_1 and C_2 are the material constants. The partial derivatives for Eq. (95) are given as follows:

$$\left.\begin{aligned}
&\frac{\partial\hat{\chi}_{c\mathrm{i}}^*}{\partial f_{tj}^*}=\frac{\partial(f_{t\mathrm{i}}^*/f_n-\beta_\mathrm{i}^*)/C_\mathrm{i}}{\partial f_{t\mathrm{i}}^*}=\frac{1}{f_nC_\mathrm{i}}\\
&\frac{\partial\hat{\chi}_{c\mathrm{i}}^*}{\partial f_n}=\frac{\partial(f_{t\mathrm{i}}^*/f_n-\beta_\mathrm{i}^*)/C_\mathrm{i}}{\partial f_n}=\frac{-f_{t\mathrm{i}}^*}{f_n^2C_\mathrm{i}}=-\frac{\hat{\chi}_{c\mathrm{i}}^*}{f_n}\\
&\frac{\partial\hat{\chi}_c^*}{\partial\hat{\chi}_{c\mathrm{i}}^*}=\frac{1}{2\hat{\chi}_c^*}2\hat{\chi}_{c\mathrm{i}}^*=\hat{\zeta}_{c\mathrm{i}}^*\\
&\frac{\partial\hat{\chi}_c^*}{\partial f_{t\mathrm{i}}^*}=\frac{\partial\hat{\chi}_c^*}{\partial\hat{\chi}_{c\mathrm{i}}^*}\frac{\partial\hat{\chi}_{c\mathrm{i}}^*}{\partial f_{t\mathrm{i}}^*}=\frac{\hat{\chi}_{c\mathrm{i}}^*}{\hat{\chi}_c^*}\frac{1}{f_nC_\mathrm{i}}=\frac{1}{f_n}\frac{\hat{\zeta}_{c\mathrm{i}}^*}{C_\mathrm{i}}\\
&\frac{\partial\hat{\chi}_c^*}{\partial f_n}=\frac{\partial\hat{\chi}_c^*}{\partial\hat{\chi}_{c1}^*}\frac{\partial\hat{\chi}_{c1}^*}{\partial f_n}+\frac{\partial\hat{\chi}_c^*}{\partial\hat{\chi}_{c2}^*}\frac{\partial\hat{\chi}_{c2}^*}{\partial f_n}=-\frac{1}{f_n}\big(\hat{\zeta}_{c1}^*\chi_{c1}^*+\hat{\zeta}_{c2}^*\chi_{c2}^*\big)
\end{aligned}\right\} \quad (98)$$

Further, it holds from Eqs. (16) and (98) that

$$\bar{\mathbf{N}}=\frac{\partial\hat{\chi}_c^*}{\partial\mathbf{f}}=\frac{1}{f_n}\Big\{\frac{\hat{\zeta}_{c1}^*}{C_1}\mathbf{e}_1^*+\frac{\hat{\zeta}_{c2}^*}{C_2}\mathbf{e}_2^*+(\hat{\zeta}_{c1}^*\chi_{c1}^*+\hat{\zeta}_{c2}^*\chi_c^*)\mathbf{n}\Big\} \quad (99)$$

$$(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\bullet\bar{\mathbf{N}}=(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})$$
$$\bullet\frac{1}{f_n}\Big\{\frac{\hat{\zeta}_{c1}^*}{C_1}\mathbf{e}_1^*+\frac{\hat{\zeta}_{c2}^*}{C_2}\mathbf{e}_2^*+(\hat{\zeta}_{c1}^*\chi_{c1}^*+\hat{\zeta}_{c2}^*\chi_c^*)\mathbf{n}\Big\}$$
$$=\frac{1}{f_n}\Big(\frac{\hat{\zeta}_{c1}^*}{C_1}\mathbf{e}_1^*+\frac{\hat{\zeta}_{c2}^*}{C_2}\mathbf{e}_2^*\Big) \quad (100)$$

$$\bar{\mathbf{t}}_n=\frac{1}{\sqrt{(\frac{\hat{\zeta}^*_{c1}}{C_1})^2+(\frac{\hat{\zeta}^*_{c2}}{C_2})^2}}(\frac{\hat{\zeta}^*_{c1}}{C_1}\mathbf{e}^*_1+\frac{\hat{\zeta}^*_{c2}}{C_2}\mathbf{e}^*_2) \tag{101}$$

$$\bar{\mathbf{N}}\bullet\mathbf{C}^e=\frac{1}{f_n}\{\frac{\hat{\zeta}^*_{c1}}{C_1}\mathbf{e}^*_1+\frac{\hat{\zeta}^*_{c2}}{C_2}\mathbf{e}^*_2+(\hat{\zeta}^*_{c1}\chi^*_{c1}+\hat{\zeta}^*_{c2}\chi^*_c)\mathbf{n}\}$$
$$\bullet\{\alpha_n\mathbf{e}^*_3\otimes\mathbf{e}^*_3+\alpha_t(\mathbf{e}^*_1\otimes\mathbf{e}^*_1+\mathbf{e}^*_2\otimes\mathbf{e}^*_2)\}$$
$$=\frac{1}{f_n}\left\{\alpha_t(\frac{\hat{\zeta}^*_{c1}}{C_1}\mathbf{e}^*_1+\frac{\hat{\zeta}^*_{c2}}{C_2})+\alpha_n(\hat{\zeta}^*_{c1}\chi^*_{c1}+\hat{\zeta}^*_{c2}\chi^*_c)\mathbf{n}\right\} \tag{102}$$

$$\bar{\mathbf{N}}\bullet\mathbf{C}^e\bullet\bar{\mathbf{t}}_n=\frac{1}{f_n}\left\{\alpha_t(\frac{\hat{\zeta}^*_{c1}}{C_1}\mathbf{e}^*_1+\frac{\hat{\zeta}^*_{c2}}{C_2})+\alpha_n(\hat{\zeta}^*_{c1}\chi^*_{c1}+\hat{\zeta}^*_{c2}\chi^*_c)\mathbf{n}\right\}$$
$$\bullet\frac{1}{\sqrt{(\frac{\hat{\zeta}^*_{c1}}{C_1})^2+(\frac{\hat{\zeta}^*_{c2}}{C_2})^2}}(\frac{\hat{\zeta}^*_{c1}}{C_1}\mathbf{e}^*_1+\frac{\hat{\zeta}^*_{c2}}{C_2}\mathbf{e}^*_2)$$
$$=\frac{\alpha_t}{f_n}\sqrt{(\frac{\hat{\zeta}^*_{c1}}{C_1})^2+(\frac{\hat{\zeta}^*_{c2}}{C_2})^2} \tag{103}$$

The substitution of Eqs. (16) and (99)-(103) into Eqs. (38) and (40) leads to the sliding velocity vs. contact traction rate and its inverse relation are given as follows:

$$\bar{\mathbf{v}}=\left\{\frac{1}{\alpha_n}\mathbf{n}\otimes\mathbf{n}+\frac{1}{\alpha_t}(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\right\}(\mathring{f}_1\mathbf{e}^*_1+\mathring{f}_2\mathbf{e}^*_2-\mathring{f}_n\mathbf{n})$$
$$\frac{\frac{1}{f_n}\left\{\frac{\hat{\zeta}^*_{c1}}{C_1}\mathbf{e}^*_1+\frac{\hat{\zeta}^*_{c2}}{C_2}\mathbf{e}^*_2+(\hat{\zeta}^*_{c1}\chi^*_1+\hat{\zeta}^*_{c2}\chi^*_c)\mathbf{n}\right\}\bullet(\mathring{f}_1\mathbf{e}^*_1+\mathring{f}_2\mathbf{e}^*_2-\mathring{f}_n\mathbf{n})-m^c}{m^p}$$
$$\frac{1}{\sqrt{(\frac{\hat{\zeta}^*_{c1}}{C_1})^2+(\frac{\hat{\zeta}^*_{c2}}{C_2})^2}}(\frac{\hat{\zeta}^*_{c1}}{C_1}\mathbf{e}^*_1+\frac{\hat{\zeta}^*_{c2}}{C_2}\mathbf{e}^*_2)$$
$$=\frac{1}{\alpha_t}(\mathring{f}_1\mathbf{e}^*_1+\mathring{f}_2\mathbf{e}^*_2)-\frac{1}{\alpha_n}\mathring{f}_n\mathbf{n}$$
$$+\frac{\frac{1}{f_n}\left\{\frac{\hat{\zeta}^*_{c1}}{C_1}\mathring{f}_1+\frac{\hat{\zeta}^*_{c2}}{C_2}\mathring{f}_2-(\hat{\zeta}^*_{c1}\chi^*_{c1}+\hat{\zeta}^*_{c2}\chi^*_c)\mathring{f}_n\right\}-m^c}{m^p}$$
$$\frac{1}{\sqrt{(\frac{\hat{\zeta}^*_{c1}}{C_1})^2+(\frac{\hat{\zeta}^*_{c2}}{C_2})^2}}(\frac{\hat{\zeta}^*_{c1}}{C_1}\mathbf{e}^*_1+\frac{\hat{\zeta}^*_{c2}}{C_2}\mathbf{e}^*_2) \tag{104}$$

$$\mathring{\mathbf{f}}=\{\alpha_n\mathbf{n}\otimes\mathbf{n}+\alpha_t(\mathbf{e}^*_1\otimes\mathbf{e}^*_1+\mathbf{e}^*_2\otimes\mathbf{e}^*_2)\}\left(\bar{v}_1\mathbf{e}^*_1+\bar{v}_2\mathbf{e}^*_2-\bar{v}_n\mathbf{n}\right.$$
$$-\left\langle\frac{\frac{1}{f_n}\left\{\alpha_t(\frac{\hat{\zeta}^*_{c1}}{C_1}\mathbf{e}^*_1+\frac{\hat{\zeta}^*_{c2}}{C_2})+\alpha_n(\hat{\zeta}^*_{c1}\chi^*_c+\hat{\zeta}^*_2\chi^*_c)\mathbf{n}\right\}\bullet(\bar{v}_1\mathbf{e}^*_1+\bar{v}_2\mathbf{e}^*_2-\bar{v}_n\mathbf{n})-m^c}{m^p+\frac{\alpha_t}{f_n}\sqrt{(\frac{\hat{\zeta}^*_{c1}}{C_1})^2+(\frac{\hat{\zeta}^*_{c2}}{C_2})^2}}\right\rangle$$
$$\frac{1}{\sqrt{(\frac{\hat{\zeta}^*_{c1}}{C_1})^2+(\frac{\hat{\zeta}^*_{c2}}{C_2})^2}}(\frac{\hat{\zeta}^*_{c1}}{C_1}\mathbf{e}^*_1+\frac{\hat{\zeta}^*_{c2}}{C_2}\mathbf{e}^*_2)$$
$$=\alpha_t(\bar{v}_1\mathbf{e}^*_1+\bar{v}_2\mathbf{e}^*_2)-\alpha_n\bar{v}_n\mathbf{n}$$
$$-\left\langle\frac{\frac{1}{f_n}\left\{\alpha_t(\frac{\hat{\zeta}^*_{c1}}{C_1}\bar{v}_1+\frac{\hat{\zeta}^*_{c2}}{C_2}\bar{v}_2)-\alpha_n(\hat{\zeta}^*_{c1}\chi^*_{c1}+\hat{\zeta}^*_{c2}\chi^*_c)\bar{v}_n\right\}-m^c}{m^p+\alpha_t g'(\hat{\chi}^*)\sqrt{(\frac{\hat{\zeta}^*_1}{M_1})^2+(\frac{\hat{\zeta}^*_2}{M_2})^2}}\right\rangle$$
$$\frac{1}{\sqrt{(\frac{\hat{\zeta}^*_{c1}}{C_1})^2+(\frac{\hat{\zeta}^*_{c2}}{C_2})^2}}(\frac{\hat{\zeta}^*_{c1}}{C_1}\mathbf{e}^*_1+\frac{\hat{\zeta}^*_{c2}}{C_2}\mathbf{e}^*_2) \tag{105}$$

The calculation for sliding with the orthotropic anisotropy has to be performed in the coordinate system with the principal axes of orthotropy, i.e. $(\mathbf{e}^*_1,\ \mathbf{e}^*_2,\ \mathbf{n})$.

5. Linear sliding phenomenon

We examine below the basic response of the present friction model by the numerical experiments and the comparison with test data for the linear sliding phenomenon (Fig. 1) without a normal sliding velocity leading to

$$\bar{\mathbf{v}}_n=\mathbf{0} \tag{106}$$

The traction rate vs. sliding velocity relation for Eq. (82) with Eq. (83) under the condition (106) is given from Eqs. (40), (90)-(92) by

$$\mathring{\mathbf{f}}=\mathring{\mathbf{f}}_t=\{\alpha_t(\mathbf{e}^*_1\otimes\mathbf{e}^*_2+\mathbf{e}^*_1\otimes\mathbf{e}^*_2)+\alpha_n\mathbf{e}^*_3\otimes\mathbf{e}^*_3\}\left\{\bar{v}_1\mathbf{e}^*_1+\bar{v}_2\mathbf{e}^*_2\right.$$
$$-\frac{1}{m^p+\alpha_t g'(\hat{\chi}^*)}\left\langle\left[\alpha_t\frac{g'(\hat{\chi}^*)}{\hat{\chi}^*}(\frac{\hat{\Gamma}_1}{M_1}\mathbf{e}^*_1+\frac{\hat{\Gamma}_2}{M_2}\mathbf{e}^*_2)+\alpha_n\left\{-g(\hat{\chi}^*)+\frac{g'(\hat{\chi}^*)}{\hat{\chi}^*}(\hat{\Gamma}_1\frac{\eta^*_1}{M_1}+\hat{\Gamma}_2\frac{\eta^*_2}{M_2})\right\}\mathbf{e}^*_3\right]\right.$$
$$\left.\left.\bullet(\bar{v}_1\mathbf{e}^*_1+\bar{v}_2\mathbf{e}^*_2)-m^c\right\rangle\frac{g'(\hat{\chi}^*)}{\hat{\chi}^*}(\frac{\hat{\Gamma}_1}{M_1}\mathbf{e}^*_1+\frac{\hat{\Gamma}_2}{M_2}\mathbf{e}^*_2)\right\}$$
$$=\alpha_t(\bar{v}_1\mathbf{e}^*_1+\bar{v}_2\mathbf{e}^*_2)$$
$$-\alpha_t\frac{g'(\hat{\chi}^*)}{\hat{\chi}^*}\left\langle\frac{\alpha_t\frac{g'(\hat{\chi}^*)}{\hat{\chi}^*}(\frac{\hat{\Gamma}_1}{M_1}\bar{v}_1+\frac{\hat{\Gamma}_2}{M_2}\bar{v}_2)-M^c}{m^p+\alpha_t g'(\hat{\chi}^*)}\right\rangle$$
$$(\frac{\hat{\Gamma}_1}{M_1}\mathbf{e}^*_1+\frac{\hat{\Gamma}_2}{M_2}\mathbf{e}^*_2) \tag{107}$$

while it holds that $\mathbf{f}_n=\text{const.}$

The traction rate for Eq. (95) with Eq. (96) is given from Eqs. (40) and (101)-(107) by

$$
\begin{aligned}
\overset{\circ}{\mathbf{f}} = \overset{\circ}{\mathbf{f}}_t = & \{\alpha_t(\mathbf{e}_1^* \otimes \mathbf{e}_2^* + \mathbf{e}_1^* \otimes \mathbf{e}_2^*) + \alpha_n \mathbf{e}_3^* \otimes \mathbf{e}_3^*\} \Big\{ \bar{v}_1 \mathbf{e}_1^* + \bar{v}_2 \mathbf{e}_2^* \\
& - \frac{1}{m^p + \dfrac{\alpha_t}{f_n}} \Big\langle \Big[\frac{1}{f_n \hat{\chi}_c^*} \Big\{ \alpha_t \Big(\frac{\hat{\Gamma}_{c1}}{C_1} \mathbf{e}_1^* + \frac{\hat{\Gamma}_{c2}}{C_2} \mathbf{e}_2^* \Big) \\
& \qquad + \alpha_n \Big(\hat{\Gamma}_{c1} \frac{\eta_1^*}{C_1} + \hat{\Gamma}_{c2} \frac{\eta_2^*}{C_2} \Big) \mathbf{e}_3^* \Big\} \Big] \\
& \qquad \bullet (\bar{v}_1 \mathbf{e}_1^* + \bar{v}_2 \mathbf{e}_2^*) - m^c \Big\rangle \frac{1}{\hat{\chi}_c^*} \Big(\frac{\hat{\Gamma}_{c1}}{C_1} \mathbf{e}_1^* + \frac{\hat{\Gamma}_{c2}}{C_2} \mathbf{e}_2^* \Big) \Big\} \\
= & \ \alpha_t (\bar{v}_1 \mathbf{e}_1^* + \bar{v}_2 \mathbf{e}_2^*) \\
& - \frac{\alpha_t}{\hat{\chi}_c^*} \Big\langle \frac{ \dfrac{\alpha_t}{f_n \hat{\chi}_c^*} \Big(\dfrac{\hat{\Gamma}_1}{C_1} \bar{v}_1 + \dfrac{\hat{\Gamma}_2}{C_2} \bar{v}_2 \Big) - m^c }{ m^p + \dfrac{\alpha_t}{f_n} } \Big\rangle \Big(\frac{\hat{\Gamma}_{c1}}{C_1} \mathbf{e}_1^* + \frac{\hat{\Gamma}_{c2}}{C_2} \mathbf{e}_2^* \Big)
\end{aligned}
\tag{108}
$$

while m^p and m^c are given by Eq. (70) and(71).

6. Concluding remarks

The constitutive model for friction is formulated by extending the subloading friction model[1)] so as to describe the isotropic hardening/softening of sliding-yield surface in this article. Fundamental features of this model are as follows:

1. The process for the rising of friction coefficient up to the static-friction and the subsequent reduction to the kinetic-friction is formulated in the unified way as the isotropic softening process due to the plastic sliding based on the concept of subloading surface describing the smooth elastic-plastic transition, although only the rising process has been discussed and it has been described as the isotropic hardening process in the past models[4)-12), 14), 17)].
2. The process for the recovery from the kinetic- to static-friction is formulated as the isotropic hardening due to the creep deformations of surface asperities, while it has been formulated by the irrational equation involving the elapsed time after the stop of sliding so far.
3. The smooth elastic-plastic transition is depicted and the cyclic sliding behavior can be described by incorporating the concept of the sliding-subloading surface in which the plastic sliding velocity due to the rate of contact traction inside the normal sliding-yield surface is described exhibiting the smooth elastic-plastic transition. It is inevitable for the prediction of the loosing of screws, bolts and piles, the smooth stress/strain distribution at contact surface and the increase of traction with slip in wheel rotation on a solid surface for instance.
4. The reduction of friction coefficient with the increase of normal contact traction is described by incorporating the nonlinear sliding-yield condition.
5. A judgment whether or not the sliding yield condition is fulfilled is not required in the loading criterion for the plastic sliding velocity. This advantage is of importance especially for the analysis of cyclic friction phenomena in which a loading and an unloading are repeated.
6. The difference of friction coefficients in the mutually opposite sliding directions and the difference of the range of friction coefficients in the different sliding directions are described by the rotational and the orthotropic anisotropy, i.e. the rotation and the orthotropy of sliding-yield surface.

The constitutive equation of friction formulated in this article would be applicable widely to friction phenomena between solids. It will be extended so as to be applicable to rubber-like material exhibiting a large nonlinear elastic behavior in the future.

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