

An improvement of the discrete method for analyzing the bending problem of plates

板の曲げ問題に関する離散的近似解法の改善

Mei HUANG *, Hironobu TAKAHASHI **, Hiroshi MATSUDA ***,
Chihiro MORITA ****

黄 美・高橋 弘展・松田 浩・森田 千尋

*Member Dr. Eng. Asst. Prof. Dept. Struct. Eng. Nagasaki Univ. (852-8521, Nagasaki)

**BE. Kyokuyo Shipyard Corporation (752-0953, Yamaguchi)

***Member Dr. Eng. Prof. Dept. Struct. Eng. Nagasaki Univ. (852-8521, Nagasaki)

****Member Dr. Eng. Assoc. Prof. Dept. Struct. Eng. Nagasaki Univ. (852-8521, Nagasaki)

An improved method is proposed to analyze the bending problem of plates. The fundamental differential equations are satisfied for the whole plate. By transforming these differential equations into integral equations in a small area, the quantities of an appointed point can be expressed by those of the other three points. By choosing the appointed point according to a regular order, the quantities of these three points can be replaced by the quantities of the boundary points. Finally, the quantities of any point can be expressed by those of the boundary points and the unknown quantities are only on the boundary. That makes the number of the unknown quantities and the computer time of the coefficient reduce greatly. The comparison of the present method with that used early is presented and the advantages of the present method are shown. Some numerical results are given by using uniform or non-uniform divisions. By comparing the numerical results obtained by the present method with those previously published, the efficiency and accuracy of the present method are investigated.

Key Words : *bending problem, plate, uniform division, non-uniform division*

1. Introduction

Plates are important components in aeronautical, mechanical and ocean structures. The analytical solutions of the plate are limited to only simple plate geometries and boundary conditions. For complex geometries and general boundary conditions, some numerical methods of analysis, such as the finite difference¹⁾, finite element²⁾, spline element method³⁾ or boundary element method⁴⁾, are used.

In this paper, an improved discrete method is proposed for analysing the bending problem of plates based on the Mindlin plate theory. The method is based on the discrete method^{5),6)}. Like the method^{5),6)}, the present method doesn't employ the prior assumption of shape of deflection, such as shape function used in the finite element method. So the phenomenon of the shear locking doesn't happen. Due to the unknown quantities are only on the boundary, the number of these quantities is fewer. That helps to save the computer storage. Compared with the method⁵⁾, the present method has two advantages. One is that the present method requires less computer

time to calculate the coefficients. Another is that non-uniform divisions can be used. Some numerical results are given to show the efficiency and accuracy of the present method for the bending problem of the plates.

2. Discrete method

In this section, the discrete method^{5),6)} for the plate bending problems will be reviewed.

2.1 Fundamental Differential Equations

Consider a rectangular plate of length a , width b , density ρ . An xyz coordinate system is used in the present study with its $x - y$ plane contained in the middle plane of the rectangular plate, the z -axis perpendicular to the middle plane of the plate and the origin at one of the corners of the plate.

In this paper, the deflection w , the rotations θ_x, θ_y , the shearing forces Q_x, Q_y , the twisting moment M_{xy} and the bending moments M_x, M_y are used as variables.

Considering the equations of equilibrium, the

strain-displacement relations, the stress-strain relations and the load-stress relations, the fundamental differential equations of the plate having uniform load $\bar{q}$ are as follows:

$$\begin{aligned}
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \bar{q} &= 0 \\
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y &= 0, \\
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x &= 0, \\
\frac{\partial \theta_x}{\partial x} + \nu \frac{\partial \theta_y}{\partial y} &= \frac{M_x}{D}, \\
\frac{\partial \theta_y}{\partial y} + \nu \frac{\partial \theta_x}{\partial x} &= \frac{M_y}{D}, \\
\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} &= \frac{2}{(1-\nu)} \frac{M_{xy}}{D}, \\
\frac{\partial w}{\partial x} + \theta_x &= \frac{Q_x}{Gt_s}, \\
\frac{\partial w}{\partial y} + \theta_y &= \frac{Q_y}{Gt_s},
\end{aligned} \quad (1)$$

where $D = Eh^3/(12(1-\nu^2))$ is the bending rigidity; E and G are modulus and shear modulus of elasticity, respectively; ν is Poisson's ratio; h is the thickness of plate; $t_s = h/1.2$ in which 1.2 is the shear correction factor.

By choosing the standard thickness and bending rigidity of the plate as h_0 and $D_0 = Eh_0^3/(12(1-\nu^2))$ and introducing the non-dimensional expressions,

$$[X_1, X_2] = \frac{a^2}{D_0(1-\nu^2)} [Q_y, Q_x],$$

$$[X_3, X_4, X_5] = \frac{a}{D_0(1-\nu^2)} [M_{xy}, M_y, M_x],$$

$$[X_6, X_7, X_8] = [\theta_y, \theta_x, w/a], [\eta, \zeta] = [x/a, y/b],$$

the simple systemized expression of fundamental differential equations of the bending problem of a rectangular plate is as follows ^{5),6)}:

$$\sum_{s=1}^8 \{F_{1ts} \frac{\partial X_s}{\partial \zeta} + F_{2ts} \frac{\partial X_s}{\partial \eta} + F_{3ts} X_s\} + q\delta_{1t} = 0, \quad (2)$$

where $t = 1 \sim 8$; $q = \bar{q}\mu a^2/D(1-\nu^2)$; q the distributed load; $\mu = b/a$; δ_{1t} Kronecker's delta; F_{1ts} , F_{2ts} and F_{3ts} are given in Appendix A.

2.2 Fundamental Solutions

As given in Ref. ^{5),6)}, by dividing a rectangular plate vertically into m equal-length parts and horizontally into n equal-length parts as shown in Fig. 1, the plate can be considered as a group of discrete points which are the intersections of the $(m+1)$ -vertical and $(n+1)$ -horizontal dividing lines. By integrating Eq. (1) over the area $0 \leq \eta \leq \eta_i$, $0 \leq \zeta \leq \zeta_j$ and applying the numerical integration method, the simultaneous

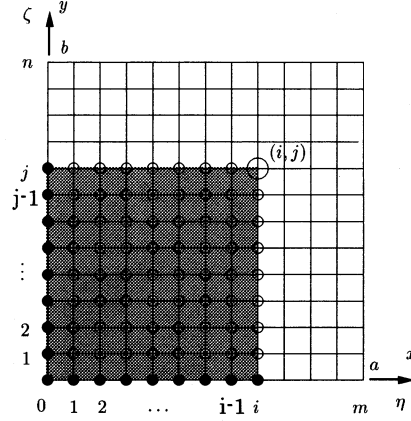


Fig. 1 Discrete points on a rectangular plate.

equation for the unknown quantities $X_{pij} = X_p(\eta_i, \zeta_j)$ at the point (i, j) is obtained as follows:

$$\begin{aligned}
X_{pij} = & \sum_{t=1}^8 \left\{ \sum_{k=0}^i \beta_{ik} A_{pt} [X_{tk0} - X_{tkj}(1-\delta_{ik})] \right. \\
& + \sum_{l=0}^j \beta_{jl} B_{pt} [X_{t0l} - X_{til}(1-\delta_{jl})] \\
& + \sum_{k=0}^i \sum_{l=0}^j \beta_{ik} \beta_{jl} C_{ptkl} X_{tkl} (1-\delta_{ik} \delta_{jl}) \left. \right\} \\
& - \sum_{k=0}^i \sum_{l=0}^j \beta_{ik} \beta_{jl} A_{p1} q_{ij},
\end{aligned} \quad (3)$$

where $p = 1 \sim 8$, A_{pt} , B_{pt} and C_{ptkl} are given in Appendix A.

By spreading the area according to the order mentioned in Ref. ^{5),6)}, the quantity X_{pij} at the point (i, j) is only related to the quantities X_{rk0} ($r=1,3,4,6,7,8$) and X_{s0l} ($s=2,3,5,6,7,8$) at the boundary dependent points. Eq. (3) is rewritten as follows.

$$X_{pij} = \sum_{d=1}^6 \left\{ \sum_{f=0}^i a_{pijfd} X_{rf0} + \sum_{g=0}^j b_{pijgd} X_{s0g} \right\} + q_{pij}, \quad (4)$$

where

$$\begin{aligned}
a_{pijfd} = & \sum_{t=1}^8 \left\{ \sum_{k=0}^i \beta_{ik} A_{pt} [a_{tk0fd} - a_{tkjfd}(1-\delta_{ki})] \right. \\
& + \sum_{l=0}^j \beta_{jl} B_{pt} [a_{t0lfd} - a_{tilfd}(1-\delta_{lj})] \\
& + \sum_{k=0}^i \sum_{l=0}^j \beta_{ik} \beta_{jl} C_{ptkl} a_{tklfd} (1-\delta_{ki} \delta_{lj}) \left. \right\} \\
b_{pijgd} = &
\end{aligned} \quad (5)$$

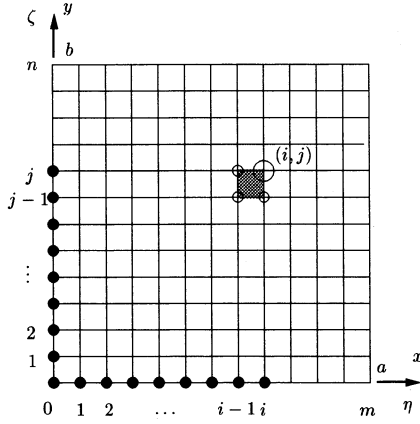


Fig. 2 Discrete points on a rectangular plate.

$$\sum_{t=1}^8 \left\{ \sum_{k=0}^i \beta_{ik} A_{pt} [b_{tk0gd} - b_{tkjgd}(1 - \delta_{ki})] + \sum_{l=0}^j \beta_{jl} B_{pt} [b_{t0lgd} - b_{tilgd}(1 - \delta_{lj})] + \sum_{k=0}^i \sum_{l=0}^j \beta_{ik} \beta_{jl} C_{ptkl} b_{tklgd}(1 - \delta_{ki} \delta_{lj}) \right\} \quad (6)$$

$$q_{pij} = \sum_{t=1}^8 \left\{ \sum_{k=0}^i \beta_{ik} A_{pt} [q_{tk0} - q_{tkj}(1 - \delta_{ki})] + \sum_{l=0}^j \beta_{jl} B_{pt} [q_{t0l} - q_{til}(1 - \delta_{lj})] + \sum_{k=0}^i \sum_{l=0}^j \beta_{ik} \beta_{jl} C_{ptkl} q_{tkl}(1 - \delta_{ki} \delta_{lj}) \right\} - \sum_{k=0}^i \sum_{l=0}^j A_{p1} q_{kl} \quad (7)$$

3. The Present Method

Basing the discrete method ^{5),6)}, an improved method for the plate bending problems is proposed.

3.1 Fundamental Differential Equations

The simple systemized expression of fundamental differential equations is as the same as Eq.(2).

3.2 Fundamental Solutions

By dividing a rectangular plate vertically into m equal-length parts and horizontally into n equal-length parts as shown in Fig. (2), the plate can be considered as a group of discrete points which are the intersections of the $(m+1)$ -vertical and $(n+1)$ -horizontal dividing lines. By integrating Eq. (2) over

the area $\eta_{(i-1)} \leq \eta \leq \eta_i$, $\zeta_{(j-1)} \leq \zeta \leq \zeta_j$, the following equation can be obtained.

$$\int_{\eta_{(i-1)}}^{\eta_i} \int_{\zeta_{(j-1)}}^{\zeta_j} \sum_{s=1}^8 \left\{ F_{1ts} \frac{\partial X_s}{\partial \zeta} + F_{2ts} \frac{\partial X_s}{\partial \eta} + F_{3ts} X_s \right\} d\eta d\zeta + \int_{\eta_{(i-1)}}^{\eta_i} \int_{\zeta_{(j-1)}}^{\zeta_j} q \delta_{1t} d\eta d\zeta = 0, \quad (8)$$

By using the Green integration and the trapezoidal integration rule, the simultaneous equation for the unknown quantities $X_{pij} = X_p(\eta_i, \zeta_j)$ at the point (i, j) is obtained as follows:

$$X_{pij} = \sum_{t=1}^8 \left\{ \beta_{ii} A_{pt} [X_{t(i-1)(j-1)} + X_{ti(j-1)} - X_{t(i-1)j}] + \beta_{jj} B_{pt} [X_{t(i-1)(j-1)} + X_{t(i-1)j} - X_{ti(j-1)}] + \beta_{ii} \beta_{jj} [C_{pt(i-1)(j-1)} X_{t(i-1)(j-1)} + C_{pt(i-1)j} X_{t(i-1)j} + C_{pti(j-1)} X_{ti(j-1)}] \right\} - A_{p1} \left\{ q_{(i-1)(j-1)} + q_{(i-1)j} + q_{i(j-1)} + q_{ij} \right\} \quad (9)$$

where $\beta_{ii} = \bar{h}_i/2$, $\bar{h}_i = \eta_i - \eta_{(i-1)}$, $\beta_{jj} = \bar{h}_j/2$, $\bar{h}_j = \zeta_j - \zeta_{(j-1)}$, $p = 1 \sim 8$, A_{pt} , B_{pt} and C_{ptkl} are given in Appendix A.

In Eq. (9), the quantity X_{pij} is related to the quantities $X_{t(i-1)(j-1)}$, $X_{t(i-1)j}$, and $X_{ti(j-1)}$ at the three internal points. By choosing the point $[i, j]$ according to the order as $[1, 1]$, $[1, 2]$, $\dots$, $[1, n]$, $[2, 1]$, $[2, 2]$, $\dots$, $[2, n]$, $\dots$, $[m, 1]$, $[m, 2]$, $\dots$, $[m, n]$ and substituting the obtained results into the corresponding terms of the right hand side of Eq. (9), the quantities $X_{t(i-1)(j-1)}$, $X_{t(i-1)j}$ and $X_{ti(j-1)}$ at the three internal points can be eliminated and the quantity X_{pij} at the point (i, j) is only related to the quantities X_{rk0} ($r=1, 3, 4, 6, 7, 8$) and X_{s0l} ($s=2, 3, 5, 6, 7, 8$) at the boundary dependent points. Eq. (9) is rewritten as follows.

$$X_{pij} = \sum_{d=1}^6 \left\{ \sum_{f=0}^i a_{pijfd} X_{rf0} + \sum_{g=0}^j b_{pijgd} X_{s0g} \right\} + q_{pij}, \quad (10)$$

where

$$a_{pijfd} = \sum_{t=1}^8 \left\{ \beta_{ii} A_{pt} [a_{t(i-1)(j-1)fd} + a_{ti(j-1)fd} - a_{t(i-1)jfd}] + \beta_{jj} B_{pt} [a_{t(i-1)(j-1)fd} + a_{t(i-1)jfd} - a_{ti(j-1)fd}] \right\}$$

$$\begin{aligned}
& + \beta_{ii}\beta_{jj}[C_{pt(i-1)(j-1)}a_{t(i-1)(j-1)}fd \\
& + C_{pt(i-1)j}a_{t(i-1)j}fd + C_{pti(j-1)}a_{ti(j-1)}fd] \} \\
(11)
\end{aligned}$$

$$\begin{aligned}
b_{pijgd} = & \sum_{t=1}^8 \left\{ \beta_{ii}A_{pt}[b_{t(i-1)(j-1)gd} + b_{ti(j-1)gd} - b_{t(i-1)jgd}] \right. \\
& + \beta_{jj}B_{pt}[b_{t(i-1)(j-1)gd} + b_{t(i-1)jgd} - b_{ti(j-1)gd}] \\
& + \beta_{ii}\beta_{jj}[C_{pt(i-1)(j-1)}b_{t(i-1)(j-1)gd} \\
& + C_{pt(i-1)j}b_{t(i-1)jgd} + C_{pti(j-1)}b_{ti(j-1)gd}] \} \\
(12)
\end{aligned}$$

$$\begin{aligned}
q_{pij} = & \sum_{t=1}^8 \left\{ \beta_{ii}A_{pt}[q_{t(i-1)(j-1)} + q_{ti(j-1)} - q_{t(i-1)j}] \right. \\
& + \beta_{jj}B_{pt}[q_{t(i-1)(j-1)} + q_{t(i-1)j} - q_{ti(j-1)}] \\
& + \beta_{ii}\beta_{jj}[C_{pt(i-1)(j-1)}q_{t(i-1)(j-1)} \\
& + C_{pt(i-1)j}q_{t(i-1)j} + C_{pti(j-1)}q_{ti(j-1)}] \} \\
& - A_{p1} \left\{ q_{(i-1)(j-1)} + q_{(i-1)j} + q_{i(j-1)} + q_{ij} \right\} \\
(13)
\end{aligned}$$

By comparing Eqs.(5) ~ (7) with Eqs.(11) ~ (13), it can be noted that the two summations have been changed to one summation. So by using the Eqs.(11) ~ (13), the computer time of the coefficients can be reduced greatly.

4. Numerical Results

To investigate the validity of the proposed method, numerical results are presented for several specific problems and comparisons are made with previously published results where possible. $\nu = 0.3$ is used. All the convergent values are obtained for the plates by using Richardson's extrapolation formula for two cases of divisional numbers $m (=n)$.

4.1 Convergence of the present method

In order to examine the convergency, numerical calculation is carried out by varying the number of divisions m and n for a square plate with four simply supported edges noted as SSSS. The deflection of the plate for different division number is shown in

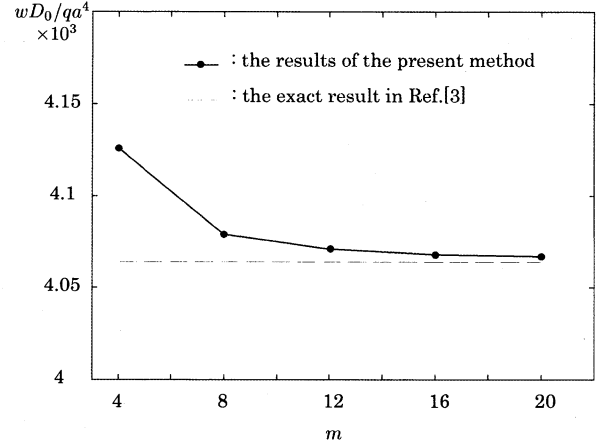


Fig. 3 The deflection versus the divisional number $m(=n)$ for the SSSS square plate.

Table 1 The quantities of SSSS square plate under uniform load ($\nu = 0.3, h/a = 0.01$)

Quantities	Pre.	Ref. ⁵⁾	Ref. ⁷⁾
$wD_0/qa^4 \times 10^3$	4.07	4.07	4.06
Q_y/qa	0.34	0.34	0.34
M_{xy}/qa^2	-0.033	-0.033	-0.033
M_x/qa^2	0.048	0.048	0.048

Pre.:Present Results

Fig. 3. It can be found the numerical results converge monotonously from above with increase of the divisional number and the results of the divisional numbers $m (=n)$ of 12 and 16 are almost same. So it is suitable to obtain the convergent result by using Richardson's extrapolation formula for two cases of divisional numbers $m (=n)$ of 12 and 16. By repeating the above procedure, the suitable number of divisions $m(=n)$ can be determined for the other plates.

4.2 Comparison of the present method with the discrete method ⁵⁾

(1) Comparison of the accuracy of the results

Table 1 shows the quantities of the deflection w , the shear force Q_y , the twisting moment M_{xy} and the moment M_x at the appointed points shown in Fig 4. The plate with four edges simply supported (SSSS) is considered. In this table, the numerical results obtained by the discrete method ⁵⁾ and the exact results obtained in Ref. ⁷⁾ are also shown. It can be seen the present method has the same accuracy as the discrete method. These results obtained by the two methods are in good agreement with exact results.

(2) Comparison of the computer time

Fig. 5 shows the flow chart of computation. The computation is divided into four parts. The ratio of

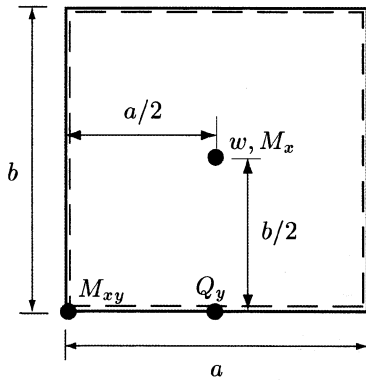


Fig. 4 The deflection w , the shear force Q_y , the moment M_x and the twisting moment M_{xy} at the appointed points of the simply supported plate.

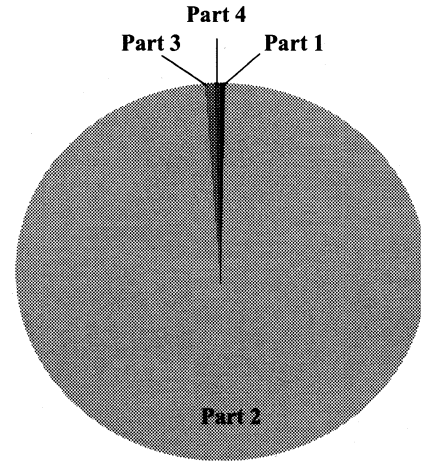


Fig. 6 The ratio of the computer time of each part.

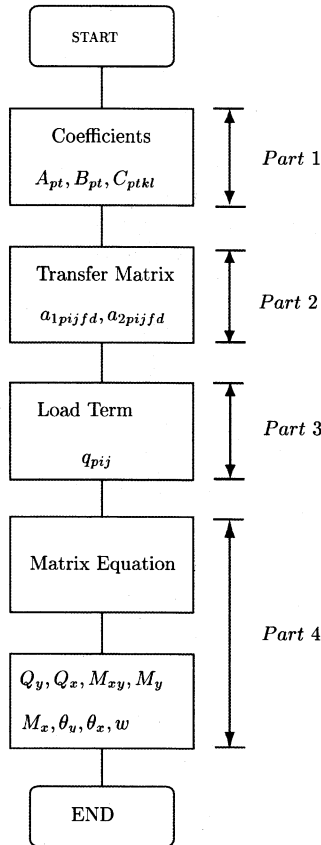


Fig. 5 Flow chart of computation.

computer time of these four parts is shown in Fig. 6 by using personal computer (Dell Pentium 4 2.8GHz 2.50GB). For divisional number $m = 12$, the computation time for the four parts is 0.1094s, 41.1406s, 0.3438s, 0.0781s by using the discrete method ⁵⁾ and 0.1094s, 4.6563s, 0.0625s, 0.0781s by using the present method, respectively. For each part, the ratio t_1/t_2 of computer time t_1 of the discrete method ⁵⁾ to computer time t_2 of the present method is shown

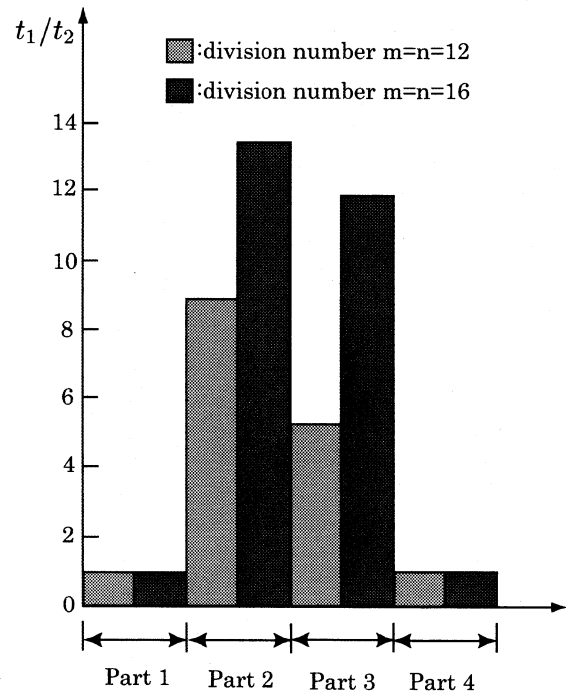


Fig. 7 The computer time ratio of the discrete approximate method to the present method at different parts.

in Fig. 7. Two divisional numbers $m(=n) = 12$ and $m(=n) = 16$ are considered. From Fig. 7, it can be noted the ratio $t_1/t_2 = 1$ for the first and the forth parts. That means the computer time of the discrete method ⁵⁾ is as the same as that of the present method. But for the second part, $t_1/t_2 = 8.8$ and $t_1/t_2 = 13.4$ for $m = 12$ and $m = 16$, and for the third part, $t_1/t_2 = 5.5$ and $t_1/t_2 = 11.9$ for $m = 12$ and $m = 16$, respectively. With increase of the divisional number, the ratio t_1/t_2 increase. As shown in Fig. 6, the computer time of the second and third parts is about 95 percentage of the whole time. So

Table 2 The number of the points used to obtain the coefficient at point (m, n)

	Division $(m \times n)$			
	4×4	8×8	12×12	16×16
Ref. ⁵⁾	24	80	168	288
Present	3	3	3	3

Table 3 The quantities of CCCC square plate under uniform load ($\nu = 0.3, h/a = 0.01$)

Quantities	Pre.	Ref. ⁷⁾
$wD_0/qa^4 \times 10^3$	1.27	1.26
Q_y/qa	0.44	—
M_y/qa^2	-0.051	-0.051
M_x/qa^2	0.023	0.023

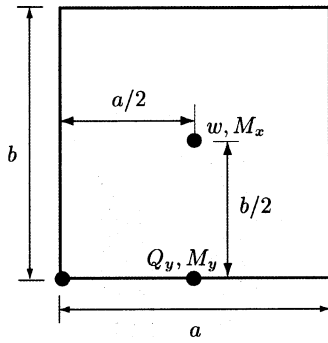


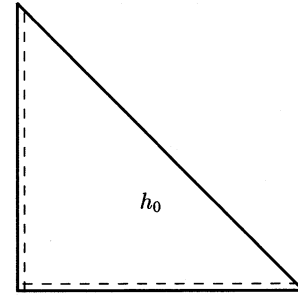
Fig. 8 The deflection w , the shear force Q_y , the moments M_x and M_y at the appointed points of CCCC square plate.

the computer time of the present method is much less than that of the discrete method ⁵⁾, especially for the larger divisional number. It can be understood by comparing Eqs.(11) ~ (13) with Eqs.(5) ~ (7). In order to obtain the coefficient at point (m, n) , $((m + 1) \times (n + 1) - 1)$ points are needed to use by using Eqs.(5) ~ (7), but only three points are needed by using Eqs.(11) ~ (13). The numbers of the points needed in the present method and the discrete method ⁵⁾ are shown in Table 2. It can be noted the number of the points needed in the present method is much less that of the discrete method ⁵⁾.

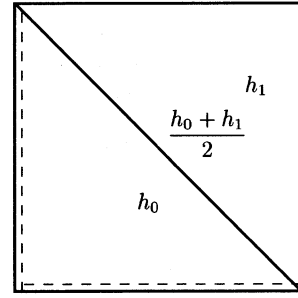
4.3 Plate with uniform divisions

Table 3 shows the quantities of the deflection w , the shear force Q_y , the twisting moment M_y and the moment M_x at the appointed points shown in Fig 8. The square plate with four clamped edges (CCCC) is considered. The results obtained by the present method are compared with the exact results of Ref. ⁷⁾. It can be seen these results agree well.

Fig 9 shows an isosceles right triangular plate and



(a)



(b)

Fig. 9 A triangular plate and its equivalent rectangular plate. (a) A triangular plate; (b) An equivalent rectangular plate of a triangular plate.

Table 4 The deflection at point $(a/4, b/4)$ of the isosceles right triangular plate with diagonal clamped and the other edges simply supported ($wD_0/qa^4 \times 10^3$, $\nu = 0.3$, $b/a = 1.0$, $h_0/a = 0.01$)

h_1/h_0	Division $(m \times n)$		Pre.	Ref. ⁸⁾
	36×36	40×40		
6	0.369	0.371	0.38	0.37
8	0.357	0.360	0.37	
10	0.353	0.356	0.37	

its equivalent rectangular plate. The thickness of the triangular plate is h_0 and the boundary conditions are the diagonal clamped and the other edges simply supported. The equivalent rectangular is obtained by adding a triangular part to the original plate. The boundary conditions of the added part are clamped and the thickness is h_1 . The thickness h_1 is much larger than h_0 . The thickness of the diagonal is chosen as $(h_0 + h_1)/2$. The numerical results for the deflection w and the moment M_x are presented in Tables 4 and 5. The numerical results obtained by Fletcher ⁸⁾ and FEM are also shown. From these tables, it can be noted that the present results have enough accuracy.

Table 5 The moment at point $(a/4, b/4)$ of the isosceles right triangular plate with diagonal clamped and the other edges simply supported (M_x/qa^2 , $\nu = 0.3$, $b/a = 1.0$, $h_0/a = 0.01$)

h_1/h_0	Division ($m \times n$)		Pre.	FEM
	36×36	40×40		
6	0.0133	0.0134	0.014	0.013
8	0.0132	0.0132	0.013	
10	0.0131	0.0131	0.013	

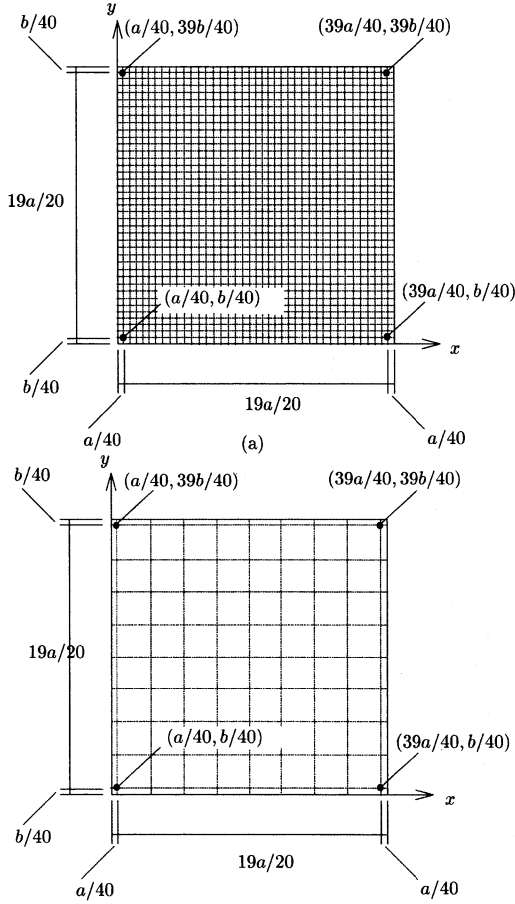


Fig. 10 Plates with divisions. (a) Non-uniform division; (b) uniform division.

4.4 Plate with non-uniform divisions

Comparing with the discrete method ⁵⁾ which is only suitable for the plate with uniform divisions, one advantage of the present method is that it can be used for the plate with non-uniform divisions. In order to show the efficient of the present method, numerical results are given for the square plates with uniform and non-uniform divisions as shown in Fig. 10. The quantities at point $(a/40, b/40)$ are presented in Table 6. In order to obtain the quantities at point $(a/40, b/40)$, divisional number must be larger than

Table 6 Numerical results at point $(a/40, b/40)$ of SSSS square plate with uniform load

Quantities	Division ($m \times n$)		
	uniform 40×40	non-uniform 10×10 20×20	
$w_1 D_0 / qa^4 \times 10^3$	0.288	0.290	0.288
Q_y / qa	0.0441	0.0444	0.0441
M_x / qa^2	0.000920	0.000925	0.000920

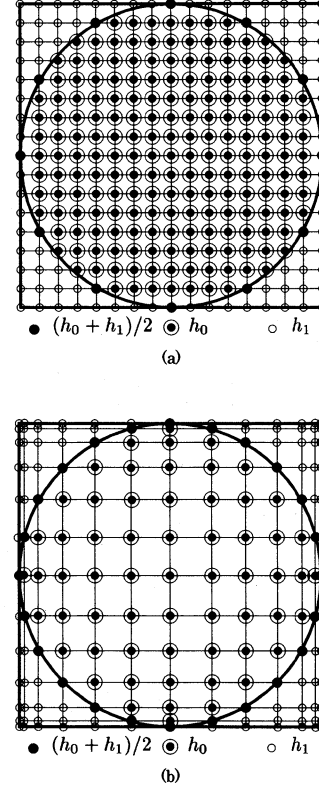


Fig. 11 The circle plate with uniform and non-uniform divisions. (a) Uniform divisions; (b) Non-uniform divisions.

40 for the plate with uniform division. But for the plate with non-uniform division, the divisional number is not limited. From Table 6, it can be noted the results decrease with increase of the non-uniform divisional number, and the results of the plate with uniform division $m = 40$ are as the same as those of the plate with non-uniform division $m = 20$. So the computer time can be saved by using non-uniform division.

In order to show the accuracy of the present method for the plate with non-uniform division, the calculation is carried out for clamped circle plate with uniform and non-uniform divisions as shown in Fig. 11. The numerical results of the deflection w and the moment M_x at point $(a/2, a/2)$ are shown in Tables 7 ~ 10. It can be noted for the plate with same

Table 7 The deflection at point $(a/2, a/2)$ of a circle plate with uniform divisions ($wD_0/qa^4 \times 10^3$, $\nu = 0.3$, $h_0/a = 0.01$)

h_1/h_0	Division($m \times n$)		Pre.	FEM
	28 $\times$ 28	32 $\times$ 32		
6.0	0.922	0.920	0.91	0.97
8.0	0.920	0.917	0.91	

Table 8 The moment at point $(a/2, a/2)$ of the circle plate with uniform divisions (M_x/qa^2 , $\nu = 0.3$, $h_0/a = 0.01$)

h_1/h_0	Division($m \times n$)		Pre.	FEM
	28 $\times$ 28	32 $\times$ 32		
6.0	0.0196	0.0198	0.021	0.020
8.0	0.0196	0.0198	0.021	

Table 9 The deflection at point $(a/2, a/2)$ of a circle with non-uniform divisions ($wD_0/qa^4 \times 10^3$, $\nu = 0.3$, $h_0/a = 0.01$)

h_1/h_0	Division($m \times n$)		Pre.	FEM
	28 $\times$ 28	32 $\times$ 32		
6.0	0.888	0.903	0.95	0.97
8.0	0.883	0.898	0.95	

Table 10 The moment at point $(a/2, a/2)$ of a circle plate with non-uniform divisions (M_x/qa^2 , $\nu = 0.3$, $h_0/a = 0.01$)

h_1/h_0	Division($m \times n$)		Pre.	FEM
	28 $\times$ 28	32 $\times$ 32		
6.0	0.0187	0.0189	0.020	0.020
8.0	0.0184	0.0187	0.020	

divisional number, the deflection at point $(a/2, a/2)$ obtained by using non-uniform division is better than that obtained by using uniform division.

5. Conclusions

An improved method is proposed for analyzing the bending problem of plate. No prior assumption of shape of deflection used in the finite element method are employed in this method. By transforming the differential equations into integral equations in a small area, the quantities of an appointed point can be expressed by those of the other three points. That makes the computer time reduce greatly. The present method is suitable for plate with uniform and non-uniform divisions. Some numerical results are given for the rectangular plate, triangular plate and circle plate. Comparison of the numerical results of the present method with those previously reported is presented. It shows that the present results have a good

convergence and satisfactory accuracy.

Appendix A

$F_{111} = F_{124} = F_{133} = F_{156} = F_{167} = F_{188} = 1$;
 $F_{146} = \nu$; $F_{212} = F_{223} = F_{235} = F_{247} = F_{266} = \mu$;
 $F_{257} = \mu\nu$; $F_{278} = 1$; $F_{321} = F_{332} = -\mu$; $F_{345} =$
 $F_{354} = -I$; $F_{363} = -J$; $F_{372} = -H$; $F_{377} = 1$; $F_{381} =$
 $-\mu H$; $F_{386} = \mu$; other $F_{kts} = 0$. $A_{p1} = \gamma_{p1}$, $A_{p2} =$
 0 , $A_{p3} = \gamma_{p2}$, $A_{p4} = \gamma_{p3}$, $A_{p5} = 0$, $A_{p6} = \gamma_{p4} + \nu\gamma_{p5}$,
 $A_{p7} = \gamma_{p6}$, $A_{p8} = \gamma_{p7}$.
 $B_{p1} = 0$, $B_{p2} = \mu\gamma_{p1}$, $B_{p3} = \mu\gamma_{p3}$, $B_{p4} = 0$, $B_{p5} =$
 $\mu\gamma_{p2}$, $B_{p6} = \mu\gamma_{p6}$, $B_{p7} = \mu(\nu\gamma_{p1} + \gamma_{p5})$, $B_{p8} = \gamma_{p8}$.
 $C_{p1kl} = \mu(\gamma_{p3} + k_{kl}\gamma_{p7})$, $C_{p2kl} = \mu\gamma_{p2} + k_{kl}\gamma_{p8}$, $C_{p3kl} =$
 $J\gamma_{p6}$, $C_{p4kl} = I_{kl}\gamma_{p4}$, $C_{p5kl} = I_{kl}\gamma_{p5}$, $C_{p6kl} = -\mu\gamma_{p7}$,
 $C_{p7kl} = -\gamma_{p8}$, $C_{p8kl} = 0$.
 $[\gamma_{pk}] = [\bar{\gamma}_{pk}]^{-1}$, $\bar{\gamma}_{11} = \beta_{ii}$, $\bar{\gamma}_{12} = \mu\beta_{jj}$, $\bar{\gamma}_{22} = -\mu\beta_{ij}$,
 $\bar{\gamma}_{23} = \beta_{ii}$, $\bar{\gamma}_{25} = \mu\beta_{jj}$, $\bar{\gamma}_{31} = -\mu\beta_{ij}$, $\bar{\gamma}_{33} = \mu\beta_{jj}$,
 $\bar{\gamma}_{34} = \beta_{ii}$, $\bar{\gamma}_{44} = -I_{ij}\beta_{ij}$, $\bar{\gamma}_{46} = \beta_{ii}$, $\bar{\gamma}_{47} = \mu\nu\beta_{jj}$,
 $\bar{\gamma}_{55} = -I_{ij}\beta_{ij}$, $\bar{\gamma}_{56} = \nu\beta_{ii}$, $\bar{\gamma}_{57} = \mu\beta_{jj}$, $\bar{\gamma}_{63} = -J_{ij}\beta_{ii}$,
 $\bar{\gamma}_{66} = \mu\beta_{jj}$, $\bar{\gamma}_{67} = \beta_{ii}$, $\bar{\gamma}_{71} = -\mu k_{ij}\beta_{ij}$, $\bar{\gamma}_{76} = \mu\beta_{ij}$,
 $\bar{\gamma}_{78} = \beta_{ii}$, $\bar{\gamma}_{82} = -H_{ij}\beta_{ij}$, $\bar{\gamma}_{87} = \beta_{ij}$, $\bar{\gamma}_{88} = \beta_{jj}$, other
 $\bar{\gamma}_{pk} = 0$, $\beta_{ij} = \beta_{ii}\beta_{jj}$

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(Received April 14, 2008)