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Extended overstress model with overstress tensor

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The elastoplastic stress is first defined as the stress which evolves as the actual strain rate is induced in an imaginary quasi-static process of elastoplastic deformation, while internal variables evolve with the viscoplastic strain rate calculated by the viscoplastic constitutive equation. Further, the novel variable "overstress tensor" reaching the current stress from the elastoplastic stress is defined. Then, the overstress model is extended so as to describe also the tangential viscoplastic strain rate induced by the overstress tensor component tangential to the yield surface. Furthermore, the viscoplastic strain rate due to the change of stress inside the yield surface is incorporated by adopting the concept of the subloading surface which falls within the framework of the unconventional elastoplasticity describing the smooth elastic-plastic transition fulfilling the smoothness and continuity conditions.

Key Words: *Overstress, plasticity, rate-dependence, subloading surface model, viscoplasticity*

1. Introduction

The *overstress* model[1)-5)] is most widely used among various viscoplastic constitutive equations. However, the existing formulations for this model possess insufficient aspects for the prediction of rate-dependent deformation behavior as follows:

1) It cannot describe the fact that the direction of viscoplastic strain rate deviates from the outward-normal to the yield surface in the general loading process involving the nonproportional stress path deviating from the outward-normal to the yield surface.

2) It is based on the conventional elastoplasticity[6)] with the yield surface enclosing a purely elastic domain. Therefore, it violates the *smoothness condition*[7)-10)] at the moment when the stress reaches the yield surface and thus it is incapable of describing the smooth elastic-plastic transition observed in real materials. Needless to say, it cannot describe the strain accumulation for the cyclic loading.

3) If the inelastic strain rate due to the stress rate tangential to the yield surface is incorporated, it is suddenly induced at the moment when the stress reaches the yield surface since the interior of yield surface is assumed to be a purely elastic domain. Then, the continuity condition[7)-10)] is also violated, and thus the uniqueness of solution does not hold for the stress path along the yield surface.

In this article a pertinent formulation without the above-mentioned defects in the existing overstress model is given as follows:

i) The elastoplastic stress is defined as the stress on the yield surface, which evolves as the actual strain rate is induced in an imaginary quasi-static process of elastoplastic deformation.

ii) The novel variable "overstress tensor" is introduced, which is defined as the tensor of stress reaching the current stress from the elastoplastic stress.

iii) The tangential viscoplastic strain rate due to the overstress tensor component tangential to the yield surface is introduced based on the concept of tangential inelasticity[11)-14)].

iv) The viscoplastic strain rate is described so as to develop gradually from the inside of yield surface by adopting the subloading surface model[15)-19)] falling within the framework of unconventional plasticity and thus the continuity and the smoothness conditions[7)-10)] are always fulfilled.

2. Extended overstress model with tangential inelasticity

The *overstress model*[1)-5)] is extended below incorporating the overstress tensor and the concept of tangential inelasticity[11)-154].

2.1 Quasi-static: elastoplastic deformation process

Denoting the current configuration of the material particle as $\mathbf{x}$ and the current velocity as $\mathbf{v}$, the velocity gradient is described as $\mathbf{L} = \partial\mathbf{v}/\partial\mathbf{x}$ from which the strain rate and the continuum spin are defined as $\mathbf{D} \equiv (\mathbf{L}+\mathbf{L}^{\mathrm{T}})/2$ and $\mathbf{W} \equiv (\mathbf{L}-\mathbf{L}^{\mathrm{T}})/2$, respectively, $(\ )^{\mathrm{T}}$ standing for the transpose. Let the strain rate $\mathbf{D}$ be additively decomposed into the elastic strain rate $\mathbf{D}^e$ and the *inelastic strain rate* $\mathbf{D}^i$, while the latter is further additively decomposed into the *(normal-)plastic strain rate* $\mathbf{D}^p_N$ and the *tangential inelastic strain rate* $\mathbf{D}^i_t$, i.e.

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^i,\quad \mathbf{D}^i = \mathbf{D}^p_N + \mathbf{D}^i_t, \tag{1}$$

where $\mathbf{D}^p_N$ and $\mathbf{D}^i_t$ are induced by the stress rate components normal and tangential, respectively, to the loading surface. Let $\mathbf{D}^e$ be related linearly to the stress rate as

$$\mathbf{D}^e = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}}. \tag{2}$$

$\boldsymbol{\sigma}$ is the Cauchy stress, $(^\circ)$ denoting the proper corotational rate with the objectivity, and the fourth-order tensor $\mathbf{E}$ is the elastic modulus. Here, limiting to an infinitesimal deformation but avoiding the influence of rigid-body rotation on the constitutive relation, the following *Zaremba-Jaumann rate* is adopted for the corotational rate.

$$\overset{\circ}{\mathbf{T}} \equiv \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W}, \tag{3}$$

where $\mathbf{T}$ is an arbitrary second-order tensor, $(^\bullet)$ denoting the material-time derivative.

2.2 Plastic strain rate

Consider the following isotropic yield condition in the simplest form.

$$f(\boldsymbol{\sigma}) = F(H), \tag{4}$$

where the scalar H is the isotropic hardening/softening variable. f is assumed to be homogeneous function of stress $\boldsymbol{\sigma}$ in degree-one fulfilling $f(s\boldsymbol{\sigma}) = sf(\boldsymbol{\sigma})$ for any nonnegative scalar s, and thus the yield surface keeps a similar shape.

The material-time derivative of Eq. (4) is given by

$$\mathrm{tr}\left(\frac{\partial f(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\boldsymbol{\sigma}}\right) = F'\dot{H}, \tag{5}$$

where

$$F' \equiv dF/dH. \tag{6}$$

Hereafter let it be assumed that the tangential-inelastic strain rate is normal to the yield surface and thus it fulfills the following equation:

$$\mathrm{tr}\left(\frac{\partial f(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}\mathbf{D}^i_t\right) = 0 \tag{7}$$

Substituting Eqs. (1) and (2) into Eq. (5) and considering Eq. (7), one has

$$\mathrm{tr}\left\{\frac{\partial f(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}\mathbf{E}(\mathbf{D}-\mathbf{D}^p_N)\right\} = F'\dot{H}, \tag{8}$$

Assume the associated flow rule

$$\mathbf{D}^p_N = \lambda\mathbf{N}\quad (\lambda>0), \tag{9}$$

where λ is a positive proportionality factor and the second-order tensor $\mathbf{N}$ denotes the normalized outward-normal to the yield surface, i.e.

$$\mathbf{N} \equiv \frac{\partial f(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}} \Big/ \left\|\frac{\partial f(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}\right\| \quad (\|\mathbf{N}\|=1), \tag{10}$$

$\|\ \|$ denoting the magnitude. The proportionality factor λ is derived by substituting Eq. (9) into Eq. (5) as follows:

$$\lambda = \frac{\mathrm{tr}\left\{\mathbf{N}\left(\mathbf{E}\mathbf{D} - \frac{F'}{F}\dot{H}\boldsymbol{\sigma}\right)\right\}}{\mathrm{tr}(\mathbf{N}\mathbf{E}\mathbf{N})}, \tag{11}$$

The following relationship due to the Euler's theorem for a homogeneous function in degree-one is used for deriving Eq. (11).

$$\frac{\partial f(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}} = \frac{\mathrm{tr}\left(\frac{\partial f(\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}\boldsymbol{\sigma}\right)}{\mathrm{tr}(\mathbf{N}\boldsymbol{\sigma})}\mathbf{N} = \frac{f(\boldsymbol{\sigma})}{\mathrm{tr}(\mathbf{N}\boldsymbol{\sigma})}\mathbf{N}. \tag{12}$$

The variation of internal structure of material is induced by the inelastic deformation and is described by the variation of internal variables. Generally speaking, if the rate of deformation increases, the viscous resistance increases and thus the elastic deformation becomes dominant so that the rate of inelastic strain rate in the strain rate decreases depressing the variation of internal variables. Therefore, the rates of internal variables have not to be calculated by the elastoplastic constitutive equation which holds only in the quasi-static deformation process but have to be calculated by the rate-dependent viscoplastic constitutive equation. Then, the rate of internal variable in the consistency condition (5) or (8) has to be calculated by the viacoplastic constitutive equation formulated later and thus the plastic flow rule (9) is not substituted to them, whilst remind that the yield surface is updated by the rate of internal variable calculated by the viscoplastic strain rate in the overstress model.

2.3 Tangential-inelastic strain rate

While the inelastic strain rate is induced also by the tangential stress rate, it depends only on the divoiatoric component of tangential stress rate[11)]. Then, let the tangential strain rate be given by the following equation[12)-14)] with the material function T of stress $\boldsymbol{\sigma}$ and internal variable H, i.e.

$$\mathbf{D}_t^i = T\,\overset{\circ}{\boldsymbol{\sigma}}{}_t^*, \quad T \equiv T(\boldsymbol{\sigma}, H), \tag{13}$$

where $\overset{\circ}{\boldsymbol{\sigma}}{}_t^*$ is the *deviatoric-tangential stress rate* given as follows:

$$\overset{\circ}{\boldsymbol{\sigma}}{}^* \equiv \bar{\mathbf{I}}^* \overset{\circ}{\boldsymbol{\sigma}} = \overset{\circ}{\boldsymbol{\sigma}}{}_n^* + \overset{\circ}{\boldsymbol{\sigma}}{}_t^*, \tag{14}$$

$$\left.\begin{aligned} \overset{\circ}{\boldsymbol{\sigma}}{}_n^* &\equiv \hat{\mathbf{n}}^* \overset{\circ}{\boldsymbol{\sigma}} = \mathrm{tr}(\mathbf{n}^* \overset{\circ}{\boldsymbol{\sigma}})\mathbf{n}^*, \\ \overset{\circ}{\boldsymbol{\sigma}}{}_t^* &\equiv \hat{\mathbf{I}}^* \overset{\circ}{\boldsymbol{\sigma}} = \overset{\circ}{\boldsymbol{\sigma}}{}^* - \overset{\circ}{\boldsymbol{\sigma}}{}_n^* \end{aligned}\right\} \tag{15}$$

with

$$\bar{I}_{ijkl} \equiv \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad \bar{\mathbf{I}}^* \equiv \bar{\mathbf{I}} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I} \tag{16}$$

$$\mathbf{N}^* \equiv \bar{\mathbf{I}}^*\mathbf{N}, \quad \mathbf{n}^* \equiv \frac{\mathbf{N}^*}{\|\mathbf{N}^*\|} \ (\|\mathbf{n}^*\| = 1), \tag{17}$$

$$\hat{\mathbf{n}}^* \equiv \mathbf{n}^* \otimes \mathbf{n}^*, \quad \hat{\mathbf{I}}^* \equiv \bar{\mathbf{I}}^* - \hat{\mathbf{n}}^*. \tag{18}$$

$\bar{\mathbf{I}}$ is the *fourth-order identity tensor* and $\bar{\mathbf{I}}^*$ might be called the *fourth-order deviatoric transformation tensor* leading to $\bar{\mathbf{I}}^*\mathbf{T} = \mathbf{T}^*$. Further, $\hat{\mathbf{n}}^*$ and $\hat{\mathbf{I}}^*$ might be called the *fourth-order norma-l* and *tangential-deviatoric transformation tensors*, respectively. It is postulated that $\mathbf{D}_t^i$ does not influence on the hardening behavior since it is tangential to the yield surface. The tangential-inelastic strain rate is schematically shown in Fig. 1 of deviatoric stress space. Besides, its response to the input of same stress rate with variation of stress state is illustrated in Fig. 2 where the violation of the continuity and smoothness conditions[7)-10)] is violated.

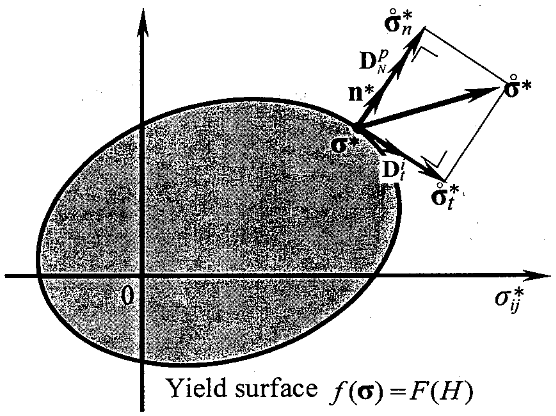


Fig. 1. Tangential-inelastic strain rate in the deviatoric stress space.

Hereafter, let the elastic modulus be given in Hooke's type of rate form, i.e.

$$\left.\begin{aligned} E_{ijkl} &= \left(K - \tfrac{2}{3}G\right)\delta_{ij}\delta_{kl} + G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \\ (\mathbf{E}^{-1})_{ijkl} &= \tfrac{1}{3}\left(\tfrac{1}{3K} - \tfrac{1}{2G}\right)\delta_{ij}\delta_{kl} + \tfrac{1}{4G}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \end{aligned}\right\} \tag{19}$$

where K and G are the elastic bulk and shear moduli, respectively, which are assumed to be the function of stress in general.

It holds from Eqs. (1), (2), (9) and (13) that

$$\mathbf{D}^* = \frac{1}{2G}\overset{\circ}{\boldsymbol{\sigma}}{}^* + \lambda\mathbf{N}^* + T\,\overset{\circ}{\boldsymbol{\sigma}}{}_t^*, \tag{20}$$

from which, noting

$$\mathbf{N}_t^* \equiv \hat{\mathbf{I}}^*\mathbf{N} = \mathbf{N}^* - \mathrm{tr}(\mathbf{n}^*\mathbf{N}^*)\mathbf{n}^* = \mathbf{0}, \tag{21}$$

we have

$$\mathbf{D}_t^* = \left(\frac{1}{2G} + T\right)\overset{\circ}{\boldsymbol{\sigma}}{}_t^*, \tag{22}$$

where

$$\mathbf{D}_t^* \equiv \hat{\mathbf{I}}^*\mathbf{D} = \mathbf{D}^* - \mathrm{tr}(\mathbf{n}^*\mathbf{D}^*)\mathbf{n}^*. \tag{23}$$

Substituting Eq. (22) into Eq. (13), the tangential strain rate is described by the strain rate as follows:

$$\mathbf{D}_t^i = \frac{1}{1 + \dfrac{1}{2GT}}\mathbf{D}_t^*. \tag{24}$$

The stress rate is given from Eqs. (1), (2), (9), (11) and (24) by the following equation, noting the positiveness of the proportionality factor λ.

$$\overset{\circ}{\boldsymbol{\sigma}} = \begin{cases} \mathbf{ED} - \mathbf{E}\left\langle \dfrac{\mathrm{tr}\{\mathbf{N}(\mathbf{ED} - \frac{F'}{F}\dot{H}\boldsymbol{\sigma})\}}{\mathrm{tr}(\mathbf{NEN})} \right\rangle \mathbf{N} \\ \quad - \dfrac{2G}{1 + \dfrac{1}{2GT}}\mathbf{D}_t^* \quad \text{if } f(\boldsymbol{\sigma}) - F(H) = 0 \\ \mathbf{ED} \quad \text{if } f(\boldsymbol{\sigma}) - F(H) < 0 \end{cases} \tag{25}$$

where $\langle\ \rangle$ is the McCauley's bracket, i.e. $\langle a \rangle = a$ for $a \geq 0$ and $\langle a \rangle = 0$ for $a < 0$ for an arbitrary scalar variable a.

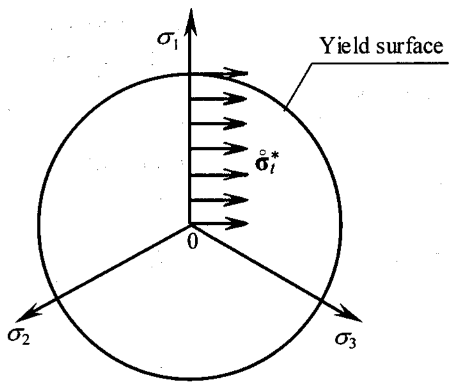
The normal-plastic strain rate of the second term in the right-hand side of Eq. $(25)_1$ has the different mathematical from from the ordinary plastic constitutive equation but fulfills the consistency condition (5) or (8) and obeys the plastic-flow rule (9) and thus exhibits the identical response as the plastic constitutive equation. Further, the denominator is always positive owing to the positive definiteness of the elastic modulus tensor $\mathbf{E}$. Then, one can performs the judgment of loading or unloading by the sign of numerator which exhibits the non-dimensional plastic relaxation stress rate as known from

$$\mathrm{tr}\left\{\frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}\mathbf{ED}\right\} - \dot{F} = \mathrm{tr}\left\{\mathbf{N}\left(\mathbf{ED} - \frac{F'}{F}\dot{H}\boldsymbol{\sigma}\right)\right\}\left\|\frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}\right\|. \tag{26}$$

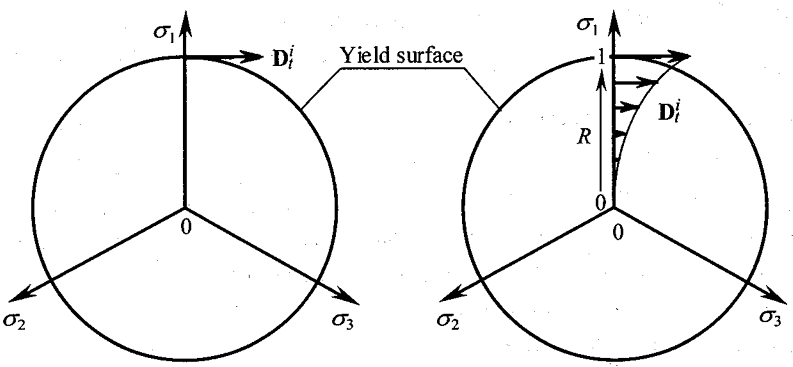
2.4 Overstress model with overstress tensor

The overstress tensor is improved by introducing the overstress tensor in this section.

Let it be assumed that the strain rate $\mathbf{D}$ is addi-



Input of tangential - deviatoric stress rate



Output of tangential-inelastic
Conventional elastoplasticiy

Output of tangential-inelastic
Subloading surface model

Fig. 2. Output of tangential-inelastic strain rate to input of identical deviatoric-tangential stress rate with increase of stress state illustrated for the Mises yield surface in the π-plane.

additively decomposed into the elastic strain rate $\mathbf{D}^e$ and the *viscoplastic strain rate* $\mathbf{D}^{vp}$ which is further additively decomposed into the *normal-viscoplastic strain rate* $\mathbf{D}_N^{vp}$ and *tangential-viscoplastic strain rate* $\mathbf{D}_t^{vp}$, i.e.

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^{vp}, \quad \mathbf{D}^{vp} = \mathbf{D}_N^{vp} + \mathbf{D}_t^{vp}. \tag{27}$$

The overstress model advocated by Bingham[1] is based on the premise that the viscoplastic strain rate is induced by the stress over the yield surface. The stress over the yield surface has been evaluated merely by the scalar quantity of the expanded quantity of the dynamic-loading surface from the yield surface, while the dynamic-loading surface passes through the current stress point and is similar to the yield surface. Here, we introduce the imaginary stress[19] defined as the stress on the yield surface, which evolves as the actual strain rate is induced in an imaginary quasi-static process of elastoplastic deformation, and let it be called the *elastoplastic stress*, denoting it by the notation $\boldsymbol{\sigma}^{ep}$. The viscoplastic strain rate could be formulated more precisely by introducing the tensor of stress reaching the current stress from the elastoplastic stress. The elastoplastic stress rate is given by the following equation with the replacement of the stress $\boldsymbol{\sigma}$ to $\boldsymbol{\sigma}^{ep}$.

$$\mathring{\boldsymbol{\sigma}}^{ep} = \begin{cases} \mathbf{ED} - \mathbf{E}\left\langle \dfrac{\mathrm{tr}\left\{\mathbf{N}^{ep}\left(\mathbf{ED} - \frac{F'}{F}\dot{H}\,\boldsymbol{\sigma}^{ep}\right)\right\}}{\mathrm{tr}(\mathbf{N}^{ep}\mathbf{E}\mathbf{N}^{ep})} \right\rangle \mathbf{N}^{ep} \\ \quad - \dfrac{2G}{1+\dfrac{1}{2GT^{ep}}}\mathbf{D}_t^{ep*} \quad \text{if } f(\boldsymbol{\sigma}) - F(H) = 0 \\ \mathbf{ED} \quad \text{if } f(\boldsymbol{\sigma}) - F(H) < 0 \end{cases} \tag{28}$$

where

$$\dot{H} = \mathrm{tr}\{\mathbf{f}_H(\boldsymbol{\sigma}^{ep}, H)\mathbf{D}_N^{vp}\}, \tag{29}$$

$$\mathbf{N}^{ep} \equiv \frac{\partial f(\boldsymbol{\sigma}^{ep})}{\partial \boldsymbol{\sigma}^{ep}} \bigg/ \left\| \frac{\partial f(\boldsymbol{\sigma}^{ep})}{\partial \boldsymbol{\sigma}^{ep}} \right\| \quad (\|\mathbf{N}^{ep}\| = 1), \tag{30}$$

$$\mathbf{N}^{ep*} \equiv \bar{\mathbf{I}}^*\mathbf{N}^{ep}, \quad \mathbf{n}^{ep*} \equiv \frac{\mathbf{N}^{ep*}}{\|\mathbf{N}^{ep*}\|} \quad (\|\mathbf{n}^{ep*}\| = 1), \tag{31}$$

$$\hat{\mathbf{n}}^{ep*} \equiv \mathbf{n}^{ep*} \otimes \mathbf{n}^{ep*}, \quad \hat{\mathbf{I}}^{ep*} \equiv \bar{\mathbf{I}}^* - \hat{\mathbf{n}}^{ep*}, \tag{32}$$

$$\mathbf{D}_t^{ep*} \equiv \hat{\mathbf{I}}^{ep}\mathbf{D} = \mathbf{D}^* - \mathrm{tr}(\mathbf{n}^{ep*}\mathbf{D}^*)\mathbf{n}^{ep*}, \tag{33}$$

$$T^{ep} \equiv T(\boldsymbol{\sigma}^{ep}, H). \tag{34}$$

$\mathbf{f}_H$ is the second-order tensor valued function of $\boldsymbol{\sigma}^{ep}, H$.

Define the *overstress tensor* to be the tensor of stress reaching the current stress from the elastoplastic stress and denote it by $\overleftarrow{\boldsymbol{\sigma}}$, i.e.

$$\overleftarrow{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \boldsymbol{\sigma}^{ep} \tag{35}$$

Then, the deviatoric part $\overleftarrow{\boldsymbol{\sigma}}^*$ of $\overleftarrow{\boldsymbol{\sigma}}$ is decomposed into the normal-deviatoric component $\overleftarrow{\boldsymbol{\sigma}}_n^*$ and the tangential-deviatoric component $\overleftarrow{\boldsymbol{\sigma}}_t^*$ as follows:

$$\overleftarrow{\boldsymbol{\sigma}}^* \equiv \bar{\mathbf{I}}^* \overleftarrow{\boldsymbol{\sigma}} = \overleftarrow{\boldsymbol{\sigma}}_n^* + \overleftarrow{\boldsymbol{\sigma}}_t^*, \tag{36}$$

$$\left.\begin{array}{l} \overleftarrow{\boldsymbol{\sigma}}_n^* \equiv \hat{\mathbf{n}}^* \overleftarrow{\boldsymbol{\sigma}} = \mathrm{tr}(\mathbf{n}^* \overleftarrow{\boldsymbol{\sigma}})\mathbf{n}^* \\ \overleftarrow{\boldsymbol{\sigma}}_t^* \equiv \hat{\mathbf{I}}_y^* \overleftarrow{\boldsymbol{\sigma}} = \overleftarrow{\boldsymbol{\sigma}}^* - \overleftarrow{\boldsymbol{\sigma}}_n^* \end{array}\right\} \tag{37}$$

which is illustrated in Fig. 3 of the deviatoric stress space.

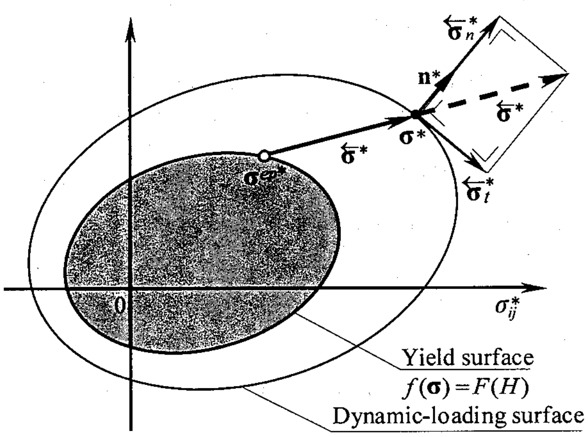


Fig. 3. Overstress tensor in conventional overstress model in the deviatoric stress space.

Then, let the normal-viscoplastic strain rate $\mathbf{D}_N^{vp}$ and the tangential-viscoplastic strain rate $\mathbf{D}_t^{vp}$ be given by

$$\mathbf{D}_N^{vp} = C_N(\boldsymbol{\sigma}, H, \mathrm{T})\left\langle \frac{f(\boldsymbol{\sigma})}{F} - 1 \right\rangle^N \mathbf{N}, \tag{38}$$

$$\mathbf{D}_t^{vp} = \begin{cases} C_t(\boldsymbol{\sigma}, H, \mathrm{T})\dfrac{\overleftarrow{\boldsymbol{\sigma}}_t^*}{F} & \text{if } f(\boldsymbol{\sigma}) - F(H) \geq 0 \\ \mathbf{0} & \text{if } f(\boldsymbol{\sigma}) - F(H) < 0 \end{cases} \tag{39}$$

where C_N and C_t are the function of $\boldsymbol{\sigma}$, H and the absolute temperature T, and N is the material constant.

The strain rate is given from Eqs. (2), (27), (38) and (39) by

$$\mathbf{D} = \begin{cases} \mathbf{E}^{-1}\mathring{\boldsymbol{\sigma}} + C_N\left\langle \dfrac{f(\boldsymbol{\sigma})}{F} - 1 \right\rangle^N \mathbf{N} \\ \quad + C_t\dfrac{\overleftarrow{\boldsymbol{\sigma}}_t^*}{F} \quad \text{if } f(\boldsymbol{\sigma}) - F(H) \geq 0 \\ \mathbf{E}^{-1}\mathring{\boldsymbol{\sigma}} \quad \text{if } f(\boldsymbol{\sigma}) - F(H) < 0 \end{cases} \tag{40}$$

from which the stress rate is given as

$$\mathring{\boldsymbol{\sigma}} = \begin{cases} \mathbf{ED} - C_N\left\langle \dfrac{f(\boldsymbol{\sigma})}{F} - 1 \right\rangle^N \mathbf{EN} \\ \quad -2GC_t\dfrac{\overleftarrow{\boldsymbol{\sigma}}_t^*}{F} \quad \text{if } f(\boldsymbol{\sigma}) - F(H) \geq 0 \\ \mathbf{ED} \quad \text{if } f(\boldsymbol{\sigma}) - F(H) < 0 \end{cases} \tag{41}$$

3. Extension by the subloading surface model

The constitutive equation formulated in the preceding sections is based on the conventional plasticity[6] with the yield surface enclosing a purely elastic domain and thus it violates the smoothness condition[7)-10)] at the moment when a stress reaches the yield surface and further it violates the continuity condition[7)-10)] as the tangential-inelastic strain rate is induced suddenly at that moment.

In what follows the constitutive equation formulated in the preceding sections is extended so as to describe the inelastic strain rate due to the rate of stress inside the yield surface by incorporating the concept of the subloading surface[15)-19)].

3.1 Subloading surface model

Here, it could be assumed in the quasi-static elastoplastic deformation process that

a) A plastic strain rate develops gradually as the stress approaches the yield surface.

b) A conventional elastoplastic constitutive equation holds when the stress lies on the yield surface,

In order to formulate an unconventional elastoplastic constitutive equation realizing these assumptions it is required to adopt an appropriate measure expressing how near the subloading stress approaches the yield surface. Then, let the following surface, called the *subloading surface* (Fig. 4), be introduced, whilst the yield surface in the conventional elastoplasticity is renamed the *normal-yield surface*.

1. It passes always through the current stress point.
2. It has the similar shape and same orientation as the normal-yield surface.

Thus, the subloading surface coincides completely with the normal-yield surface when the stress reaches the normal-yield surface.

The normal-yield and subloading surfaces with similar shape and positioning possess the following geometrical properties.

i) All lines connecting a point inside the subloading surface and the *conjugate point* inside the normal-yield surface join at a specified point, called the *similarity-center* and denoted by $\mathbf{s}$, which is simply fixed to the origin of stress apace in this section.

ii) Ratio of a length of an arbitrary line-element inside the subloading surface and that of the conjugate line-element inside the normal-yield surface is constant, which is also identical to the ratio of the sizes of these surfaces. The ratio is called the *normal-yield ratio* and is denoted as $R\ (0 \le R \le 1)$.

It should be noted that the subloading surface coincides completely with the normal-yield surface when the stress reaches the normal-yield surface, i.e. $R=1$. On the other hand, the subyield surface(s) in the multi surface model[22), 21)] or the two surface model[22), 23)] never coincides with the outmost or bounding surface but contact to them at a point. This geometrical property brings about the singularity in the field of the elastoplastic modulus which leads to the discontinuous response of these models.

The subloading surface is described as

$$f(\boldsymbol{\sigma}) = RF(H). \tag{42}$$

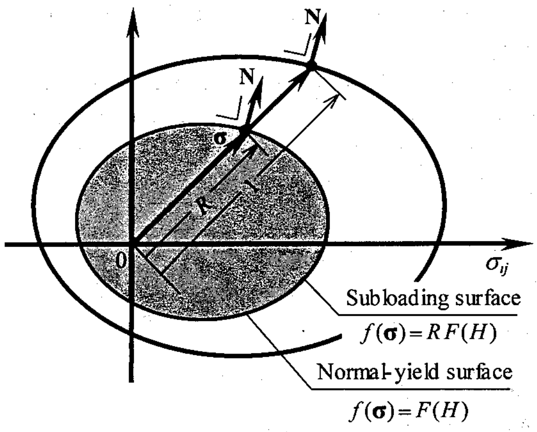


Fig. 4. Normal-yield and subloading and dynamic-loading surfaces.

The material-time derivative of Eq. (42) is given by

$$\mathrm{tr}\{\frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}\mathbf{E}(\mathbf{D}-\mathbf{D}_N^p)\} = \dot{R}F + RF'\dot{H}, \tag{43}$$

considering Eqs. (1) and (2). Eq. (43) as it is cannot play the role of the consistency condition for the derivation of plastic strain rate since it contains rate variable $\dot{R}$ which is not related to the plastic strain rate yet.

It is observed in experiments that the stress increases almost elastically when it is zero and thereafter it gradually increases approaching the normal-yield surface in the plastic loading process. Then, let the evolution rule of the subloading ratio be given by

$$\dot{R} = U(R)\|\mathbf{D}_N^p\| \quad \text{for } \mathbf{D}_N^p \neq \mathbf{0}, \tag{44}$$

where U is a monotonically decreasing function of R, fulfilling the following conditions.

$$U(R) = \begin{cases} \infty & \text{for } R=0, \\ 0 & \text{for } R=1, \\ <0 & \text{for } R>1. \end{cases} \tag{45}$$

Let the function U satisfying Eq. (45) be simply given by

$$U = -u\ln R, \tag{46}$$

where u is a material constant. The function $U(R)$ is schematically shown in Fig. 5.

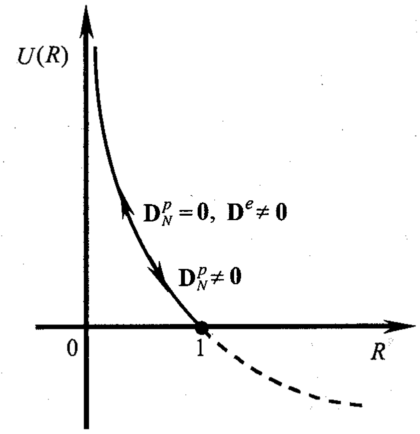


Fig. 5. Function $U(R)$ in the evolution rule of normal-yield ratio R.

The substitution of Eq. (44) into Eq. (43) leads to the following consistency condition for the subloading surface.

$$\mathrm{tr}\{\frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}\mathbf{E}(\mathbf{D}-\mathbf{D}_N^p)\} = U\|\mathbf{D}_N^p\|F + RF'\dot{H}. \tag{47}$$

Further, substituting the associated flow rule

$$\mathbf{D}_N^p = \breve{\lambda}\mathbf{N} \quad (\breve{\lambda}>0) \tag{48}$$

into Eq. (43), the positive proportionality factor is derived as

$$\breve{\lambda} = \frac{\mathrm{tr}\{\mathbf{N}(\mathbf{E}\mathbf{D}-\frac{F'}{F}\dot{H}\boldsymbol{\sigma})\}}{\frac{U}{R}\mathrm{tr}(\mathbf{N}\boldsymbol{\sigma})+\mathrm{tr}(\mathbf{NEN})}, \tag{49}$$

In this model, since the plastic strain rate is induced even inside the normal-yield surface and develops gradually as the stress approaches that surface, the distinction of constitutive equation on and inside the yield surface is not required. Then, considering Eq. (49) into Eq. (25)1, the strain rate is given as follows:

$$\mathring{\boldsymbol{\sigma}} = \mathbf{E}\mathbf{D} - \mathbf{E}\left\langle\frac{\mathrm{tr}\{\mathbf{N}(\mathbf{E}\mathbf{D}-\frac{F'}{F}\dot{H}\boldsymbol{\sigma})\}}{\frac{U}{R}\mathrm{tr}(\mathbf{N}\boldsymbol{\sigma})+\mathrm{tr}(\mathbf{NEN})}\right\rangle\mathbf{N} - \frac{2G}{1+\frac{1}{2G\breve{T}}}\mathbf{D}_t^*, \tag{50}$$

while $\breve{T}$ is the function of stress $\boldsymbol{\sigma}$, internal variable H and the normal-yield surface ratio R in a monotonically-increasing form and let it be given explicitly by the equation with the function ξ of $\boldsymbol{\sigma}$ and H, i.e.

$$\breve{T} \equiv \xi R^{\tau}, \quad \xi \equiv \xi(\boldsymbol{\sigma}, H). \tag{51}$$

ξ is the function of $\boldsymbol{\sigma}, H$ and τ is the material constant. The continuity and smoothness conditions[7)-10)] are

are fulfilled in this model as illustrated in Fig. 2 since the tangential-inelastic strain rate develops gradually as the normal-yield ratio increases.

3.2 Subloading-overstress model

The extension of the overstress model by the incorporation of the subloading surface model described above is given below, while the extended model is called the *subloading-overstress model.*

The surface (42) passing through the current stress point and similar to the normal-yield surface is to be the dynamic-loading surface. Since the distinction of constitutive relations on and inside the yield surface is not required in the subloading surface model, the elastoplastuc stress rate is given in the following equation by replacing of $\boldsymbol{\sigma}$ to $\boldsymbol{\sigma}^{ep}$.

$$\overset{\circ}{\boldsymbol{\sigma}}{}^{ep}=\mathbf{ED}-\mathbf{E}\left\langle \frac{\mathrm{tr}\left\{\mathbf{N}^{ep}\left(\mathbf{ED}-\frac{F'}{F}\dot{H}\boldsymbol{\sigma}^{ep}\right)\right\}}{\frac{U^{ep}}{R^{ep}}\mathrm{tr}(\mathbf{N}^{ep}\boldsymbol{\sigma}^{ep})+\mathrm{tr}(\mathbf{N}^{ep}\mathbf{E}\mathbf{N}^{ep})}\right\rangle \mathbf{N}^{ep} - \frac{2G}{1+\frac{1}{2G\breve{T}^{ep}}}\mathbf{D}_t^{ep*}, \quad (52)$$

where

$$\dot{H}=\mathrm{tr}\{\mathbf{f}_H(\boldsymbol{\sigma}^{ep},H)\mathbf{D}_N^{vp}\}, \quad (53)$$

$$R^{ep}\equiv\frac{f(\boldsymbol{\sigma}^{ep})}{F(H)}, \quad (54)$$

$$U^{ep}\equiv U(R^{ep}),\quad U^{ep}=-u\ln R^{ep} \quad (55)$$

$$\breve{T}^{ep}\equiv\xi^{ep}R^{ep\tau},\quad \xi^{ep}\equiv\xi(\boldsymbol{\sigma}^{ep},H), \quad (56)$$

where R^{ep} $(0\le R^{ep}\le 1)$ is called the *subloading ratio.*

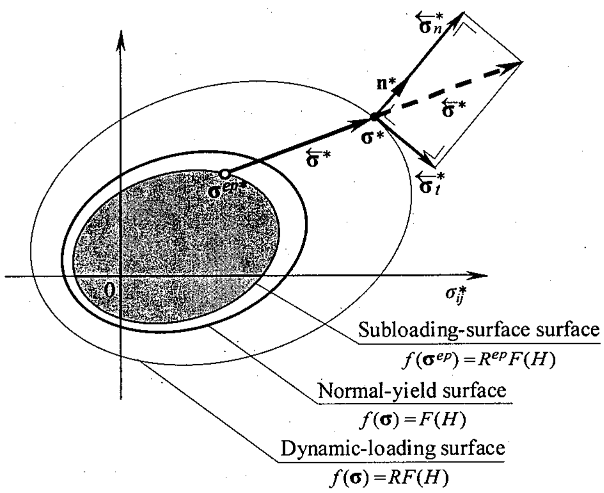


Fig. 6. Overstress tensor in subloading-overstress model in the deviatoric stress space.

Noting that the distinction of constitutive equation inside and on the yield surface is not required in the subloading surface model, the strain rate and the stress rate are given by extending Eqs. (40) and (41) as follows:

$$\mathbf{D}=\mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}}+C_N\langle R-R^{ep}\rangle^N\mathbf{N}+\breve{C}_t\frac{\overset{\leftarrow *}{\boldsymbol{\sigma}}_t}{F}, \quad (57)$$

$$\overset{\circ}{\boldsymbol{\sigma}}=\mathbf{ED}-C_N\langle R-R^{ep}\rangle^N\mathbf{EN}-2G\breve{C}_t\frac{\overset{\leftarrow *}{\boldsymbol{\sigma}}_t}{F}, \quad (58)$$

where $\breve{C}_t$ is the function of stress $\boldsymbol{\sigma}$, internal variable H, absolute temperature T and the normal-yield ratio R in a monotonically-increasing form and let it be assumed to be given as

$$\breve{C}_t\equiv\zeta_t(\boldsymbol{\sigma},H,\mathrm{T})\,R^{\kappa_t}. \quad (59)$$

κ_t is the material constant. The overstress tensor for this model is illustrated in Fig. 6 of the deviatoric stress space.

4. Further extension by the extended subloading surface model

The subloading-overstress model formulated in the preceding section **3** is incapable of describing the cyclic loading behavior exhibiting the open hysteresis loop for unloading-reloading process since the similarity-center $\mathbf{s}$ is fixed in the origin of stress space. The overstress model is further extended below so as to describe the cyclic loading behavior by introducing the *extended subloading surface model*[19] in which the similarity-center moves with the plastic deformation. Besides, the anisotropy is also introduced in the formulation.

4.1 Extended subloading surface model

Introduce the yield condition with the anisotropy:

$$f(\hat{\boldsymbol{\sigma}},\mathbf{H})=F(H), \quad (60)$$

where

$$\hat{\boldsymbol{\sigma}}\equiv\boldsymbol{\sigma}-\boldsymbol{\alpha}. \quad (61)$$

The second-order tensor $\boldsymbol{\alpha}$ is the kinematic hardening variable, i.e. the back stress for metals and the second-order tensor $\mathbf{H}$ is the anisotropic hardening variable inducing the rotation of yield surface in soils for example. The function f is assumed to be homogeneous of degree one in $\hat{\boldsymbol{\sigma}}$ satisfying $f(s\hat{\boldsymbol{\sigma}},\mathbf{H})=sf(\hat{\boldsymbol{\sigma}},\mathbf{H})$ for any nonnegative scalar s, and thus the yield surface keeps a similar shape for $\mathbf{H}=\mathbf{0}$.

The yield stress $\boldsymbol{\sigma}_y$ on the normal-yield surface the outward-normal at which is same as that on the subloading surface at the current stress $\boldsymbol{\sigma}$ is described by

$$\boldsymbol{\sigma}_y=\frac{1}{\bar{R}}\{\boldsymbol{\sigma}-(1-\bar{R})\mathbf{s}\}\quad(\boldsymbol{\sigma}-\mathbf{s}=\bar{R}(\boldsymbol{\sigma}_y-\mathbf{s})), \quad (62)$$

where $\bar{R}$ is the *extended normal-yield ratio*, i.e. the ratio of the size of extended subloading surface to that of the extended normal-yield surface of Eq. (60).

Substituting Eq. (62) into Eq. (60) in which $\boldsymbol{\sigma}_y$ in Eq. (62) is regarded to be $\boldsymbol{\sigma}$, the extended subloading surface for the extended normal-yield surface of Eq. (60) is described as

$$f(\bar{\boldsymbol{\sigma}},\mathbf{H})=\bar{R}F(H), \tag{63}$$

where

$$\bar{\boldsymbol{\sigma}}\equiv\boldsymbol{\sigma}-\bar{\boldsymbol{\alpha}}\ (=\bar{R}\hat{\boldsymbol{\sigma}}_y), \tag{64}$$

$$\hat{\boldsymbol{\sigma}}_y\equiv\boldsymbol{\sigma}_y-\boldsymbol{\alpha}, \tag{65}$$

$$\bar{\boldsymbol{\alpha}}\equiv\bar{R}\boldsymbol{\alpha}+(1-\bar{R})\mathbf{s}\ \ (\bar{\boldsymbol{\alpha}}-\mathbf{s}=\bar{R}(\boldsymbol{\alpha}-\mathbf{s})). \tag{66}$$

$\bar{\boldsymbol{\alpha}}$ is the conjugate point in the extended subloading surface for the back stress $\boldsymbol{\alpha}$ in the normal-yield surface. In calculation first $\bar{R}$ is determined from Eq. (63) with Eq. (64) by substituting values of H, $\boldsymbol{\alpha}$, $\mathbf{H}$, $\mathbf{s}$ and $\boldsymbol{\sigma}$ and thereafter $\bar{\boldsymbol{\alpha}}$ is found from Eq.(66). The four plastic internal variables H, $\boldsymbol{\alpha}$, $\mathbf{H}$ and $\mathbf{s}$ are involved in the present model. The normal-yield and the extended subloading surfaces are illustrated in Fig. 7.

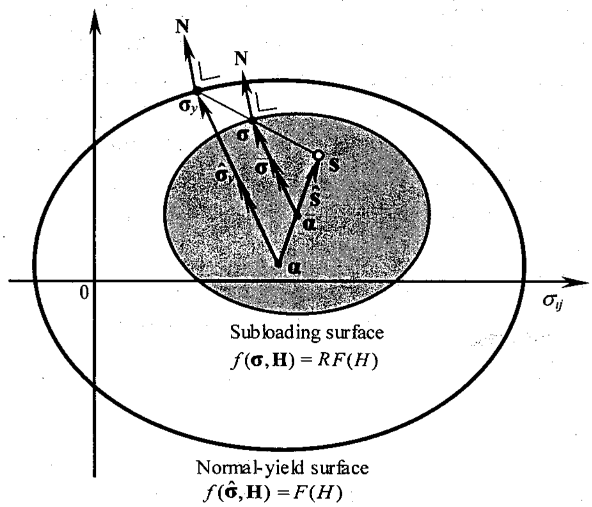


Fig. 7. Normal-yield and subloading surfaces in the extended subloading surface model.

The material-time derivative of Eq. (63) is given by

$$\mathrm{tr}\Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\boldsymbol{\sigma}}\Big)-\mathrm{tr}\Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\bar{\boldsymbol{\alpha}}}\Big)+\mathrm{tr}\Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial\mathbf{H}}\overset{\circ}{\mathbf{H}}\Big)=\dot{\bar{R}}F+\bar{R}F'\dot{H}, \tag{67}$$

The following inequality must hold since the similarity-center $\mathbf{s}$ has to exist inside the normal-yield surface.

$$f(\hat{\mathbf{s}},\mathbf{H})\leq F(H), \tag{68}$$

where

$$\hat{\mathbf{s}}\equiv\mathbf{s}-\boldsymbol{\alpha}. \tag{69}$$

Let the ultimate state $f(\hat{\mathbf{s}},\mathbf{H})=F(H)$ be considered, in which the similarity-center exists just on the normal-yield surface and thus the risk that the similarity-center goes out from the normal-yield surface has to be avoided. The time-differentiation of Eq. (68) in the ultimate state leads to:

$$\mathrm{tr}\Big[\frac{\partial f(\hat{\mathbf{s}},\mathbf{H})}{\partial\mathbf{s}}\Big(\overset{\circ}{\mathbf{s}}+\frac{1}{F}\Big\{\mathrm{tr}\Big(\frac{\partial f(\hat{\mathbf{s}},\mathbf{H})}{\partial\mathbf{H}}\overset{\circ}{\mathbf{H}}\Big)-\dot{F}\Big\}\hat{\mathbf{s}}\Big)\Big]\leq 0 \quad \text{for } f(\hat{\mathbf{s}},\mathbf{H})=F(H). \tag{70}$$

The inequality (68) or (70) is called the *enclosing condition of similarity-center*.

In the ultimate state $f(\hat{\mathbf{s}},\mathbf{H})=F(H)$, the vector $\boldsymbol{\sigma}_y-\mathbf{s}$ makes an obtuse angle with the outward-normal vector $\partial f(\hat{\mathbf{s}},\mathbf{H})/\partial\mathbf{s}$ to the normal-yield surface at the similarity-center, provided that the normal-yield surface is convex. Noting this fact and considering the fact that the similarity-center moves only with the normal-plastic deformation, let the following equation be assumed so as to fulfill the inequality (70):

$$\overset{\circ}{\mathbf{s}}+\frac{1}{F}\Big\{\mathrm{tr}\Big(\frac{\partial f(\hat{\mathbf{s}},\mathbf{H})}{\partial\mathbf{H}}\overset{\circ}{\mathbf{H}}\Big)-\dot{F}\Big\}\hat{\mathbf{s}}=c\|\mathbf{D}_N^p\|\frac{\tilde{\boldsymbol{\sigma}}}{\bar{R}}, \tag{71}$$

where c is a material constant influencing the translating rate of the similarity-center and

$$\tilde{\boldsymbol{\sigma}}\equiv\boldsymbol{\sigma}-\mathbf{s}\ (=\bar{R}(\boldsymbol{\sigma}_y-\mathbf{s})) \tag{72}$$

The translation rule of the similarity-center is now derived from Eq. (71) as follows:

$$\overset{\circ}{\mathbf{s}}=c\|\mathbf{D}_n^p\|\frac{\tilde{\boldsymbol{\sigma}}}{\bar{R}}+\overset{\circ}{\boldsymbol{\alpha}}+\frac{1}{F}\Big\{F'\dot{H}-\mathrm{tr}\Big(\frac{\partial f(\hat{\mathbf{s}},\mathbf{H})}{\partial\mathbf{H}}\overset{\circ}{\mathbf{H}}\Big)\Big\}\hat{\mathbf{s}}. \tag{73}$$

It is conceivable that the similarity-center $\mathbf{s}$ approaches $\boldsymbol{\sigma}_y$ as can be seen from the simple case of the nonhardening state ($\overset{\circ}{\boldsymbol{\alpha}}=\overset{\circ}{\mathbf{H}}=\mathbf{0}$, $\dot{H}=0$), although the evolution rule (73) is derived on the fulfillment of the requirement (70) in the ultimate state $f(\hat{\mathbf{s}},\mathbf{H})=F(H)$.

The evolution rule of the extended normal-yield ratio $\bar{R}$ is given by the following equation in the identical form of Eq. (44) for R.

$$\dot{\bar{R}}=\bar{U}(\bar{R})\|\mathbf{D}_N^p\| \quad \text{for } \mathbf{D}_N^p\neq\mathbf{0}, \tag{74}$$

where $\bar{U}$ is the function of $\bar{R}$ having the same form as U in Eq. (45) and thus the simplest explicit function is given by

$$\bar{U}=-u\ln\bar{R}. \tag{75}$$

The substitution of Eqs. (73) and (74) into Eq. (67) leads to the consistency condition for the subloading surface:

$$\mathrm{tr}\Big\{\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\mathbf{E}(\mathbf{D}-\mathbf{D}_N^p)\Big\}-\mathrm{tr}\Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\bar{\boldsymbol{\alpha}}}\Big)+\mathrm{tr}\Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial\mathbf{H}}\overset{\circ}{\mathbf{H}}\Big)=\bar{U}\|\mathbf{D}_N^p\|F+\bar{R}F'\dot{H} \tag{76}$$

with

$$\overset{\circ}{\bar{\boldsymbol{\alpha}}}=\overset{\circ}{\boldsymbol{\alpha}}+\frac{1-\bar{R}}{F}\left\{F'\dot{H}-\mathrm{tr}\left(\frac{\partial f(\hat{\mathbf{s}},\mathbf{H})}{\partial\mathbf{H}}\overset{\circ}{\mathbf{H}}\right)\right\}\hat{\mathbf{s}}$$
$$-\|\mathbf{D}_N^p\|\left\{\bar{U}\hat{\mathbf{s}}-c\left(\frac{1}{\bar{R}}-1\right)\tilde{\boldsymbol{\sigma}}\right\} \quad (77)$$

Substituting the associated flow rule

$$\mathbf{D}_N^p=\bar{\lambda}\bar{\mathbf{N}} \quad (\bar{\lambda}>0), \quad (78)$$

where $\bar{\lambda}$ is a positive proportionality factor and the second-order tensor $\bar{\mathbf{N}}$ denotes the normalized outward-normal to the subloading surface, i.e.

$$\bar{\mathbf{N}}\equiv\frac{f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\Big/\left\|\frac{f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\right\| \quad (\|\bar{\mathbf{N}}\|=1), \quad (79)$$

it is obtained that

$$\bar{\lambda}=\mathrm{tr}\Big[\bar{\mathbf{N}}\Big\{\mathbf{E}\mathbf{D}-\frac{F'}{F}\dot{H}\bar{\boldsymbol{\sigma}}-\bar{R}\overset{\circ}{\boldsymbol{\alpha}}-(1-\bar{R})\overset{\circ}{\hat{\mathbf{s}}}\Big\}$$
$$+\frac{1}{\bar{R}F}\mathrm{tr}\left(\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial\mathbf{H}}\overset{\circ}{\mathbf{H}}\right)\bar{\boldsymbol{\sigma}}\Big\}\Big]$$
$$\Big/\Big[\bar{U}\mathrm{tr}\Big\{\bar{\mathbf{N}}\Big(\frac{1}{\bar{R}}\bar{\boldsymbol{\sigma}}-\hat{\mathbf{s}}\Big)\Big\}+\mathrm{tr}(\bar{\mathbf{N}}\mathbf{E}\bar{\mathbf{N}})\Big], \quad (80)$$

Referring to Eq. (50) for the subloading surface model and substituting Eq. (80), the stress rate is given as follows:

$$\overset{\circ}{\boldsymbol{\sigma}}=\mathbf{E}\mathbf{D}-$$
$$\mathbf{E}\Big\langle\mathrm{tr}\Big[\bar{\mathbf{N}}\Big\{\mathbf{E}\mathbf{D}-\frac{F'}{F}\dot{H}\bar{\boldsymbol{\sigma}}-\bar{R}\overset{\circ}{\boldsymbol{\alpha}}-(1-\bar{R})\overset{\circ}{\hat{\mathbf{s}}}$$
$$+\frac{1}{\bar{R}F}\mathrm{tr}\left(\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial\mathbf{H}}\overset{\circ}{\mathbf{H}}\right)\bar{\boldsymbol{\sigma}}\Big\}\Big]$$
$$\Big/\Big[\bar{U}\mathrm{tr}\Big\{\bar{\mathbf{N}}\Big(\frac{1}{\bar{R}}\bar{\boldsymbol{\sigma}}-\hat{\mathbf{s}}\Big)\Big\}+\mathrm{tr}(\bar{\mathbf{N}}\mathbf{E}\bar{\mathbf{N}})\Big]\Big\rangle\bar{\mathbf{N}}$$
$$-\frac{2G}{1+\dfrac{1}{2G\bar{T}}}\bar{\mathbf{D}}_t^*, \quad (81)$$

where

$$\bar{\mathbf{N}}^*\equiv\bar{\mathbf{I}}^*\bar{\mathbf{N}}, \quad \bar{\mathbf{n}}^*\equiv\frac{\bar{\mathbf{N}}^*}{\|\bar{\mathbf{N}}^*\|}, \quad (82)$$

$$\hat{\bar{\mathbf{n}}}\equiv\bar{\mathbf{n}}^*\otimes\bar{\mathbf{n}}^*, \quad \hat{\bar{\mathbf{I}}}^*\equiv\bar{\mathbf{I}}^*-\hat{\bar{\mathbf{n}}}^*, \quad (83)$$

$$\bar{\mathbf{D}}_t^*\equiv\hat{\bar{\mathbf{I}}}^*\mathbf{D}=\mathbf{D}^*-\mathrm{tr}(\bar{\mathbf{n}}^*\mathbf{D}^*)\bar{\mathbf{n}}^* \quad (84)$$

where $\bar{T}$ is given by the following equation, replacing the normal-yield ratio R to the extended normal-yield ratio $\bar{R}$.

$$\bar{T}\equiv\bar{\xi}\bar{R}^{\tau}, \quad \bar{\xi}\equiv\xi(\boldsymbol{\sigma},H,\boldsymbol{\alpha},\mathbf{H}). \quad (85)$$

4.2 Extended subloading-overstress model

Replacing the stress $\boldsymbol{\sigma}$ to the elastoplastic stress $\boldsymbol{\sigma}^{ep}$ in Eq. (81), the elastoplastic stress in the *extended subloading-overstress model* is given by

$$\overset{\circ}{\boldsymbol{\sigma}}{}^{ep}=\mathbf{E}\mathbf{D}-$$
$$\mathbf{E}\Big\langle\mathrm{tr}\Big[\bar{\mathbf{N}}^{ep}\Big\{\mathbf{E}\mathbf{D}-\frac{F'}{F}\dot{H}\bar{\boldsymbol{\sigma}}^{ep}-\bar{R}^{ep}\overset{\circ}{\boldsymbol{\alpha}}-(1-\bar{R}^{ep})\overset{\circ}{\hat{\mathbf{s}}}$$
$$+\frac{1}{\bar{R}^{ep}F}\mathrm{tr}\left(\frac{\partial f(\bar{\boldsymbol{\sigma}}^{ep},\mathbf{H})}{\partial\mathbf{H}}\overset{\circ}{\mathbf{H}}\right)\bar{\boldsymbol{\sigma}}^{ep}\Big\}\Big]$$
$$\Big/\Big[\bar{U}^{ep}\mathrm{tr}\Big\{\bar{\mathbf{N}}^{ep}\Big(\frac{1}{\bar{R}^{ep}}\bar{\boldsymbol{\sigma}}^{ep}-\hat{\mathbf{s}}\Big)\Big\}+\mathrm{tr}(\bar{\mathbf{N}}^{ep}\mathbf{E}\bar{\mathbf{N}}^{ep})\Big]\Big\rangle\bar{\mathbf{N}}^{ep}$$
$$-\frac{2G}{1+\dfrac{1}{2G\bar{T}^{ep}}}\bar{\mathbf{D}}_t^{ep*}, \quad (86)$$

where

$$\left.\begin{aligned}\dot{H}&=\mathrm{tr}\{\mathbf{f}_H(\boldsymbol{\sigma}^{ep},H,\boldsymbol{\alpha},\mathbf{H})\mathbf{D}_N^{vp}\}\\ \overset{\circ}{\boldsymbol{\alpha}}&=\mathbf{f}_{\alpha}(\boldsymbol{\sigma}^{ep},H,\boldsymbol{\alpha},\mathbf{H})\mathbf{D}_N^{vp}\\ \overset{\circ}{\mathbf{H}}&=\mathbf{f}_{\mathbf{H}}(\boldsymbol{\sigma}^{ep},H,\boldsymbol{\alpha},\mathbf{H})\mathbf{D}_N^{vp}\end{aligned}\right\} \quad (87)$$

$$\bar{\boldsymbol{\sigma}}^{ep}\equiv\boldsymbol{\sigma}^{ep}-\bar{\boldsymbol{\alpha}}, \quad (88)$$

$$\bar{\mathbf{N}}^{ep}\equiv\frac{\partial f(\bar{\boldsymbol{\sigma}}^{ep},\mathbf{H})}{\partial\boldsymbol{\sigma}^{ep}}\Big/\left\|\frac{\partial f(\bar{\boldsymbol{\sigma}}^{ep},\mathbf{H})}{\partial\boldsymbol{\sigma}^{ep}}\right\| \quad (\|\bar{\mathbf{N}}^{ep}\|=1), \quad (89)$$

$$\bar{\mathbf{N}}^{ep*}\equiv\bar{\mathbf{I}}^*\bar{\mathbf{N}}^{ep}, \quad \bar{\mathbf{n}}^{ep*}\equiv\frac{\bar{\mathbf{N}}^{ep*}}{\|\bar{\mathbf{N}}^{ep*}\|}, \quad (90)$$

$$\hat{\bar{\mathbf{n}}}^{ep*}\equiv\bar{\mathbf{n}}^{ep*}\otimes\bar{\mathbf{n}}^{ep*}, \quad \hat{\bar{\mathbf{I}}}^{ep*}\equiv\bar{\mathbf{I}}^*-\hat{\bar{\mathbf{n}}}^{ep*}, \quad (91)$$

$$\bar{\mathbf{D}}_t^{ep*}\equiv\hat{\bar{\mathbf{I}}}^{ep*}\mathbf{D}=\mathbf{D}^*-\mathrm{tr}(\bar{\mathbf{n}}^{ep*}\mathbf{D}^*)\bar{\mathbf{n}}^{ep*} \quad (92)$$

$$\bar{R}^{ep}\equiv\frac{f(\bar{\boldsymbol{\sigma}}^{ep},\mathbf{H})}{F(H)}, \quad (93)$$

$$\bar{U}^{ep}=U(\bar{R}^{ep}), \quad \bar{U}^{ep}=-u\ln\bar{R}^{ep}, \quad (94)$$

$$\tilde{\boldsymbol{\sigma}}^{ep}\equiv\boldsymbol{\sigma}^{ep}-\mathbf{s}, \quad (95)$$

$$\overset{\circ}{\mathbf{s}}=c\|\mathbf{D}_N^{vp}\|\frac{\tilde{\boldsymbol{\sigma}}^{ep}}{\bar{R}^{ep}}+\overset{\circ}{\boldsymbol{\alpha}}+\frac{1}{F}\left\{F'\dot{H}-\mathrm{tr}\left(\frac{\partial f(\hat{\mathbf{s}},\mathbf{H})}{\partial\mathbf{H}}\overset{\circ}{\mathbf{H}}\right)\right\}\hat{\mathbf{s}}, \quad (96)$$

The second-order tensor $\mathbf{f}_H$ and the fourth-order tensor $\mathbf{f}_{\alpha}$, $\mathbf{f}_{\mathbf{H}}$ are the functions of $\boldsymbol{\sigma}^{ep}$, H, $\boldsymbol{\alpha}$ and $\mathbf{H}$. $\bar{R}_s$ $(0\le\bar{R}_s\le1)$ is called the *extended subloading stress.*

Noting that the distinction of constitutive equation inside and on the yield surface is not required in this model, the strain rate and the stress rate are given by extending Eqs. (40) and (41) as follows:

$$\mathbf{D}=\mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}}+\hat{C}_N\langle\bar{R}-\bar{R}^{ep}\rangle^N\bar{\mathbf{N}}+\bar{C}_t\frac{\overleftarrow{\boldsymbol{\sigma}}_t^*}{F}, \quad (97)$$

$$\overset{\circ}{\boldsymbol{\sigma}}=\mathbf{E}\mathbf{D}-\hat{C}_N\langle\bar{R}-\bar{R}^{ep}\rangle^N\mathbf{E}\bar{\mathbf{N}}-2G\bar{C}_t\frac{\overleftarrow{\boldsymbol{\sigma}}_t^*}{F}, \quad (98)$$

where

$$\overleftarrow{\boldsymbol{\sigma}}=\overleftarrow{\boldsymbol{\sigma}}_n^*+\overleftarrow{\boldsymbol{\sigma}}_t^* \quad (99)$$

$$\left.\begin{aligned}\overleftarrow{\boldsymbol{\sigma}}_n^*&\equiv\hat{\bar{\mathbf{n}}}^*\overleftarrow{\boldsymbol{\sigma}}=\mathrm{tr}(\bar{\mathbf{n}}^*\overleftarrow{\boldsymbol{\sigma}})\bar{\mathbf{n}}^*\\ \overleftarrow{\boldsymbol{\sigma}}_t^*&\equiv\hat{\bar{\mathbf{I}}}^*\overleftarrow{\boldsymbol{\sigma}}=\overleftarrow{\boldsymbol{\sigma}}^*-\overleftarrow{\boldsymbol{\sigma}}_n^*\end{aligned}\right\} \quad (100)$$

$\hat{C}_N$ and $\bar{C}_t$ are the functions of stress $\boldsymbol{\sigma}$, internal variables $H, \boldsymbol{\alpha}, \mathbf{H}$, absolute temperature T, $\bar{C}_t$ is further the monotonically-increasing function of $\bar{R}$ as follows:

$$\left.\begin{array}{l}\hat{C}_N \equiv \hat{C}_N(\boldsymbol{\sigma}, H, \boldsymbol{\alpha}, \mathbf{H}, \mathrm{T}), \\ \bar{C}_t \equiv \hat{\zeta}_t(\boldsymbol{\sigma}, H, \boldsymbol{\alpha}, \mathbf{H}, \mathrm{T}) \bar{R}^{\kappa_t}.\end{array}\right\} \quad (101)$$

The equation (86) was derived by the different method that the associated flow rule is substituted into Eq. (67) in the form of stress rate without the transformation to Eq. (76) in the form of strain rate.

5. Concluding remarks

The novel variable "overstress tensor" is proposed, which has been the missing link in the overstress model and the constitutive formulation of overstress model is improved based on it in this article. It is capable of describing the viscoplastic strain rate due to the rate of stress inside the yield surface and the tangential-viscoplastic strain rate due to the overstress component tangential to the yield surface. The qualitative response of this model and its comparison with test data will b described in detail in subsequent papers.

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