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Constitutive equation for friction with static-kinetic friction transition

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A high friction coefficient is first observed as a sliding between bodies commences, which is called the *static-friction*. Then, the friction coefficient decreases approaching the lowest stationary value, which is called the *kinetic-friction*. Thereafter, if the sliding stops for a while and then it starts again, the friction coefficient recovers and a similar behavior as that in the first sliding is reproduced. These are fundamental characteristics in the friction phenomenon, which have been widely recognized for a long time. In this article the *subloading-friction model* with a smooth elastic-plastic sliding transition is extended so as to describe these facts by formulating the rate-dependent hardening/softening rule of sliding-yield surface adequately.

Key Words: *Constitutive law, friction, hardening/softening, subloading, time-dependency*

1. Introduction

Description of the friction phenomenon as a constitutive equation has been attained first as a rigid-plasticity[1),2)]. Further, they have been extended to an elastoplasticity[3)~16)] in which the penalty concept, i.e. the fictitious springs between contact surfaces is incorporated and the isotropic hardening is taken into account so as to describe the test results[17)] exhibiting the smooth contact traction vs. sliding displacement curve reaching the static-friction. However, the interior of the sliding-yield surface has been assumed to be an elastic domain and thus the plastic-sliding velocity due to the rate of traction inside the sliding-yield surface and the accumulation of plastic-sliding due to the cyclic loading of contact traction cannot be described by these models. They could be called the conventional friction model in accordance with the classification of plastic constitutive models by Drucker[18)]. On the other hand, the first author of the present article has proposed the subloading surface model[19)~21)] within the framework of unconventional plasticity, which is capable of describing the plastic strain rate by the rate of stress inside the yield surface. Further, the authors proposed the *subloading-friction model*[22)] describing the smooth transition from the elastic- to the plastic-sliding state by incorporating the concept of subloading surface. Besides, in this model the decrease of friction coefficient with the increase of normal contact traction observed in experiments[23),24)] is described by incorporating the nonlinear sliding-yield surface, while the decrease has not been taken into account in Coulomb friction law which has been adopted widely in friction models.

It is widely known that when bodies at rest begin to slide to each other, a high friction coefficient appears first, which is called the *static-friction*, and then it decreases approaching a stationary value, called the *kinetic-friction*. However, this process has not been formulated so far, although the increase of friction coefficient up the peak has been described as the hardening process[3)~16)]. Further, it has been found that if the sliding ceases for a while and then it starts again, the static-friction recovers and the identical behavior as that in the initial sliding is reproduced[24)~36)]. The recovery has been formulated by equations[24)~36)] including the time elapsed after the stop of sliding. However, the inclusion of time itself leads to the loss of objectivity in constitutive equations as known from the fact that the elapsed time varies depending on the judgment of time as the stop of sliding, which is accompanied with the arbitrariness especially when the sliding velocity fluctuates. Here, it should be noted that the variation of material property has to be described by internal variables and their rates.

The decrease of friction coefficient from the static- to kinetic-friction and the recovery of friction coefficient mentioned above are to be the fundamental behavior of friction between bodies, which have been recognized widely. Difference of the static- and the kinetic-friction often reaches up to several tens percents. Therefore, the formulation of the transition from the static- to the kinetic-friction and vice versa is of importance for the development of mechanical design in the field of engineering. However, the rational formulation has not been attained so far.

In this article the subloading-friction model[22] is extended so as to describe the decrease of friction coefficient from the static- to kinetic-friction by the softening due to the plastic-sliding and the recovery of friction coefficient by the hardening due to the creep deformation under the contact pressure. The extended model would be called the *time-dependent subloading-friction model.*

2. Formulation of the constitutive equation for friction

The subloading-friction model proposed recently by the authors[22] is extended below to the time-dependent friction model describing the *static-kinetic friction transition*, i.e., the transition from static- to kinetic-frictions, and vice versa.

2.1 Decomposition of sliding velocity

The sliding (relative) velocity $\mathbf{r}$ between contact surfaces is additively decomposed into the normal component $\mathbf{r}_n$ and the tangential component $\mathbf{r}_t$ as follows:

$$\mathbf{r} = \mathbf{r}_n + \mathbf{r}_t, \tag{1}$$

which are given as

$$\left.\begin{aligned} \mathbf{r}_n &= (\mathbf{r} \bullet \mathbf{n})\mathbf{n} = (\mathbf{n} \otimes \mathbf{n})\mathbf{r}, \\ \mathbf{r}_t &= \mathbf{r} - \mathbf{r}_n = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{r}, \end{aligned}\right\} \tag{2}$$

where $\mathbf{n}$ is the unit outward-normal vector at the contact surface, $(\bullet)$ and $\otimes$ denote the scalar and the tensor products, respectively, and $\mathbf{I}$ is the identity tensor. On the other hand, it is assumed that $\mathbf{r}$ is additively decomposed into the elastic (penalty)-sliding velocity $\mathbf{r}^e$ and the plastic-sliding velocity $\mathbf{r}^p$, i.e.,

$$\mathbf{r} = \mathbf{r}^e + \mathbf{r}^p \tag{3}$$

with

$$\left.\begin{aligned} \mathbf{r}_n &= \mathbf{r}_n^e + \mathbf{r}_n^p, \\ \mathbf{r}_t &= \mathbf{r}_t^e + \mathbf{r}_t^p. \end{aligned}\right\} \tag{4}$$

First, let the elastic-sliding velocity be given by

$$\left.\begin{aligned} \mathring{\mathbf{f}}_n &= -\alpha_n \mathbf{r}_n^e, \\ \mathring{\mathbf{f}}_t &= -\alpha_t \mathbf{r}_t^e, \end{aligned}\right\} \tag{5}$$

where $\mathbf{f}_n$ and $\mathbf{f}_t$ are the normal component and tangential component, respectively, of the traction vector $\mathbf{f}$ applied to a unit area of contact surface, i.e.

$$\left.\begin{aligned} \mathbf{f}_n &\equiv (\mathbf{n} \bullet \mathbf{f})\mathbf{n} = (\mathbf{n} \otimes \mathbf{n})\mathbf{f}, \\ \mathbf{f}_t &\equiv \mathbf{f} - \mathbf{f}_n = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{f}, \end{aligned}\right\} \tag{6}$$

and, $(\circ)$ denoting the corotational rate, $\mathring{\mathbf{f}}_n$ and $\mathring{\mathbf{f}}_t$ are the normal component and tangential component, respectively, of the corotational rate $\mathring{\mathbf{f}}$ of the traction vector $\mathbf{f}$, i.e.,

$$\left.\begin{aligned} \mathring{\mathbf{f}}_n &\equiv (\mathbf{n} \bullet \mathring{\mathbf{f}})\mathbf{n} = (\mathbf{n} \otimes \mathbf{n})\mathring{\mathbf{f}}, \\ \mathring{\mathbf{f}}_t &\equiv \mathring{\mathbf{f}} - \mathring{\mathbf{f}}_n = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathring{\mathbf{f}}, \end{aligned}\right\} \tag{7}$$

which are related to the material-time derivative denoted by $(\dot{\ })$ as follows:

$$\left.\begin{aligned} \mathring{\mathbf{f}} &= \dot{\mathbf{f}} - \mathbf{\Omega}\mathbf{f}, \\ \mathring{\mathbf{f}}_n &= \dot{\mathbf{f}}_n - \mathbf{\Omega}\mathbf{f}_n, \\ \mathring{\mathbf{f}}_t &= \dot{\mathbf{f}}_t - \mathbf{\Omega}\mathbf{f}_t, \end{aligned}\right\} \tag{8}$$

where the skew-symmetric tensor $\mathbf{\Omega}$ designates the rigid-body rotation of the contact surface. α_n and α_t are the contact penalty parameters representing the fictitious contact elastic moduli in the normal and the tangential directions to the contact surface. Thus, it follows from Eq. (5) that

$$\mathring{\mathbf{f}} = \mathring{\mathbf{f}}_n + \mathring{\mathbf{f}}_t = \mathbf{C}^e \mathbf{r}^e, \tag{9}$$

where the second-order tensor $\mathbf{C}^e$ is the fictitious contact elastic modulus tensor between contact surfaces and is decomposed into the normal and tangential components, i.e.,

$$\mathbf{C}^e = \mathbf{C}_n^e + \mathbf{C}_t^e \tag{10}$$

with

$$\left.\begin{aligned} \mathbf{C}_n^e &= -\alpha_n \mathbf{n} \otimes \mathbf{n}, \\ \mathbf{C}_t^e &= -\alpha_t (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}). \end{aligned}\right\} \tag{11}$$

2.2 Normal-sliding and sliding-subloading surfaces

Assume the following *sliding-yield surface* with isotropic hardening/softening, which describes the *sliding-yield condition.*

$$f(\|\mathbf{f}_n\|, \|\mathbf{f}_t\|) = F, \tag{12}$$

where $\| \ \|$ designates the magnitude, and F is the isotropic hardening/softening function denoting the variation of the size of sliding-yield surface. In what follows, we assume that the interior of the sliding-yield surface is not a purely elastic domain but that the plastic-sliding velocity is induced by the variation of traction inside that surface. Therefore, let the surface described by Eq. (12) be renamed the *normal-sliding surface.*

Next, in accordance with the concept of subloading surface[19)~21)], we introduce the *sliding-subloading surface*, which always passes through the current traction $\mathbf{f}$ and keeps a similar shape and a same orientation to the normal-sliding surface with respect to the zero traction point $\mathbf{f} = \mathbf{0}$. Then, the sliding-subloading surface fulfills the following geometrical characteristics.

i) All lines connecting an arbitrary point inside the sliding-subloading surface and its conjugate point inside the normal-sliding surface join at a unique point, called the *similarity-center*, which is the origin of the contact traction space in the present model.

ii) All ratios of length of an arbitrary line-element connecting two points inside the sliding-subloading surface to that of an arbitrary conjugate line-element connecting two conjugate points inside the normal-sliding surface are identical. The ratio is called the *similarity-ratio,* which coincides with the ratio of the sizes of these surfaces.

Let the similarity-ratio of the sliding-subloading surface to the normal-sliding surface be called the *normal-sliding ratio*, denoted by $R\ (0 \le R \le 1)$, where $R = 0$ corresponds to the null traction state ($f = 0$) as the most elastic state, $0 < R < 1$ to the *subsliding state* ($0 < f < F$), and $R = 1$ to the *normal-sliding state* in which the traction lies on the normal-sliding surface ($f = F$). Therefore, the normal-sliding ratio R plays the role of three-dimensional measure of the degree of approach to the normal-sliding state. Then, the sliding-subloading surface is described by

$$f(\|\mathbf{f}_n\|,\ \|\mathbf{f}_t\|) = RF\,. \tag{13}$$

The material-time derivative of Eq. (13) leads to

$$\frac{\partial f}{\partial\|\mathbf{f}_n\|}\mathbf{n}\bullet\dot{\mathbf{f}}_n+\frac{\partial f}{\partial\|\mathbf{f}_t\|}\mathbf{t}\bullet\dot{\mathbf{f}}_t=\dot{R}F+R\dot{F}\,, \tag{14}$$

where

$$\mathbf{n}\equiv\frac{\mathbf{f}_n}{\|\mathbf{f}_n\|},\quad \mathbf{t}\equiv\frac{\mathbf{f}_t}{\|\mathbf{f}_t\|}\,. \tag{15}$$

Eq. (14) is obviously transformed to

$$\frac{\partial f}{\partial\|\mathbf{f}_n\|}\mathbf{n}\bullet\mathring{\mathbf{f}}_n+\frac{\partial f}{\partial\|\mathbf{f}_t\|}\mathbf{t}\bullet\mathring{\mathbf{f}}_t=\dot{R}F+R\dot{F} \tag{16}$$

since the function f is a scalar variable, while we can confirm easily the rationality of this transformation by substituting Eq. (8) into Eq. (14), noting the equation

$$\mathbf{a}\bullet(\mathbf{\Omega a})=0 \tag{17}$$

for an arbitrary vector $\mathbf{a}$.

2.3 Evolution rules of the hardening function and the normal-sliding ratio

It could be stated from experiments that

1) If the sliding commences, the friction coefficient reaches first the highest value of static-friction and then it reduces to the lowest stationary value of kinetic-friction. Physically, this phenomenon could be interpreted to be caused by the separation of adhesion of surface asperities between contact bodies due to the plastic-sliding. Then, let it be assumed that the reduction is caused by the contraction of the normal-sliding surface, i.e. the isotropic softening due to the plastic-sliding.

2) If the sliding ceases after the reduction of friction coefficient, the friction coefficient recovers gradually with the elapse of time and the static-friction is reproduced after a sufficient time. Physically, this phenomenon could be interpreted to be caused by the reconstruction of the adhesion of surface asperities between contact bodies subjected to the contact pressure. Then, let it be assumed that the recovery is caused by the expansion of the normal-sliding surface, i.e. the isotropic hardening due to the creep deformation under the contact pressure.

Taking account of these facts, let the evolution rule of the isotropic hardening function F be postulated as follows:

$$\dot{F}=-\kappa\left(\frac{F}{F_k}-1\right)^m\|\mathbf{r}^p\|+\eta\left(1-\frac{F}{F_s}\right)^n, \tag{18}$$

where F_s and F_k $(F_s > F_k)$ are the ultimate values of F for the static- and the kinetic-friction, respectively. κ and m are the material constants influencing the decreasing rate of F with the plastic-sliding, and η and n are the material constants influencing the recovering rate of F with the elapse of time, while they would be functions of absolute temperature in general. The first and the second terms in Eq. (18) stand for the deterioration and the formation, respectively, of the adhesion between surface asperities.

It is observed in experiments that the tangential traction increases almost elastically when it is zero and thereafter it gradually increases approaching the normal-sliding surface in the plastic-sliding process. Then, we assume the evolution rule of the normal-sliding ratio as follows:

$$\dot{R}=U(R)\|\mathbf{r}^p\|\quad \text{for } \mathbf{r}^p\neq\mathbf{0}\,, \tag{19}$$

where U is a monotonically decreasing function of R fulfilling the following conditions.

$$\left.\begin{array}{l} U=+\infty \quad \text{for } R=0,\\ U=0 \quad \text{for } R=1,\\ (U<0 \quad \text{for } R>1). \end{array}\right\} \tag{20}$$

The simplest function U fulfilling Eq. (20) is given by

$$U=-u\ln R\,, \tag{21}$$

where u is the material constant.

2.4 Sliding velocity

The substitution of Eqs. (18) and (19) into Eq. (16) gives rise to the *consistency condition* for the sliding-subloading surface:

$$\frac{\partial f}{\partial\|\mathbf{f}_n\|}\mathbf{n}\bullet\mathring{\mathbf{f}}_n+\frac{\partial f}{\partial\|\mathbf{f}_t\|}\mathbf{t}\bullet\mathring{\mathbf{f}}_t$$
$$=U\|\mathbf{r}^p\|F+R\left\{-\kappa\left(\frac{F}{F_k}-1\right)^m\|\mathbf{r}^p\|+\eta\left(1-\frac{F}{F_s}\right)^n\right\}. \tag{22}$$

Assume the following *sliding-plastic flow rule.*

$$\mathbf{r}^p=-\lambda\mathbf{t}\ (\lambda>0)\,, \tag{23}$$

where λ is a positive proportionality factor, where the sign minus in the right-hand side is added since the plastic-sliding rate is induced in the opposite direction of the tangential contact traction. Substituting Eq. (23) into Eq. (22), the proportionality factor λ is derived as follows:

$$\lambda = \frac{-\left\{\frac{\partial f}{\partial\|\mathbf{f}_n\|}(\mathbf{n}\bullet\overset{\circ}{\mathbf{f}})+\frac{\partial f}{\partial\|\mathbf{f}_t\|}(\mathbf{t}\bullet\overset{\circ}{\mathbf{f}})\right\}+\eta R\left(1-\frac{F}{F_s}\right)^n}{R\kappa\left(\frac{F}{F_k}-1\right)^m-UF} . \quad (24)$$

2.5 Contact traction rate-sliding velocity relation

The expression of the sliding velocity in terms of the contact traction rate is given from Eqs. (1), (4), (5), (23), and (24) as follows:

$$\begin{aligned}\mathbf{r} &= (\mathbf{r}_n^e+\mathbf{r}_n^p)+(\mathbf{r}_t^e+\mathbf{r}_t^p)\\ &= -\frac{1}{\alpha_n}\overset{\circ}{\mathbf{f}}_n-\frac{1}{\alpha_t}\overset{\circ}{\mathbf{f}}_t-\lambda\mathbf{t}\\ &= -\frac{1}{\alpha_n}\overset{\circ}{\mathbf{f}}_n-\frac{1}{\alpha_t}\overset{\circ}{\mathbf{f}}_t\\ &\quad -\frac{-\left\{\frac{\partial f}{\partial\|\mathbf{f}_n\|}(\mathbf{n}\bullet\overset{\circ}{\mathbf{f}})+\frac{\partial f}{\partial\|\mathbf{f}_t\|}(\mathbf{t}\bullet\overset{\circ}{\mathbf{f}})\right\}+\eta R\left(1-\frac{F}{F_s}\right)^n}{R\kappa\left(\frac{F}{F_k}-1\right)^m-UF}\mathbf{t}.\end{aligned} \quad (25)$$

On the other hand, the substitution of Eqs. (4), (5), and (23) into Eq. (22) leads to

$$\begin{aligned}&-\alpha_n\frac{\partial f}{\partial\|\mathbf{f}_n\|}\mathbf{n}\bullet(\mathbf{r}_n-\mathbf{r}_n^p)-\alpha_t\frac{\partial f}{\partial\|\mathbf{f}_t\|}\mathbf{t}\bullet(\mathbf{r}_t-\mathbf{r}_t^p)\\ &= U\|\mathbf{r}^p\|\,F+R\left\{-\kappa\left(\frac{F}{F_k}-1\right)^m\|\mathbf{r}^p\|+\eta\left(1-\frac{F}{F_s}\right)^n\right\},\end{aligned}$$

which is further rewritten as

$$\begin{aligned}&-\alpha_n\frac{\partial f}{\partial\|\mathbf{f}_n\|}\mathbf{n}\bullet\mathbf{n}\otimes(\mathbf{n}\bullet\mathbf{r})-\alpha_t\frac{\partial f}{\partial\|\mathbf{f}_t\|}(\mathbf{r}+\lambda\mathbf{t})\\ &= U\lambda F+R\left\{-\kappa\left(\frac{F}{F_k}-1\right)^m\lambda+\eta\left(1-\frac{F}{F_s}\right)^n\right\}\end{aligned} \quad (26)$$

by Eqs. (2), (4), and (23). The proportionality factor λ expressed in terms of the relative velocity, rewritten as Λ, is given by

$$\Lambda = \frac{-\left(\alpha_n\frac{\partial f}{\partial\|\mathbf{f}_n\|}\mathbf{n}+\alpha_t\frac{\partial f}{\partial\|\mathbf{f}_t\|}\mathbf{t}\right)\bullet\mathbf{r}-\eta R\left(1-\frac{F}{F_s}\right)^n}{-R\kappa\left(\frac{F}{F_k}-1\right)^m+UF+\alpha_t\frac{\partial f}{\partial\|\mathbf{f}_t\|}} . \quad (27)$$

The rate of contact traction is given from Eqs. (1)-(5), (23) and (27) as follows:

$$\begin{aligned}\overset{\circ}{\mathbf{f}} &= -\alpha_n(\mathbf{r}_n-\mathbf{r}_n^p)-\alpha_t(\mathbf{r}_t-\mathbf{r}_t^p)\\ &= -\alpha_n(\mathbf{n}\bullet\mathbf{r})\mathbf{n}-\alpha_t\{\mathbf{r}-(\mathbf{n}\bullet\mathbf{r})\mathbf{n}+\Lambda\mathbf{t}\}\\ &= -\alpha_t\mathbf{r}-(\alpha_n-\alpha_t)(\mathbf{n}\bullet\mathbf{n})\mathbf{r}\\ &\quad -\alpha_t\frac{-\left(\alpha_n\frac{\partial f}{\partial\|\mathbf{f}_n\|}\mathbf{n}+\alpha_t\frac{\partial f}{\partial\|\mathbf{f}_t\|}\mathbf{t}\right)\bullet\mathbf{r}-\eta R\left(1-\frac{F}{F_s}\right)^n}{-R\kappa\left(\frac{F}{F_k}-1\right)^m+UF+\alpha_t\frac{\partial f}{\partial\|\mathbf{f}_t\|}}\mathbf{t},\end{aligned}$$

which results in

$$\begin{aligned}\overset{\circ}{\mathbf{f}} = -\Bigg\{&\alpha_t\mathbf{I}+(\alpha_n-\alpha_t)(\mathbf{n}\otimes\mathbf{n})\\ &-\alpha_t\frac{\mathbf{t}\otimes\left(\alpha_n\frac{\partial f}{\partial\|\mathbf{f}_n\|}\mathbf{n}+\alpha_t\frac{\partial f}{\partial\|\mathbf{f}_t\|}\mathbf{t}\right)}{-R\kappa\left(\frac{F}{F_k}-1\right)^m+UF+\alpha_t\frac{\partial f}{\partial\|\mathbf{f}_t\|}}\Bigg\}\mathbf{r}\\ &+\alpha_t\frac{\eta R\left(1-\frac{F}{F_s}\right)^n}{-R\kappa\left(\frac{F}{F_k}-1\right)^m+UF+\alpha_t\frac{\partial f}{\partial\|\mathbf{f}_t\|}}\mathbf{t}.\end{aligned} \quad (28)$$

The loading criterion[22] for the plastic-sliding velocity is given as follows:

$$\left.\begin{aligned}\mathbf{r}^p\neq\mathbf{0}:\Lambda>0,\\ \mathbf{r}^p=\mathbf{0}:\Lambda\leq 0\end{aligned}\right\} \quad (29)$$

due to the requirement of positiveness for the proportionality factor Λ.

3. Concrete constitutive equations

Formulated below are concrete constitutive equations for friction, in which concrete forms of normal-sliding surface are adopted. It can be stated from experiments that the friction coefficient decreases with the increase of contact pressure. Therefore, the sliding-yield surface would not be described appropriately by the Coulomb friction law in which the tangential contact traction and the normal contact traction are linearly related to each other using the angle of friction and the adhesion. In what follows, nonlinear relation of tangential contact traction and normal contact traction is assumed below, by which the reduction of friction coefficient the normal contact traction is described.

We assume first the following quadric surface[22] for the normal-sliding surface:

$$f(\|\mathbf{f}_n\|,\ \|\mathbf{f}_t\|)=\|\mathbf{f}_n\|A, \quad (30)$$

where

$$\eta\equiv\frac{\|\mathbf{f}_t\|}{\|\mathbf{f}_n\|},\quad \chi\equiv\frac{\eta}{M},\quad A\equiv\frac{1}{1-\chi/2}. \quad (31)$$

M is a material constant depending on the frictional property. It holds for Eq. (30) that $\chi<2$ and

$$\frac{\partial f}{\partial\|\mathbf{f}_n\|}=(1-\chi)A^2,\quad \frac{\partial f}{\partial\|\mathbf{f}_t\|}=\frac{A^2}{2M} \quad (32)$$

for which Eq. (28) becomes

$$\overset{\circ}{\mathbf{f}} = -\left[\alpha_t\mathbf{I}+(\alpha_n-\alpha_t)(\mathbf{n}\otimes\mathbf{n}) -\alpha_t\frac{\mathbf{t}\otimes\left\{\alpha_n(1-\chi)A^2\mathbf{n}+\alpha_t\frac{A^2}{2M}\mathbf{t}\right\}}{-R\kappa\left(\frac{F}{F_k}-1\right)^m+UF+\alpha_t\frac{A^2}{2M}}\right]\mathbf{r}$$

$$+\alpha_t\frac{\eta R\left(1-\frac{F}{F_s}\right)^n}{-R\kappa\left(\frac{F}{F_k}-1\right)^m+UF+\alpha_t\frac{1}{2MA^2}}\mathbf{t}. \tag{33}$$

Furthermore, we adopt the following tear-shaped surface for the normal-sliding surface[22] (Fig. 1), which was first proposed for the yield surface of soils by Hashiguchi[37),38)].

$$f(\|\mathbf{f}_n\|,\|\mathbf{f}_t\|)=\|\mathbf{f}_n\|B, \tag{34}$$

where

$$B\equiv\exp\left(\frac{\chi^2}{2}\right). \tag{35}$$

It holds for Eq. (34) that

$$\frac{\partial f}{\partial\|\mathbf{f}_n\|}=(1-\chi^2)B,\quad \frac{\partial f}{\partial\|\mathbf{f}_t\|}=\frac{\chi}{M}B \tag{36}$$

for which Eq. (28) becomes

$$\overset{\circ}{\mathbf{f}}=-\left[\alpha_t\mathbf{I}+(\alpha_n-\alpha_t)(\mathbf{n}\otimes\mathbf{n})-\alpha_t\frac{\mathbf{t}\otimes\left\{\alpha_n(1-\chi^2)B\mathbf{n}+\alpha_t\frac{\chi}{M}B\mathbf{t}\right\}}{-R\kappa\left(\frac{F}{F_k}-1\right)^m+UF+\alpha_t\frac{\chi}{M}B}\right]\mathbf{r}$$
$$+\alpha_t\frac{\eta R\left(1-\frac{F}{F_s}\right)^n}{-R\kappa\left(\frac{F}{F_k}-1\right)^m+UF+\alpha_t\frac{\chi}{M}B}\mathbf{t}. \tag{37}$$

While Coulomb law involves the angle of friction and the cohesion as the material constants, the above-mentioned sliding-yield surface could describe the decrease of friction coefficient with the increase of normal-tangential traction without increasing the number of material parameters (two: M, F). They are identical to material parameters included in the normal-yield surface of soils. That is, the height of friction coefficient is described mainly by M and its continuation with the increase of normal traction is described by F.

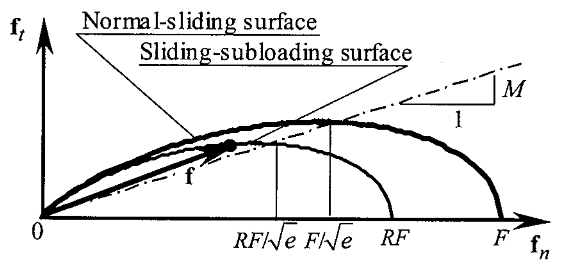

Fig. 1 Tear-shaped normal-sliding and sliding-subloading Surfaces of Eq. (34).

4. Basic response of the present model

4.1 Numerical experiments

Below, we examine the basic response of the present friction model by the numerical experiments of the linear sliding phenomenon without a rigid-body rotation under a constant normal traction as follows:

$$\mathbf{r}_n=\mathbf{0},\ \mathbf{f}_n=\text{const.},\ \mathbf{\Omega}=\mathbf{0} \tag{38}$$

as shown in Fig. 2 in which the sliding velocity of the body A relative to the body B is denoted by $\mathbf{r}$ and the tangential traction acting to the body A from the body B is denoted by $\mathbf{f}$. In this figure the two-dimensional coordinate system (t,n) is adopted, where the axes t and n is directed toward $-\mathbf{t}$ and $\mathbf{n}$ respectively, i.e.

$$\mathbf{t}=\begin{Bmatrix}-1\\0\end{Bmatrix},\quad \mathbf{n}=\begin{Bmatrix}0\\1\end{Bmatrix}. \tag{39}$$

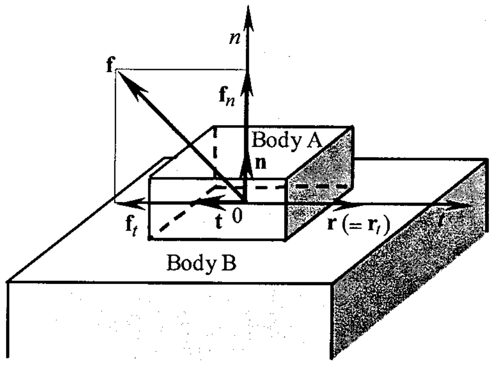

Fig. 2. Sliding velocity **r** of body A relative to body B and traction **f** acting to body A from body B in the coordinate system (t, n).

For the quadric sliding surface (30), it holds that

$$R=\frac{f_n}{F}A, \tag{40}$$

$$df_t=-\alpha_t\left\{1-\alpha_t\frac{\frac{A^2}{2M}}{-\kappa R\left(\frac{F}{F_k}-1\right)^m+UF+\alpha_t\frac{A^2}{2M}}\right\}du_t$$
$$+\alpha_t\frac{\eta R\left(1-\frac{F}{F_s}\right)^n}{-\kappa R\left(\frac{F}{F_k}-1\right)^m+UF+\alpha_t\frac{A^2}{2M}}dt \tag{41}$$

or

$$du_t=-\left\{\frac{1}{\alpha_t}+\frac{\frac{A^2}{2M}}{-\kappa R\left(\frac{F}{F_k}-1\right)^m+UF}\right\}$$
$$\left\{df_t-\alpha_t\frac{\eta R\left(1-\frac{F}{F_s}\right)^n}{-\kappa R\left(\frac{F}{F_k}-1\right)^m+UF+\alpha_t\frac{A^2}{2M}}dt\right\}, \tag{42}$$

where $df_t=\overset{\circ}{f}_t dt$ $(=\dot{f}_t dt)$ and $du_t=r_t dt$ (t: time).

For the tear-shaped sliding surface (34), it holds that

$$R=\frac{f_n}{F}B, \tag{43}$$

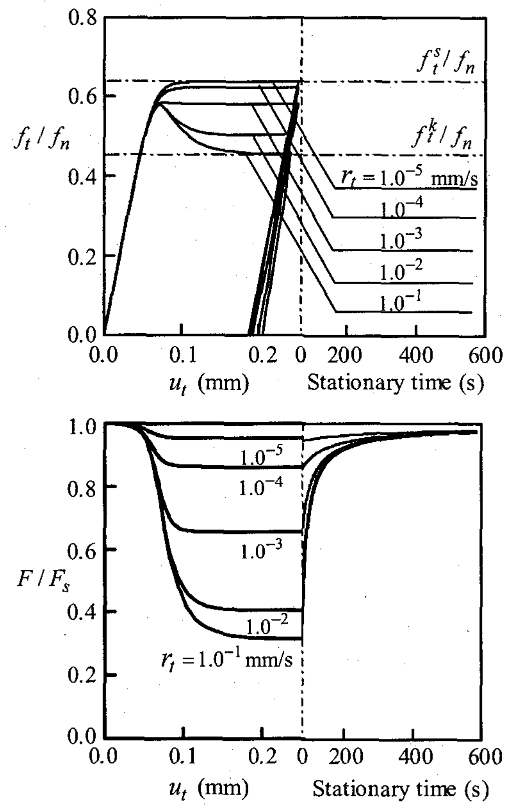

Fig. 3. Variations of friction coefficient and hardening function with tangential sliding distance and stationary time for various tangential sliding velocity (u=100).

$$df_t=-\alpha_t\left\{1-\alpha_t\frac{\frac{\chi}{M}B}{-R\kappa\left(\frac{F}{F_k}-1\right)^m+UF+\alpha_t\frac{\chi}{M}B}\right\}du_t$$

$$+\alpha_t\frac{\eta R\left(1-\frac{F}{F_s}\right)^n}{-R\kappa\left(\frac{F}{F_k}-1\right)^m+UF+\alpha_t\frac{\chi}{M}B}dt \qquad (44)$$

or

$$du_t=-\left\{\frac{1}{\alpha_t}+\frac{\frac{\chi}{M}B}{-R\kappa\left(\frac{F}{F_k}-1\right)^m+UF}\right\}$$

$$\left\{df_t-\alpha_t\frac{\eta R\left(1-\frac{F}{F_s}\right)^n}{-R\kappa\left(\frac{F}{F_k}-1\right)^m+UF+\alpha_t\frac{\chi}{M}B}dt\right\}. \qquad (45)$$

For the numerical experiments the tear-shaped sliding surface (34) is adopted and the material parameters and the normal traction are selected as follows:

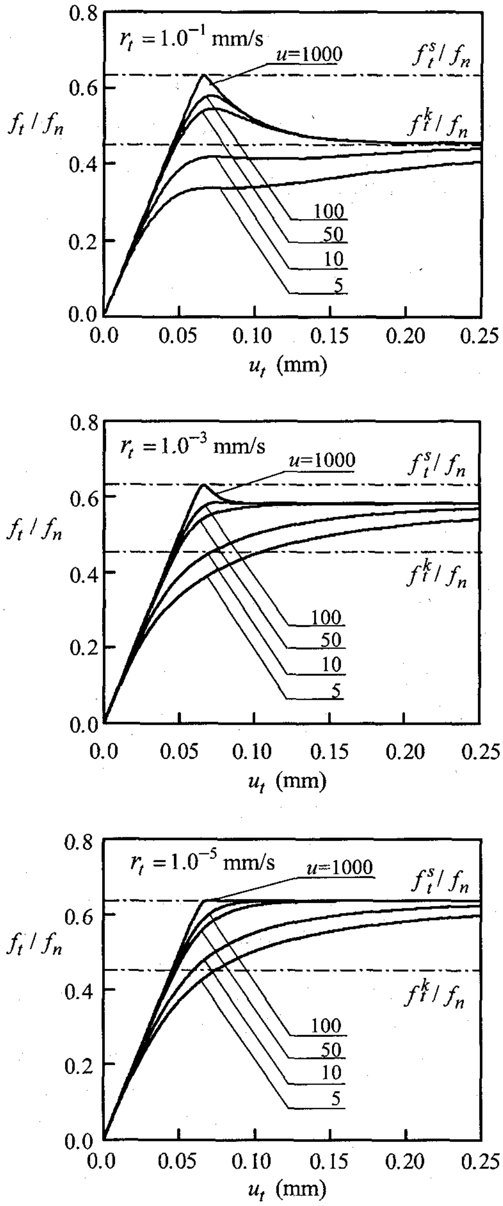

Fig. 4. Influences of the material constant u on the relation of friction coefficient vs. tangential sliding displacement for three levels of tangential sliding rate.

$F_0 = F_s$=100 MPa, F_k=30 MPa,
κ=1000 MPa/mm, m=1, η=10 MPa/s, n=2,
u=5, 10, 50, 100 and 1000 mm^{-1},
α_n=100 MPa/mm, α_t=100 MPa/mm,
f_n=10 MPa,

where F_0 is the initial value of F.

Variations of the contact traction ratio f_t / f_n and the tangential sliding displacement u_t and stationary time for various tangential sliding velocity are shown in Fig. 3, while f_t^s and f_t^k are the values of f_t calculated from the normal-sliding condition choosing $F = F_s$ and F_k, respectively. It is observed that the peak tangential traction, i.e. the static-friction and the lowest stationary tangential traction, i.e. the kinetic-friction decrease with the increase of tangential sliding velocity r_t. The reduction of the hardening function F with the tangential sliding distance and its recovery from the minimum value in the kinetic-friction with the time t elapsed after the complete unloading are also shown in this figure. Besides, the influence of the value of u for the evolution rate of normal-sliding ratio R is shown in Fig. 4, while it is observed that the curve becomes smoother for the smaller vale of u.

The influence of the stationary time on the recovery of friction coefficient is shown in Fig. 5. It is observed that the recovery is larger for a longer stationary time, while almost complete recover is realized for the stationary time 200s.

4.2 Comparison with test data

The validity of the present theory for the prediction of friction behavior of real material is verified comparing with some test data for linear sliding behavior in this section.

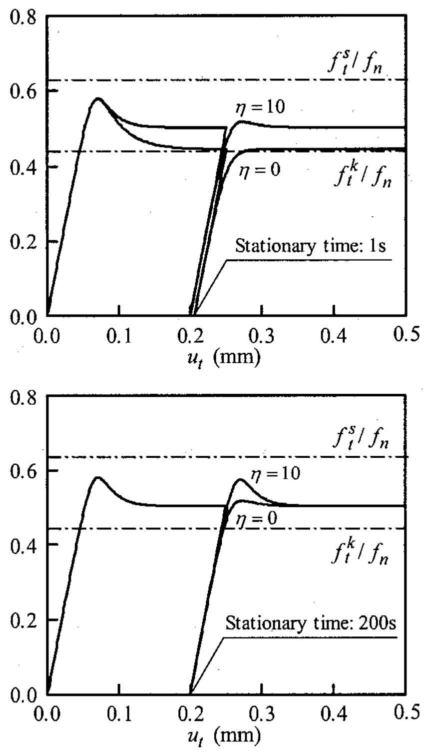

Fig. 5. Relationships between contact traction ratio and tangential relative displacement with stationary contact time ($r_t = 1.0^{-2}$ mm/s , $u = 100$ mm/s).

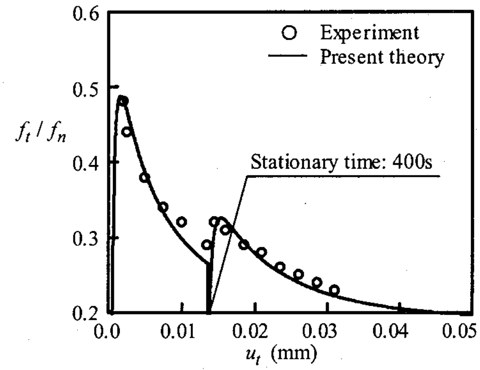

Fig. 6. Recovery of friction coefficient by the stop of sliding in the reduction process from the static- to kinetic-friction under the infinitesimal sliding velocity (Test data after Ferrero and Barrau, 1997).

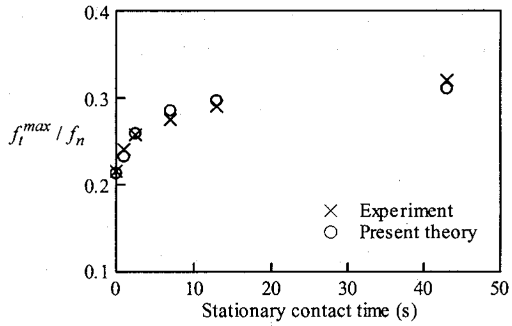

Fig. 7. Recovery process of friction coefficient with the elapsed time after the stop of sliding (Test data after Brockley and Davis, 1968).

Reduction process of friction coefficient from the static- to kinetic-friction and recovery of friction coefficient by the stop of sliding is shown in Fig. 6. The test curve[35] under the infinitesimal sliding velocity $r_t \leq 2\times10^{-4}$ mm/s and the stationary time 400s is simulated sufficiently well by the present model, where the material parameters are selected as follows:

$F_0=F_s=120$ MPa, $F_k=25$ MPa, $M=0.28$,
$\kappa=3000$ MPa/mm, $m=2$, $\eta=0.1$ MPa/s, $n=2$,
$u=1500$ mm^{-1},
$\alpha_n=10$ GPa/mm, $\alpha_t=10$ GPa/mm,
$f_n=10$ MPa, $r_t=2.0\times10^{4}$mm/s.

Recovery process of friction coefficient is shown in Fig.7. The sliding was first given reaching the kinetic-friction and then the tangential contact traction was unloaded to zero. Further, after the stop of sliding for a while the sliding was given again. The relations of the non-dimensional maximum value of frictional coefficient f_t^{max}/f_n vs. the time elapsed after the stop of sliding are plotted in this figure. The test curve[31] is simulated sufficiently well by the present model, where the material parameters are selected as follows:

$F_0 = F_s$=1200 kPa, F_k=200 kPa, M=0.16,
κ=10 MPa/mm, m=2, η=230 kPa/s, n=2,
u= 3000 mm^{-1},
α_n=10 MPa/mm, α_t=10 MPa/mm,
f_n =138 kPa, r_t=1.0^{-1}mm/s (before stop of sliding).

5. Concluding remarks

The time-dependent subloading-friction model is formulated in this article in order to describe the reduction of frictional coefficient from the static- to kinetic-friction and the recovery of frictional coefficient, and the basic characteristics of this model is examined by the numerical experiments of linear sliding phenomenon. It would be capable of describing the transition from the static- to the kinematic-friction and vice versa adequately, which have not been formulated as a constitutive equation up to present. Further, the quantitative predictability of the present model was be verified by comparing with the various basic test results.

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