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Generalized subloading surface model with tangential stress rate effect

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Abstract - The traditional elastoplastic constitutive equation, which is independent of the stress rate component tangential to the yield surface predicts an unrealistically stiff mechanical response for the nonproportional loading process in which the stress rate has a component tangential to the yield surface. In this article, the generalized constitutive equation is then formulated by incorporating the inelastic strain rate due to the stress rate tangential to the subloading surface into the subloading surface model exhibiting a smooth elastic-plastic transition.
Key Words: Constitutive equation; Elastoplaticity; Nonproportional loading; Subloading surface model; Tangential strain rate

1. Introduction

The following facts are generally observed in the inelastic deformation behavior of real materials.

1) The magnitude of the inelastic strain rate depends not only on the component of the stress rate normal to the yield surface, called the *normal stress rate*, but also on the component of the stress rate tangential to the yield surface, called the *tangential stress rate*.

2) The direction of the inelastic strain rate depends not only on the stress but also on the stress rate.

3) Thus, the *non-coaxiality*, i.e. the discordance of the principal axes of the inelastic strain rate and the stress is induced.

However, the traditional elastoplastic constitutive equation, which has a single smooth yield surface and in which the plastic strain rate is derived from the consistency condition with a plastic potential flow rule, is incapable of describing these facts since the plastic strain rate is independent of the tangential stress rate. It then has problems in the analysis of the deformation behavior under the nonproportional loading process with a significant tangential stress rate. The stress path often deviates significantly from that of proportional loading in plastic instability phenomena with the bifurcation of deformation and often with the localization of the deformation; the traditional elastoplastic constitutive equation tends to predict an unrealistically stiff mechanical response leading to an excessively high limit load. Consequently, an extended constitutive equation needs to be formulated, in which the above-mentioned facts 1)-3) and the following facts are also taken into account.

4) It was evidenced by Rudnicki and Rice[1] that "no vertex can result from hydrostatic stress increments", based on consideration of the sliding mechanism in a fissure model. Thus, it might be assumed that only the deviatoric part of the tangential stress rate, called the *deviatoric-tangential stress rate*, influences the inelastic deformation behavior.

5) The direction of the *tangential strain rate* induced by the deviatoric-tangential stress rate has components not only tangential but also outward-normal to the yield surface, as has been found in various experimental and theoretical studies: test data of metals[2] and soils[3)-7)] numerical experiments for metals based on the *KBW model*[8),9)] by Ito[10)] and the *Taylor polycrystalline*

model[11)-13)] by Kuroda and Tvergaard[14)] and numerical experiments for granular media based on the *discrete element method* by Bardet et al.[15),16)], Kishino and Wu[17),18)].

6) The tangential strain rate would also cause the hardening of yield surface by the outward-normal component to the yield surface as well as the plastic strain rate.

A brief overview of the existing constitutive models extended to overcome the above-mentioned limitations of the traditional constitutive equation is given below.

Extensions of the associated flow rule incorporating the directional tensor of stress rate and that of strain rate have been proposed by Mroz[19)], Dafalias and Popov[20)], Dafalias[21)], Wang *et al.*[22)], Hashiguchi[23)], etc. and by Hill[24)], Hashiguchi[25)], Kuroda and Tvergaard[14),26)], Kuroda[27)], etc., respectively. However, in these models the plastic strain rate is induced only by the normal stress rate since the positive proportionality factors in their flow rules are derived from the consistency condition of the yield surface as is done in the traditional plastic constitutive formulation.

Approaches that more drastically modify the traditional elastoplasticity, i.e. the intersection of multiple yield surfaces[28)-34)] and the corner theory[35)-42)] have also been proposed.

Approaches using the *intersection of multiple yield surfaces* have difficulty in describing the mutual influences between the hardenings of various yield surfaces, i.e. latent hardening. While a practical calculation of deformation using this method has been performed by Sewell[33),34)], it was restricted to base states of uniaxial stress. Experimental measurements of hardening moduli have proven difficult, as described by Storen and Rice[43)], and a computational practical extension of this model to general stress states is not obvious, as was indicated by Christoffersen and Hutchinson[36)].

The *corner theory* assumes the existence of a corner (vertex, cone) on the yield surface, inducing a geometrical singularity in the field of the outward-normal vectors to the yield surface. However, an evolution rule for the cone due to plastic deformation has not so far been given. Perhaps it cannot be formulated rationally, especially if the stress rate is directed outward from the yield surface but more than 90° from the outward central axis of the cone, whilst the cone has to contract. Thus, this model is not applicable to the general loading process including unloading, reloading and reverse loading but applicable only to monotonic loading near the proportional loading process.

Models incorporating the tangential strain rate into the traditional elastoplastic constitutive equation have been proposed by Rudnicki and Rice[1)], Papamichos *et al.*[44)] and Hashiguchi[45)] or Hashiguchi and Tsutsumi[46)]. These models have been widely applied to the analysis of plastic instability phenomena by various researchers[43),46)-59)]. However, the models of Rudnicki and Rice[1)] and Papamichos *et al.*[44)] are applicable only to materials having a yield surface with a circular π-section. Further, they predict the tangential strain rate only at the moment when the stress reaches the yield surface and thus violates not only the smoothness but also the continuity conditions[23),25),60),61)], since they are based on the *conventional elastoplasticity*[62)] which assumes the interior of the yield surface to be an elastic domain. Therefore, the deformation predicted by them for the stress path along the yield surface does not fulfill the uniqueness of solution, which leads to the serious defect in the analysis of boundary value problems. Only the model of Hashiguchi[45),46)] based on the *subloading surface model*[63)-66)] falling within the unconventional plasticity exhibiting a smooth elastic-plastic transition, fulfills the continuity and smoothness conditions. The tangential strain rates in all the above-mentioned models formulated in this approach is directed merely towards the tangent to the yield/subloading surface, ignoring the fact 5). Further, the hardening due to the tangential strain rate is not taken account into the evolution of yield condition in these models.

In this article, the generalized constitutive equation, referred to as the *tangential-subloading surface model,* is formulated by incorporating the tangential strain rate into the subloading surface model *model*[63)-66)] so as to fulfill all the aforementioned facts 1)-6), while the facts 5) and 6) have not been taken into account in the previous formulation[45),46)].

2. Constitutive Equation

The subloading surface model[63)-66)] will be extended so as to incorporate the tangential srain rate and the hardening due to it in this section.

Denoting the current configuration of the material particle as $\mathbf{x}$ and the current velocity as $\mathbf{v}$, the velocity gradient is described as $\mathbf{L} = \partial \mathbf{v}/\partial \mathbf{x}$ from which the strain rate and the continuum spin are defined as $\mathbf{D} \equiv (\mathbf{L}+\mathbf{L}^{\mathrm{T}})/2$ and $\mathbf{W} \equiv (\mathbf{L}-\mathbf{L}^{\mathrm{T}})/2$, respectively, $(\ )^{\mathrm{T}}$ standing for the transpose. Let the strain rate $\mathbf{D}$ be additively decomposed into the elastic strain rate $\mathbf{D}^e$ and the inelastic strain rate $\mathbf{D}^i$, while the latter is further additively decomposed into the plastic strain rate $\mathbf{D}^p$ and the tangential strain rate $\mathbf{D}^t$, i.e.

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^i, \quad \mathbf{D}^i = \mathbf{D}^p + \mathbf{D}^t, \tag{1}$$

where $\mathbf{D}^p$ and $\mathbf{D}^t$ are induced by the stress rate components normal and tangential, respectively, to the subloading surface. Let $\mathbf{D}^e$ be related linearly to the stress rate as

$$\mathbf{D}^e = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}}, \tag{2}$$

$\boldsymbol{\sigma}$ is the Cauchy stress, $(^\circ)$ denoting the proper corotational rate with the objectivity, and the fourth-order tensor $\mathbf{E}$ is the elastic modulus. Here, limiting to an infinitesimal deformation but avoiding the influence of rigid-body rotation on the constitutive relation, the following *Zaremba-Jaumann rate* is adopted for the corotational rate.

$$\overset{\circ}{\mathbf{A}} \equiv \dot{\mathbf{A}} - \mathbf{WA} + \mathbf{AW}, \tag{3}$$

where $\mathbf{A}$ is an arbitrary second-order tensor, $(^\bullet)$ denoting the material-time derivative.

2.1 Plastic strain rate

Let the following yield condition be assumed.

$$f(\hat{\boldsymbol{\sigma}}, \mathbf{H}) = F(H), \tag{4}$$

where

$$\hat{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \boldsymbol{\alpha}. \tag{5}$$

The scalar H is the isotropic hardening variable inducing the expansion/contraction of yield surface, the second-order tensor $\mathbf{H}$ is the anisotropic hardening variable inducing the rotation of yield surface in soils for example and the second-order tensor $\boldsymbol{\alpha}$ is the kinematic hardening variable, i.e. the back stress. The function f of the tensor $\hat{\boldsymbol{\sigma}}$ is assumed to be homogeneous of degree one of $\hat{\boldsymbol{\sigma}}$ satisfying $f(s\hat{\boldsymbol{\sigma}}, \mathbf{H}) = sf(\hat{\boldsymbol{\sigma}}, \mathbf{H})$ for any nonnegative scalar s, and thus the yield surface keeps a similar shape for $\mathbf{H}=\mathbf{0}$. The evolution of internal structure of materials would be caused by the inelastic strain rate $\mathbf{D}^i$. Then, let it be assumed that the rates of internal variables H, $\mathbf{H}$ and $\boldsymbol{\alpha}$ are linear functions of $\mathbf{D}^i$, i.e.

$$\left.\begin{aligned} \dot{H} &= \mathrm{tr}\{\mathbf{f}_H(\boldsymbol{\sigma}, H, \mathbf{H}, \boldsymbol{\alpha})\,\mathbf{D}^i\}, \\ \overset{\circ}{\mathbf{H}} &= \mathbf{f}_{\mathbf{H}}(\boldsymbol{\sigma}, H, \mathbf{H}, \boldsymbol{\alpha})\,\mathbf{D}^i, \\ \overset{\circ}{\boldsymbol{\alpha}} &= \mathbf{f}_{\alpha}(\boldsymbol{\sigma}, H, \mathbf{H}, \boldsymbol{\alpha})\,\mathbf{D}^i, \end{aligned}\right\} \tag{6}$$

where $\mathbf{f}_H$ is the second-order tensor function, and $\mathbf{f}_{\mathbf{H}}$ and $\mathbf{f}_\alpha$ are the fourth-order tensor functions of $\boldsymbol{\sigma}, H, \mathbf{H}$ and $\boldsymbol{\alpha}$, $\mathrm{tr}(\,)$ denoting the trace. $\dot{H}$, $\overset{\circ}{\mathbf{H}}$ and $\overset{\circ}{\boldsymbol{\alpha}}$ are additively decomposed into the plastic and the tangential parts by Eq.(1), viz.,

$$\left.\begin{aligned} \dot{H} &= \dot{H}^p + \dot{H}^t, \\ \overset{\circ}{\mathbf{H}} &= \overset{\circ}{\mathbf{H}}{}^p + \overset{\circ}{\mathbf{H}}{}^t, \\ \overset{\circ}{\boldsymbol{\alpha}} &= \overset{\circ}{\boldsymbol{\alpha}}{}^p + \overset{\circ}{\boldsymbol{\alpha}}{}^t, \end{aligned}\right\} \tag{7}$$

where

$$\left.\begin{aligned} \dot{H}^p &\equiv \mathrm{tr}\{\mathbf{f}_H(\boldsymbol{\sigma}, H, \mathbf{H}, \boldsymbol{\alpha})\mathbf{D}^p\}, \\ \dot{H}^t &\equiv \mathrm{tr}\{\mathbf{f}_H(\boldsymbol{\sigma}, H, \mathbf{H}, \boldsymbol{\alpha})\,\mathbf{D}^t\}, \\ \overset{\circ}{\mathbf{H}}{}^p &= \mathbf{f}_{\mathbf{H}}(\boldsymbol{\sigma}, H, \mathbf{H}, \boldsymbol{\alpha})\,\mathbf{D}^p, \\ \overset{\circ}{\mathbf{H}}{}^t &= \mathbf{f}_{\mathbf{H}}(\boldsymbol{\sigma}, H, \mathbf{H}, \boldsymbol{\alpha})\,\mathbf{D}^t, \\ \overset{\circ}{\boldsymbol{\alpha}}{}^p &\equiv \mathbf{f}_{\alpha}(\boldsymbol{\sigma}, H, \mathbf{H}, \boldsymbol{\alpha})\,\mathbf{D}^p, \\ \overset{\circ}{\boldsymbol{\alpha}}{}^t &\equiv \mathbf{f}_{\alpha}(\boldsymbol{\sigma}, H, \mathbf{H}, \boldsymbol{\alpha})\,\mathbf{D}^t. \end{aligned}\right\} \tag{8}$$

noting the time-independence and the fact that these rate variables are not zero only when the inelastic strain rate is not zero.

In what follows let the conventional elastoplastic constitutive model[62] with the yield surface enclosing a purely elastic domain be extended to the unconventional elastoplastic constitutive model describing the plastic strain rate due to the rate of stress inside the yield surface. Here, it could be assumed that

a) A plastic strain rate develops gradually as the stress approaches the yield surface.

b) A conventional elastoplastic constitutive equation holds when the stress lies on the yield surface.

In order to formulate an unconventional elastoplastic constitutive equation realizing these assumptions it is required to adopt the relevant measure expressing how near the stress approaches the yield surface. Then, let the following surface, called the *subloading surface*, be introduced.

1. It passes always the current stress point.
2. It has the similar shape and same orientation as the normal-yield surface. Thus, the subloading surface coincides completely with the normal-yield surface when the stress reaches the normal-yield surface.
3. The similarity-center $\mathbf{s}$ of theses surfaces moves

with a plastic deformation,

whilst the yield surface in the conventional elastoplasticity is renamed the *normal-yield surface*.

Then, let the ratio of the size of subloading surface to that of normal-yield surface, called the *normal-yield ratio* and denoted by the notation $R\,(0 \le R \le 1)$, be introduced as the three dimensional measure for the approaching degree of stress to the normal-yield surface.

Now, it holds that

$$\boldsymbol{\sigma}_y = \frac{1}{R}\{\boldsymbol{\sigma}-(1-R)\mathbf{s}\} \quad (\boldsymbol{\sigma}-\mathbf{s}=R(\boldsymbol{\sigma}_y-\mathbf{s})), \tag{9}$$

where $\boldsymbol{\sigma}_y$ on the normal yield surface is the *conjugate stress* of the current stress $\boldsymbol{\sigma}$ on the subloading surface.

It should be noted that the subloading surface coincides completely with the normal-yield surface when the stress reaches the normal-yield surface, i.e. R=1. On the other hand, the subyield surface(s) in the multi[67] and the two surface models[68] never coincide with the outmost or the bounding surface but contact to them at a point. These geometrical properties bring about the singularity of the elastoplastic moduli leading to the discontinuous response in the multi and the two surface models.

By substituting Eq. (9) into Eq. (4) (regarding $\boldsymbol{\sigma}$ in Eq. (4) as $\boldsymbol{\sigma}_y$), the subloading surface (see Fig. 1) is described as

$$f(\bar{\boldsymbol{\sigma}},\mathbf{H}) = RF(H), \tag{10}$$

where

$$\bar{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \bar{\boldsymbol{\alpha}} \ (=R\hat{\boldsymbol{\sigma}}_y), \tag{11}$$

$$\hat{\boldsymbol{\sigma}}_y \equiv \boldsymbol{\sigma}_y - \boldsymbol{\alpha}, \tag{12}$$

$$\bar{\boldsymbol{\alpha}} \equiv R\boldsymbol{\alpha} + (1-R)\mathbf{s} \quad (\bar{\boldsymbol{\alpha}} - \mathbf{s} = R(\boldsymbol{\alpha} - \mathbf{s})). \tag{13}$$

$\bar{\boldsymbol{\alpha}}$ in the subloading surface is the conjugate point of $\boldsymbol{\alpha}$ in the normal yield surface. In the calculation, R has to be calculated first by substituting current values of $\boldsymbol{\sigma}, H, \boldsymbol{\alpha}, \mathbf{H}, \mathbf{s}$ into Eq. (10), and thereafter $\bar{\boldsymbol{\alpha}}$ is calculated using Eq. (13).

The material-time derivative of Eq. (10) is given by

$$\mathrm{tr}\Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial \boldsymbol{\sigma}}\overset{\circ}{\boldsymbol{\sigma}}\Big) - \mathrm{tr}\Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial \boldsymbol{\sigma}}\overset{\circ}{\bar{\boldsymbol{\alpha}}}\Big) + \mathrm{tr}\Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial \mathbf{H}}\overset{\circ}{\mathbf{H}}\Big) = \dot{R}F + RF'\dot{H}, \tag{14}$$

where

$$F' \equiv \frac{dF}{dH}. \tag{15}$$

Eq. (14) as it is cannot play the role of the consistency condition for the derivation of plastic strain rate since it contains rate variable $\dot{R}$ which is not explicitly related to the plastic strain rate. In order to embody Eq. (14) as the consistency condition let the evolution rule of R, i.e. $\dot{R}$ be formulated so as to fulfill the following conditions.

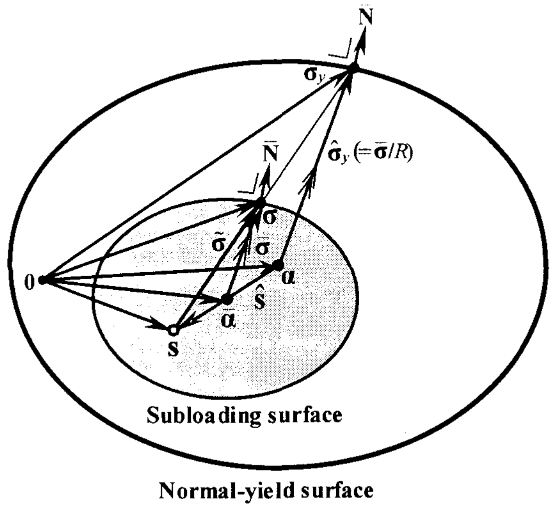

Fig. 1. Normal-yield and subloading surfaces.

1) R increases infinitely for the inelastic strain rate component outward-normal to the subloading surface when the surface contracts to a point coinciding with the center of similarity-center $\mathbf{s}$, i.e. $\dot{R}/\mathrm{tr}(\mathbf{N}\mathbf{D}^i) = \infty$ for R=0, where the second-order tensor $\mathbf{N}$ denotes the normalized outward-normal to the subloading surface, i.e.

$$\mathbf{N} \equiv \frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial \boldsymbol{\sigma}} \Big/ \Big\|\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial \boldsymbol{\sigma}}\Big\| \quad (\|\mathbf{N}\|=1), \tag{16}$$

$\|\ \|$ denoting the magnitude.

2) R increases with the inelastic strain rate component outward-normal to the subloading surface, i.e. $\dot{R} > 0$ for $\mathrm{tr}(\mathbf{N}\mathbf{D}^i) > 0$.

3) The subloading surface does not expand over the normal-yield surface, i.e. $\dot{R} = 0$ for R=1 and $\dot{R} < 0$ for R>1.

Thus, the following evolution equation of the normal-yield ratio R is assumed.

$$\dot{R} = U(R)\,\mathrm{tr}(\mathbf{N}\mathbf{D}^i) \ \text{ for } \mathbf{D}^i \neq \mathbf{0}, \tag{17}$$

where U is a monotonically decreasing function of R, fulfilling the following conditions.

$$U(R) = \begin{cases} \infty & \text{for } R = 0, \\ 0 & \text{for } R = 1, \\ <0 & \text{for } R > 1. \end{cases} \tag{18}$$

Let the function U satisfying Eq. (18) be simply given by

$$U = -u \ln R, \tag{19}$$

where u is a material constant.

The similarity-center $\mathbf{s}$ must lie inside the normal-yield surface since the subloading surface plays the role of loading and plastic potential surfaces and thus is not allowed to intersect the normal-yield surface. Then, the following inequality must be fulfilled.

$$f(\hat{\mathbf{s}},\ \mathbf{H}) \le F(H), \tag{20}$$

where

$$\hat{\mathbf{s}} \equiv \mathbf{s} - \boldsymbol{\alpha}\cdot \tag{21}$$

The time-differentiation of Eq. (20) in the ultimate state $f(\hat{\mathbf{s}},\ \mathbf{H}) = F(H)$, where $\mathbf{s}$ lies just on the normal-yield surface, leads to:

$$\mathrm{tr}\Big[\frac{\partial f(\hat{\mathbf{s}},\ \mathbf{H})}{\partial \mathbf{s}}\Big(\overset{\circ}{\mathbf{s}} - \overset{\circ}{\boldsymbol{\alpha}} + \frac{1}{F}\{\mathrm{tr}\big(\frac{\partial f(\hat{\mathbf{s}},\ \mathbf{H})}{\partial \mathbf{H}}\overset{\circ}{\mathbf{H}}\big) - \dot{F}\}\hat{\mathbf{s}}\Big)\Big] \le 0 \quad \text{for } f(\hat{\mathbf{s}},\ \mathbf{H}) = F(H), \tag{22}$$

while the relation $\mathrm{tr}[\{\partial f(\hat{\mathbf{s}},\ \mathbf{H})/\partial\mathbf{s}\}\hat{\mathbf{s}}] = F$ due to Euler's theorem for homogeneous functions is used to derive Eq. (22). The inequality (20) or (22) is called the *enclosing condition for the similarity-center.* In the ultimate state $f(\hat{\mathbf{s}},\ \mathbf{H}) = F(H)$, the vector $\boldsymbol{\sigma}_y - \mathbf{s}\ (=(\boldsymbol{\sigma}-\mathbf{s})/R)$ makes an obtuse angle with vector $\partial f(\hat{\mathbf{s}},\ \mathbf{H})/\partial\mathbf{s}$ which is the outward-normal to the similarity-center surface $f(\hat{\mathbf{s}},\ \mathbf{H}) = F(H)$ coinciding with the normal-yield surface, on the premise that the normal-yield surface is convex. Noting this fact and considering the fact that the similarity-center moves only with the plastic deformation, let the following equation be assumed to fulfill the inequality (22):

$$\overset{\circ}{\mathbf{s}} - \overset{\circ}{\boldsymbol{\alpha}} + \frac{1}{F}\{\mathrm{tr}(\frac{\partial f(\hat{\mathbf{s}},\ \mathbf{H})}{\partial \mathbf{H}}\overset{\circ}{\mathbf{H}}) - \dot{F}\}\hat{\mathbf{s}} = c\,\mathrm{tr}(\mathbf{N}\mathbf{D}^i)\frac{\tilde{\boldsymbol{\sigma}}}{R}$$
$$(= c\,\mathrm{tr}(\mathbf{N}\mathbf{D}^i)(\boldsymbol{\sigma}_y - \mathbf{s})) \tag{23}$$

from which the *translation rule of similarity-center* is given as follows:

$$\overset{\circ}{\mathbf{s}} = c\,\mathrm{tr}(\mathbf{N}\mathbf{D}^i)\frac{\tilde{\boldsymbol{\sigma}}}{R} + \overset{\circ}{\boldsymbol{\alpha}} + \frac{1}{F}\{F'\dot{H} - \mathrm{tr}(\frac{\partial f(\hat{\mathbf{s}},\ \mathbf{H})}{\partial \mathbf{H}}\overset{\circ}{\mathbf{H}})\}\hat{\mathbf{s}}, \tag{24}$$

where c is a material constant influencing the translating rate of the similarity-center and

$$\tilde{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \mathbf{s}. \tag{25}$$

It would be conceivable that the similarity-center $\mathbf{s}$ approaches the current stress $\boldsymbol{\sigma}$ as can be seen from the simple case of the nonhardening state ($\dot{H} = 0$, $\overset{\circ}{\boldsymbol{\alpha}} = \overset{\circ}{\mathbf{H}} = \mathbf{0}$) leading to $\overset{\circ}{\mathbf{s}} = c\,\mathrm{tr}(\mathbf{N}\mathbf{D}^i)(\boldsymbol{\sigma}-\mathbf{s})/R$, although the translation rule (24) is derived so as to fulfill the requirement (22) in the ultimate state $f(\hat{\mathbf{s}},\ \mathbf{H}) = F(H)$.

Substitution of Eqs. (13), (17) and (24) into Eq. (14) leads to the *consistency condition* for the subloading surface:

$$\mathrm{tr}(\frac{\partial f(\bar{\boldsymbol{\sigma}},\ \mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\boldsymbol{\sigma}}) - \mathrm{tr}(\frac{\partial f(\bar{\boldsymbol{\sigma}},\ \mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\bar{\boldsymbol{\alpha}}}) + \mathrm{tr}(\frac{\partial f(\bar{\boldsymbol{\sigma}},\ \mathbf{H})}{\partial\mathbf{H}}\mathbf{f}_{\mathrm{H}}\mathbf{D}^i)$$
$$+\mathrm{tr}(\frac{\partial f(\bar{\boldsymbol{\sigma}},\ \mathbf{H})}{\partial\mathbf{H}}\mathbf{f}_{\mathrm{H}}\mathbf{D}^i) = U\,\mathrm{tr}(\mathbf{N}\mathbf{D}^i)F + RF'\,\mathrm{tr}(\mathbf{f}_H\mathbf{D}^i) \tag{26}$$

with

$$\overset{\circ}{\bar{\boldsymbol{\alpha}}} = R\mathbf{f}_\alpha\mathbf{D}^i - \hat{\mathbf{s}}\,U\mathrm{tr}(\mathbf{N}\mathbf{D}^i) + (1-R)\Big[c\,\mathrm{tr}(\mathbf{N}\mathbf{D}^i)\frac{\tilde{\boldsymbol{\sigma}}}{R} + \mathbf{f}_\alpha\mathbf{D}^i$$
$$+\frac{1}{F}\{F'\,\mathrm{tr}(\mathbf{f}_H\mathbf{D}^i) - \mathrm{tr}(\frac{\partial f(\hat{\mathbf{s}},\ \mathbf{H})}{\partial\mathbf{H}}\mathbf{f}_{\mathrm{H}}\mathbf{D}^i)\}\hat{\mathbf{s}}\Big] \tag{27}$$

Adopt the associated flow rule

$$\mathbf{D}^p = \lambda\mathbf{N}, \tag{28}$$

where λ is a positive proportionality factor.

Let the evolution rules of the state variables be given by the following forms as usual.

$$\left.\begin{aligned}
\dot{H}^p &= \mathrm{tr}\{(a\mathbf{I} + b\mathbf{n}^*)\mathbf{D}^p\} = a\mathrm{tr}\mathbf{D}^p + b\,\|\mathbf{D}^{p*}\| \\
&= \lambda(a\mathrm{tr}\mathbf{N} + b\|\mathbf{N}^*\|) = \lambda h, \\
\overset{\circ}{\mathbf{H}}^p &= \mathbf{h}\otimes\frac{\mathbf{n}^*}{\|\mathbf{N}^*\|}\mathbf{D}^p = \mathbf{h}\frac{\|\mathbf{D}^{p*}\|}{\|\mathbf{N}^*\|} = \lambda\mathbf{h}, \\
\overset{\circ}{\boldsymbol{\alpha}}^p &= \mathbf{a}\otimes\frac{\mathbf{n}^*}{\|\mathbf{N}^*\|}\mathbf{D}^p = \mathbf{a}\frac{\|\mathbf{D}^{p*}\|}{\|\mathbf{N}^*\|} = \lambda\mathbf{a},
\end{aligned}\right\} \tag{29}$$

( )* denoting the deviatoric part, and thus one has

$$\left.\begin{aligned}
&\mathbf{f}_H = a\mathbf{I} + b\mathbf{n}^*,\ h \equiv \mathrm{tr}(\mathbf{f}_H\mathbf{N}) = a\mathrm{tr}\mathbf{N} + b\|\mathbf{N}^*\|, \\
&\mathbf{f}_{\mathrm{H}} = \mathbf{h}\otimes\frac{\mathbf{n}^*}{\|\mathbf{N}^*\|}, \\
&\mathbf{f}_\alpha = \mathbf{a}\otimes\frac{\mathbf{n}^*}{\|\mathbf{N}^*\|},
\end{aligned}\right\} \tag{30}$$

where a and h are the scalar functions and $\mathbf{h}$ and $\mathbf{a}$ are the second-order tensor functions of stress and internal variables and $\mathbf{N}$ in homogenous degree one and

$$\mathbf{n}^* \equiv (\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}})^* \Big/ \Big\|(\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}})^*\Big\| = \frac{\mathbf{N}^*}{\|\mathbf{N}^*\|}\quad (\|\mathbf{n}^*\| = 1) \tag{31}$$

The substitution of Eq. (28) with Eq. $(1)_2$ into Eq. (26) leads to

$$\mathrm{tr}(\frac{\partial f(\bar{\boldsymbol{\sigma}},\ \mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\boldsymbol{\sigma}}) - \mathrm{tr}(\frac{\partial f(\bar{\boldsymbol{\sigma}},\ \mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\bar{\boldsymbol{\alpha}}})$$
$$+\mathrm{tr}\{\frac{\partial f(\bar{\boldsymbol{\sigma}},\ \mathbf{H})}{\partial\mathbf{H}}\mathbf{f}_{\mathrm{H}}(\lambda\mathbf{N} + \mathbf{D}^i)\}$$
$$= U\{\lambda + \mathrm{tr}(\mathbf{N}\mathbf{D}^t)\}F + RF'\,\mathrm{tr}\{\mathbf{f}_H(\lambda\mathbf{N} + \mathbf{D}^t)\} \tag{32}$$

with

$$\overset{\circ}{\tilde{\boldsymbol{\alpha}}} = \mathbf{f}_{\alpha}(\lambda\mathbf{N}+\mathbf{D}^{t}) + c\Big(\frac{1}{R}-1\Big)\{\lambda+\mathrm{tr}(\mathbf{N}\mathbf{D}^{t})\}\tilde{\boldsymbol{\sigma}}$$
$$+\Big[\frac{1-R}{F}\Big\{F'\,\mathrm{tr}\,\{\mathbf{f}_{H}(\lambda\mathbf{N}+\mathbf{D}^{t})\}$$
$$-\mathrm{tr}\Big(\frac{\partial f(\hat{\mathbf{s}},\,\mathbf{H})}{\partial\mathbf{H}}\mathbf{f}_{\mathrm{H}}(\lambda\mathbf{N}+\mathbf{D}^{t})\Big)\Big\} - U\{\lambda+\mathrm{tr}(\mathbf{N}\mathbf{D}^{t})\}\Big]\hat{\mathbf{s}} \tag{33}$$

which is obtained from Eqs. $(1)_2$, (28) and (27).

The proportionality factor λ is derived from Eq. (32) as follows:

$$\lambda = \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}}) - \tilde{D}^{t}}{M^{p}}, \tag{34}$$

where

$$\tilde{D}^{t} = \mathrm{tr}\,(\mathbf{N}\mathbf{P}\mathbf{D}^{t}), \tag{35}$$

$$M^{p} \equiv \mathrm{tr}\,(\mathbf{N}\mathbf{P}\mathbf{N}) \tag{36}$$

by putting

$$\mathbf{P} \equiv \bar{\boldsymbol{\sigma}} \otimes \Big\{\frac{F'}{F}\mathbf{f}_{H} + \frac{U}{R}\mathbf{N} - \frac{1}{RF}\frac{\partial f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\partial\mathbf{H}}\mathbf{f}_{\mathrm{H}}\Big\} + \mathbf{f}_{\alpha}$$
$$+c\Big(\frac{1}{R}-1\Big)\tilde{\boldsymbol{\sigma}}\otimes\mathbf{N} + \hat{\mathbf{s}}\otimes\Big\{(1-R)\frac{F'}{F}\mathbf{f}_{H} - U\mathbf{N}$$
$$-\frac{1-R}{F}\frac{\partial f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\partial\mathbf{H}}\mathbf{f}_{\mathrm{H}}\Big\}. \tag{37}$$

The following relationship due to the Euler's theorem for a homogeneous function is used for deriving Eq. (34) with Eqs. (35) and (36).

$$\frac{\partial f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\partial\boldsymbol{\sigma}} = \frac{\partial f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\partial\bar{\boldsymbol{\sigma}}} = \frac{\mathrm{tr}\Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\partial\bar{\boldsymbol{\sigma}}}\bar{\boldsymbol{\sigma}}\Big)}{\mathrm{tr}(\mathbf{N}\bar{\boldsymbol{\sigma}})}\mathbf{N}$$
$$= \frac{f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\mathrm{tr}(\mathbf{N}\bar{\boldsymbol{\sigma}})}\mathbf{N} = \frac{RF}{\mathrm{tr}(\mathbf{N}\bar{\boldsymbol{\sigma}})}\mathbf{N}, \tag{38}$$

noting $\partial\bar{\boldsymbol{\sigma}}/\partial\boldsymbol{\sigma} = \mathbf{I}$, where $\mathbf{I}$ is the second-order identity tensor having the components of Kronecker's delta δ_{ij} fulfilling $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$.

2.2 *Tangential strain rate*

Noting the fact 5) that the tangential strain rate has the direction not only tangential but also outward-normal to the subloading surface as described in the introduction, let the tangential strain rate be assumed as follows:

$$\mathbf{D}^{t} = \frac{1}{M^{t}}(\overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{t} + d_{n}\|\overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{t}\|\mathbf{n}^{*}), \tag{39}$$

where $\overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{t}$ is the *deviatoric-tangential stress rate* given as follows:

$$\overset{\circ}{\boldsymbol{\sigma}}{}^{*} = \overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{n} + \overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{t}, \tag{40}$$

where the following notations are used.

$$\left.\begin{aligned}\mathbf{A}^{*}_{n} &\equiv \hat{\mathbf{n}}^{*}\mathbf{A} = \mathrm{tr}(\mathbf{n}^{*}\mathbf{A})\mathbf{n}^{*},\\ \mathbf{A}^{*}_{t} &\equiv \hat{\mathbf{I}}^{*}\mathbf{A} = \mathbf{A}^{*} - \mathbf{A}^{*}_{n}\end{aligned}\right\} \tag{41}$$

for an arbitrary second-order tensor $\mathbf{A}$ with the notations

$$\bar{I}_{ijkl} \equiv \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{42}$$

$$\bar{\mathbf{I}}^{*} \equiv \bar{\mathbf{I}} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}, \tag{43}$$

$$\hat{\mathbf{n}}^{*} \equiv \mathbf{n}^{*}\otimes\mathbf{n}^{*}, \tag{44}$$

$$\hat{\mathbf{I}}^{*} \equiv \bar{\mathbf{I}}^{*} - \hat{\mathbf{n}}^{*}. \tag{45}$$

$\bar{\mathbf{I}}$ is the *fourth-order identity tensor* and $\bar{\mathbf{I}}^{*}$ might be called the *fourth-order deviatoric transformation tensor* leading to $\bar{\mathbf{I}}^{*}\mathbf{A} = \mathbf{A}^{*}$. Further, $\hat{\mathbf{n}}^{*}$ and $\hat{\mathbf{I}}^{*}$ might be called the *fourth-order deviatoric-normal* and *-tangential transformation tensors*, respectively. The material function M^{t}, called the *tangential-inelastic modulus*, is a monotonically decreasing function of R and is given simply by

$$M^{t} = \frac{1}{\xi R^{n}}, \tag{46}$$

where n is a material constant and ξ is a material function of stress and plastic internal variables in general. Besides, d_n is a material constant.

2.3 *Elastoplastic-tangential constitutive equation*

The strain rate is expressed in terms of the stress rate from Eqs.(1), (28), (34) and (39) as

$$\mathbf{D} = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} + \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}}) - \frac{d_{n}}{M^{t}}\mathrm{tr}\,(\mathbf{N}\mathbf{P}\mathbf{n}^{*})\|\overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{t}\|}{M^{p}}\mathbf{N}$$
$$+\frac{1}{M^{t}}(\overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{t} + d_{n}\|\overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{t}\|\mathbf{n}^{*}), \tag{47}$$

noting that $\tilde{D}^{t}$ is described by the stress rate as

$$\tilde{D}^{t} = \frac{d_{n}}{M^{t}}\mathrm{tr}\,(\mathbf{N}\mathbf{P}\mathbf{n}^{*})\|\overset{\circ}{\boldsymbol{\sigma}}{}^{*}_{t}\| \tag{48}$$

by substituting Eq. (39) into Eq. (35). Eq. (47) does not fulfill the exact differential form, i.e. the complete integrability condition with respect to the stress rate[66] and is further rate-nonlinear resulting in the so-called *hypo-plasticity*, while the elastic strain rate equation (2) is required to fulfill the exact differential form leading to the hyper-elasticity.

Hereafter, assume that the elastic modulus tensor $\mathbf{E}$ is given by Hooke's type, i.e.

$$\left.\begin{aligned}E_{ijkl} &= \Big(K - \frac{2}{3}G\Big)\delta_{ij}\delta_{kl} + G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),\\ (\mathbf{E}^{-1})_{ijkl} &= \frac{1}{3}\Big(\frac{1}{3K} - \frac{1}{2G}\Big)\delta_{ij}\delta_{kl} + \frac{1}{4G}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\end{aligned}\right\}$$

(49)

where K and G are the elastic bulk modulus and the elastic shear modulus, respectively, which leads to $(\mathbf{EA})^* = 2G\mathbf{A}^*$ for an arbitrary second-order tensor $\mathbf{A}$. Then, it is obtained from Eq. (47) that

$$\mathbf{D}_t^* = \frac{M^t + 2G}{2GM^t}\overset{\circ}{\boldsymbol{\sigma}}_t^*, \tag{50}$$

where

$$\mathbf{D}_t^* \equiv \hat{\mathbf{I}}^*\mathbf{D} = \mathbf{D}^* - \mathrm{tr}(\mathbf{n}^*\mathbf{D}^*)\mathbf{n}^*, \tag{51}$$

noting

$$\mathbf{N}_t^* \equiv \hat{\mathbf{I}}^*\mathbf{N} = \mathbf{N}^* - \mathrm{tr}(\mathbf{n}^*\mathbf{N}^*)\mathbf{n}^* = \mathbf{N}^* - \mathrm{tr}\left(\frac{\mathbf{N}^*}{\|\mathbf{N}^*\|}\mathbf{N}^*\right)\frac{\mathbf{N}^*}{\|\mathbf{N}^*\|} = \mathbf{0}. \tag{52}$$

Substituting Eq. (50) into Eq.(39), the tangential strain rate is described by the strain rate as follows:

$$\mathbf{D}^t = \frac{2G}{M^t + 2G}(\mathbf{D}_t^* + d_n\|\mathbf{D}_t^*\|\mathbf{n}^*). \tag{53}$$

It is noticeable that the linear relationship (50) between the deviatoric-tangential stress rate $\overset{\circ}{\boldsymbol{\sigma}}_t^*$ and the *deviatoric-tangential strain rate* $\mathbf{D}_t^*$ in Eq. (51) holds in spite of the rate-nonlinearity of Eq. (47).

Substituting Eq. (50) into Eq. (47) with Eq. (48) and noting

$$\mathrm{tr}(\mathbf{NE}\overset{\circ}{\boldsymbol{\sigma}}_t^*)\ (= 2G\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}}_t^*)) = 0, \tag{54}$$

the proportionality factor λ can be rewritten in terms of the strain rate, rewriting λ as Λ, as follows:

$$\Lambda = \frac{1}{M^p + \mathrm{tr}(\mathbf{NEN})}\Big[\mathrm{tr}(\mathbf{NED}) - d_n\frac{2G}{M^t + 2G}\{\mathrm{tr}(\mathbf{NPn}^*) + 2G\,\mathrm{tr}(\mathbf{Nn}^*)\}\|\mathbf{D}_t^*\|\Big] \tag{55}$$

since $\tilde{D}^t$ is written in terms of strain rate by substituting Eq. (53) into Eq. (48), rewriting as $\tilde{D}^t$ as $\hat{D}^t$, as follows:

$$\hat{D}^t \equiv d_n\frac{2G}{M^t + 2G}\mathrm{tr}(\mathbf{NPn}^*)\|\mathbf{D}_t^*\|. \tag{56}$$

The inverse expression, i.e. the analytical expression of the stress rate in terms of the strain rate is derived from Eqs. (1), (2), (28), (53) and (55) as follows:

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{ED} - \Big\langle\frac{1}{M^p + \mathrm{tr}(\mathbf{NEN})}\Big[\mathrm{tr}(\mathbf{NED}) - d_n\frac{2G}{M^t + 2G}\{\mathrm{tr}(\mathbf{NPn}^*) + 2G\,\mathrm{tr}(\mathbf{Nn}^*)\}\|\mathbf{D}_t^*\|\Big]\Big\rangle\mathbf{EN} - \frac{(2G)^2}{M^t + 2G}(\mathbf{D}_t^* + d_n\|\mathbf{D}_t^*\|\mathbf{n}^*). \tag{57}$$

The loading criterion is given by

$$\left.\begin{aligned}&\mathbf{D}^p \neq \mathbf{0}: \mathrm{tr}(\mathbf{NED}) - d_n\frac{2G}{M^t + 2G}\{\mathrm{tr}(\mathbf{NPn}^*) + 2G\,\mathrm{tr}(\mathbf{Nn}^*)\}\|\mathbf{D}_t^*\| > 0,\\&\mathbf{D}^p = \mathbf{0}: \text{otherwise}.\end{aligned}\right\} \tag{58}$$

For $d_n = 0$ leading to $\tilde{D}^t = \hat{D}^t = 0$ the set of constitutive equations (47), (57) and (58) reduces to the quite simple forms:

$$\mathbf{D} = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} + \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^p}\mathbf{N} + \frac{1}{M^t}\overset{\circ}{\boldsymbol{\sigma}}_t^*, \tag{59}$$

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{ED} - \frac{\mathrm{tr}(\mathbf{NED})}{M^p + \mathrm{tr}(\mathbf{NEN})}\mathbf{EN} - \frac{(2G)^2}{M^t + 2G}\mathbf{D}_t^*, \tag{60}$$

$$\left.\begin{aligned}&\mathbf{D}^p \neq \mathbf{0}: \mathrm{tr}(\mathbf{NED}) > 0,\\&\mathbf{D}^p = \mathbf{0}: \text{otherwise}.\end{aligned}\right\} \tag{61}$$

Eq. (60) is of the quite simple form, comparing with the equation shown in the previous papers[46].

The influence of tangential strain rate to the hardening is reflected in the variable $\tilde{D}^t$ or $\hat{D}^t$ and thus the constitutive equation without the hardening due to the tangential strain rate is given as the following equations by putting $\tilde{D}^t = \hat{D}^t = 0$ in Eqs. (47), (57) and (58).

$$\mathbf{D} = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} + \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^p}\mathbf{N} + \frac{1}{M^t}(\overset{\circ}{\boldsymbol{\sigma}}_t^* + d_n\|\overset{\circ}{\boldsymbol{\sigma}}_t^*\|\mathbf{n}^*), \tag{62}$$

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{ED} - \Big\langle\frac{\mathrm{tr}(\mathbf{NED}) - d_n\frac{(2G)^2}{M^t + 2G}\mathrm{tr}(\mathbf{Nn}^*)\|\mathbf{D}_t^*\|}{M^p + \mathrm{tr}(\mathbf{NEN})}\Big\rangle\mathbf{EN} - \frac{(2G)^2}{M^t + 2G}(\mathbf{D}_t^* + d_n\|\mathbf{D}_t^*\|\mathbf{n}^*), \tag{63}$$

$$\left.\begin{aligned}&\mathbf{D}^p \neq \mathbf{0}: \mathrm{tr}(\mathbf{NED}) - d_n\frac{(2G)^2}{M^t + 2G}\mathrm{tr}(\mathbf{Nn}^*)\|\mathbf{D}_t^*\| > 0,\\&\mathbf{D}^p = \mathbf{0}: \text{otherwise}.\end{aligned}\right\} \tag{64}$$

In the present formulation the plastic strain rate (28) with Eq. (34) is derived by substituting the associated flow rule (28) into the consistency condition (26) which is obtained by incorporating the evolution rule (17) of the normal-yield ratio R into the time-derivative (14) of the subloading surface equation (10). Then, the plastic strain rate develops gradually as the stress approaches the normal-yield surface, exhibiting a *smooth elastic-plastic transition.* In addition, the tangential-inelastic strain rate (39) also develops gradually as the stress approaches the normal-yield surface. Then, Eq. (57) of the stress rate is expressed by the continuous function of the stress and the strain rate. Then, the tangential-subloading surface model possesses the following mechanical properties.

1. The *continuity* condition[23),25),60),62)] defined as "the stress rate changes continuously for a continuous change of the strain rate" is fulfilled, which is expressed mathematically as follows:

$$\lim_{\delta\mathbf{D}\to\mathbf{0}} \overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma},\ \mathbf{H}_i, \mathbf{D}+\delta\mathbf{D}) = \overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma},\ \mathbf{H}_i, \mathbf{D}), \qquad (65)$$

where $\mathbf{H}_i$ (i=1, 2, 3, •••, m) denotes collectively scalar- or tensor-valued internal state variables which describe the alteration of the mechanical response due to the irreversible deformation, $\delta(\ )$ stands for an infinitesimal variation and the response of the stress rate to the strain rate in the current state of stress and internal variables is designated by $\overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma},\ \mathbf{H}_i,\ \mathbf{D})$.

2. The *smoothness condition*[23),25),60),62)] defined as "the stress rate induced by the identical strain rate, changes continuously for a continuous change of stress state", is fulfilled, which can be expressed mathematically as follows:

$$\lim_{\delta\boldsymbol{\sigma}\to\mathbf{0}} \overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma}+\delta\boldsymbol{\sigma},\ \mathbf{H}_i, \mathbf{D}) = \overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma},\ \mathbf{H}_i, \mathbf{D}) \cdot \qquad (66)$$

The rate-linear constitutive equation is generally described as

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{M}^{ep}(\boldsymbol{\sigma},\ \mathbf{H}_i)\mathbf{D}, \qquad (67)$$

where the fourth-order tensor $\mathbf{M}^{ep}$ is the elastoplastic modulus which is a function of the stress and internal variables and can be described generally as

$$\mathbf{M}^{ep} = \frac{\partial\overset{\circ}{\boldsymbol{\sigma}}}{\partial\mathbf{D}}. \qquad (68)$$

Therefore, Eq. (66) can be rewritten as

$$\lim_{\delta\boldsymbol{\sigma}\to\mathbf{0}} \mathbf{M}^{ep}(\boldsymbol{\sigma}+\delta\boldsymbol{\sigma},\ \mathbf{H}_i) = \mathbf{M}^{ep}(\boldsymbol{\sigma},\ \mathbf{H}_i) \cdot \qquad (69)$$

Thus, the subloading surface model and its extension to the tangential relaxation, i.e. the tangential-subloading surface model have notable advantages as follows:

i) It predicts a smooth response (a smooth relationship of axial stress and axial logarithmic strain in the uniaxial loading for instance) for a smooth monotonic loading process. By contrast, a nonsmooth response is predicted by the conventional constitutive model which assumes the yield surface enclosing a purely elastic domain and thus violates the smoothness condition.

ii) The stress always lies on the subloading surface which plays the role of loading surface. Therefore, only the judgment for the sign of the proportionality factor Λ is required in the loading criterion for the subloading surface model. On the other hand, the judgment as to whether or not the stress lies on the yield surface, in addition to the judgment for the sign of Λ, is required in the conventional plasticity violating the smoothness condition.

iii) A stress is automatically drawn back to the normal-yield surface even if it goes out from that surface since it is formulated that $\dot{R}>0$ for $R<1$ (subyield state) and $\dot{R}<0$ for $R>1$ (over normal-yield state) in Eq. (17) with the condition (18). Thus, a stable calculation is performed even by rough loading steps compared with the conventional models when the explicit method is adopted in numerical calculations.

On the other hand, the tangential-inelastic strain rate is induced suddenly at the moment when the stress reaches the yield surface in the other tangential-plasticity models violating the smoothness condition, e.g. Rudnicki and Rice's model[1)] and Papamichos et al.'s model[44)], and thus the continuity condition is also violated in these models. Further, unconventional models other than the subloading surface model, e.g. the multi surface model and the two surface model, also violate the smoothness and continuity conditions if the tangential-inelastic strain rate is incorporated. Therefore, they lead to the serious defect that the uniqueness of solution is violated for the stress path along the yield surface.

3. Verification of the present model

First, let the concrete constitutive equation of metals be formulated based on the tangential-subloading surface model formulated in the preceding section. In what follows, the mechanical response of the present model will be shown briefly.

3. 1 Constitutive equation of metals

The von Mises yield condition with isotropic-kinematic hardening is described for the normal-yield/subloading surface as follows[69)]:

$$f(\overline{\boldsymbol{\sigma}}) = \sqrt{\frac{3}{2}}\left\|\overline{\boldsymbol{\sigma}}^*\right\|, \qquad (70)$$

$$\mathbf{N} = \mathbf{N}^* = \mathbf{n}^* = \frac{\overline{\boldsymbol{\sigma}}^*}{\left\|\overline{\boldsymbol{\sigma}}^*\right\|}, \qquad (71)$$

$$F(H) = F_0[1+h_1\{1-\exp(-h_2 H)\}], \tag{72}$$

$$F' = F_0\, h_1\, h_2 \exp(-h_2 H), \tag{73}$$

$$\overset{\bullet}{H} = \sqrt{\frac{2}{3}}\mathrm{tr}(\mathbf{N}\mathbf{D}^{i}),\ \mathbf{f}_H = \sqrt{\frac{2}{3}}\mathbf{N}, \tag{74}$$

$$\overset{\bullet}{H}{}^{p} = \sqrt{\frac{2}{3}}\lambda = \sqrt{\frac{2}{3}}\left\|\mathbf{D}^{p}\right\|,\ h = \sqrt{\frac{2}{3}}\ \left(a=0,\ b=\sqrt{\frac{2}{3}}\right), \tag{75}$$

$$\overset{\bullet}{H}{}^{t} = \sqrt{\frac{2}{3}}\,\mathrm{tr}(\mathbf{N}\mathbf{D}^{t}) = \sqrt{\frac{2}{3}}\frac{d_n}{M^t}\|\mathring{\boldsymbol{\sigma}}_t^{*}\| = \sqrt{\frac{2}{3}}d_n\frac{2GM^t}{M^t+2G}\|\mathbf{D}_t^{*}\|, \tag{76}$$

$$\mathbf{H}^p = \mathbf{H}^t = \mathbf{0}, \tag{77}$$

$$\mathring{\boldsymbol{\alpha}} = \mathrm{tr}(\mathbf{N}\mathbf{D}^{i})\mathbf{a}\ \ (\mathbf{f}_\alpha = \mathbf{a}\otimes\mathbf{N}),\ \ \mathbf{a} \equiv k_1\frac{\bar{\boldsymbol{\sigma}}^{*}}{\|\bar{\boldsymbol{\sigma}}^{*}\|} - k_2\boldsymbol{\alpha}, \tag{78}$$

$$\mathring{\boldsymbol{\alpha}}^{p} = \mathrm{tr}(\mathbf{N}\mathbf{D}^{p})\mathbf{a} = \lambda\,\mathbf{a}, \tag{79}$$

$$\mathring{\boldsymbol{\alpha}}^{t} = \mathrm{tr}(\mathbf{N}\mathbf{D}^{t})\mathbf{a} = \frac{d_n}{M^t}\|\mathring{\boldsymbol{\sigma}}_t^{*}\|\mathbf{a} = d_n\frac{2GM^t}{M^t+2G}\|\mathbf{D}_t^{*}\|\mathbf{a}. \tag{80}$$

The variables k_1, k_2, h_1 and h_2 are material constants, and F_0 is the initial value of *F*. It holds from Eqs. (36), (48), (56) and (70)-(78) that

$$M^p = \mathrm{tr}\Big[\frac{\bar{\boldsymbol{\sigma}}^{*}}{\|\bar{\boldsymbol{\sigma}}^{*}\|}\Big\{\Big(\sqrt{\frac{2}{3}}\frac{F'}{F}+\frac{U}{R}\Big)\bar{\boldsymbol{\sigma}} + k_1\frac{\bar{\boldsymbol{\sigma}}^{*}}{\|\bar{\boldsymbol{\sigma}}^{*}\|} - k_2\boldsymbol{\alpha} + c\Big(\frac{1}{R}-1\Big)\tilde{\boldsymbol{\sigma}} + \Big(\sqrt{\frac{2}{3}}(1-R)\frac{F'}{F}-U\Big)\hat{\mathbf{s}}\Big\}\Big], \tag{81}$$

$$\tilde{D}^t \equiv d_n\frac{M^p}{M^t}\|\mathring{\boldsymbol{\sigma}}_t^{*}\|, \tag{82}$$

$$\hat{D}^t = d_n\frac{2GM^p}{M^t+2G}\|\mathbf{D}_t^{*}\|. \tag{83}$$

For the isotropy, i.e. $\boldsymbol{\alpha}=\mathbf{0}$ ($k_1=k_2=0$), $\mathbf{s}=\mathbf{0}$ (c=0) leading to $\bar{\boldsymbol{\alpha}}=\mathbf{0}$, the plastic modulus in Eq. (81) reduces to

$$M^p = \sqrt{\frac{2}{3}}\Big(\sqrt{\frac{2}{3}}RF'+UF\Big). \tag{84}$$

3. 2 Mechanical response

The mechanical features of the constitutive equation of metals described in the preceding subsection is examined by analyzing the response on the π-plane. In what follows, the description of only principal components, i.e. the principal space representation is adopted, in which corresponding components of the stress rate and strain rate are taken in the same directions, since all the input and output variables have only principal components because of the mechanical isotropy.

Consider the response of the strain rate $\mathbf{D}(=\mathbf{D}^{*})$ to the input of the deviatoric-stress rate $\mathring{\boldsymbol{\sigma}}\ (=\mathring{\boldsymbol{\sigma}}^{*})$ with the constant magnitude, i.e. $\|\mathring{\boldsymbol{\sigma}}\| = \mathrm{const.}$ on the π-plane, while $\mathring{\boldsymbol{\sigma}}$ is given as

$$\begin{Bmatrix}\mathring{\sigma}_1\\ \mathring{\sigma}_2\\ \mathring{\sigma}_3\end{Bmatrix} = \sqrt{\frac{2}{3}}\|\mathring{\boldsymbol{\sigma}}\|\begin{Bmatrix}\cos\theta\\ \cos\{\theta-(2/3)\pi\}\\ \cos\{\theta+(2/3)\pi\}\end{Bmatrix}, \tag{85}$$

from the state of stress

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{Bmatrix} = \frac{1}{3}RF\begin{Bmatrix}2\\ -1\\ -1\end{Bmatrix}, \tag{86}$$

resulting in

$$\begin{Bmatrix}N_1\\ N_2\\ N_3\end{Bmatrix} = \frac{1}{\sqrt{6}}\begin{Bmatrix}2\\ -1\\ -1\end{Bmatrix},\ \begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}0\\ 1/\sqrt{2}\\ -1/\sqrt{2}\end{Bmatrix}, \tag{87}$$

$$\begin{Bmatrix}\mathring{\sigma}_N\\ \mathring{\sigma}_T\end{Bmatrix} = \|\mathring{\boldsymbol{\sigma}}\|\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix},\ \begin{Bmatrix}(\mathring{\boldsymbol{\sigma}}_t^{*})_N\\ (\mathring{\boldsymbol{\sigma}}_t^{*})_T\end{Bmatrix} = \begin{Bmatrix}0\\ \mathring{\sigma}_T\end{Bmatrix}, \tag{88}$$

$$\|\mathring{\boldsymbol{\sigma}}_t^{*}\| = \|\mathring{\boldsymbol{\sigma}}\||\sin\theta|, \tag{89}$$

where

$$\cos 3\theta \equiv \sqrt{6}\,\mathrm{tr}\Big(\frac{\mathring{\boldsymbol{\sigma}}^{*}}{\|\mathring{\boldsymbol{\sigma}}^{*}\|}\Big)^3, \tag{90}$$

θ standing for the angle measured in a clock-wise direction from the direction of $\mathbf{N}$ to the direction of the stress rate $\mathring{\boldsymbol{\sigma}}$ on the π-plane. The unit vector $\mathbf{T}$ has the direction rotated $\pi/2$ in a clock-wise direction from $\mathbf{N}$ and thus is tangential to the subloading surface. Consider the two-dimensional orthogonal coordinate system with unit base vectors $\mathbf{N}$ and $\mathbf{T}$ in which corresponding components of the stress rate and strain rate are taken in the same directions, and the components in directions $\mathbf{N}$ and $\mathbf{T}$ are denoted by the notations

$$\left.\begin{aligned}A_N &\equiv \mathrm{tr}(\mathbf{N}\mathbf{A}),\\ A_T &\equiv \mathrm{tr}(\mathbf{T}\mathbf{A})\end{aligned}\right\} \tag{91}$$

for an arbitrary second-order tensor $\mathbf{A}$. Then, it is written from Eqs. (47) and (85)-(89) that

$$\begin{Bmatrix}D_N^e\\ D_T^e\end{Bmatrix} = \frac{1}{2G}\|\mathring{\boldsymbol{\sigma}}\|\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix}, \tag{92}$$

$$\begin{Bmatrix}D_N^p\\ D_T^p\end{Bmatrix} = \|\mathring{\boldsymbol{\sigma}}\|\begin{Bmatrix}\Big\langle\frac{1}{M^p}\cos\theta - \frac{d_n}{M^t}|\sin\theta|\Big\rangle\\ 0\end{Bmatrix}, \tag{93}$$

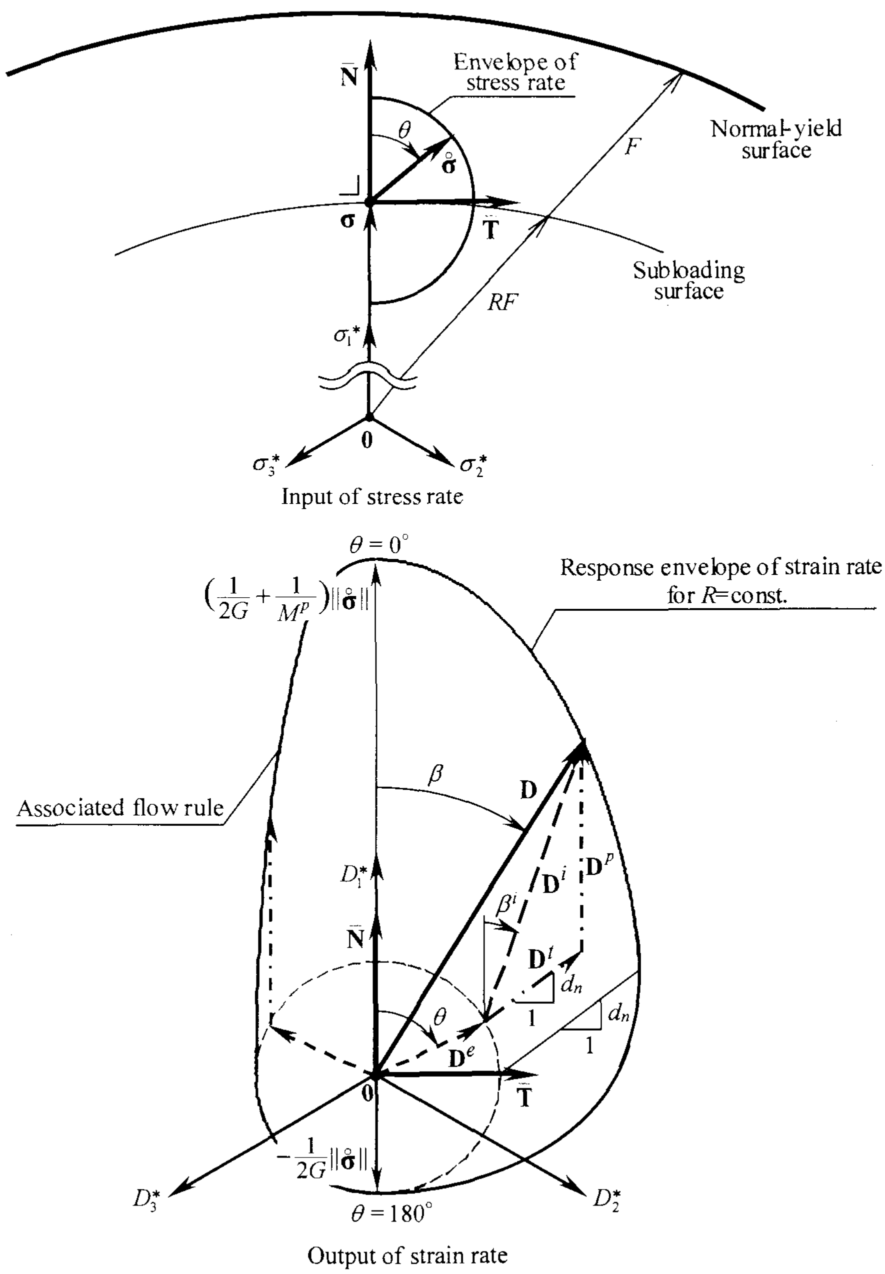

Fig. 2. Illustration of response envelope of strain rate $\mathbf{D}$ (= $\mathbf{D}^*$) to the input of the stress rate $\mathring{\boldsymbol{\sigma}}$ (= $\mathring{\boldsymbol{\sigma}}^*$) with a constant magnitude, i.e. $\|\mathring{\boldsymbol{\sigma}}\|$=const.

$$\begin{Bmatrix} D_N^t \\ D_T^t \end{Bmatrix} = \frac{\|\mathring{\boldsymbol{\sigma}}\|}{M^t} \begin{Bmatrix} d_n|\sin\theta| \\ \sin\theta \end{Bmatrix}. \quad (94)$$

The strain rate response envelope to the input of the stress rate calculated by Eqs. (92)-(92) is illustrated in Fig. 2, where β and β^i are the angles measured in a clockwise direction from the axis $\mathbf{N}$ to the strain rate and inelastic strain rate, respectively, in the ($\mathbf{N}$, $\mathbf{T}$) plane, i.e.

$$\left.\begin{aligned} \cos\beta &\equiv \frac{\|\mathbf{D}\|}{\|\mathbf{D}\|_{\theta=0}} = \frac{D_N}{\sqrt{D_N^2 + D_T^2}}, \\ \cos\beta^i &\equiv \frac{\|\mathbf{D}^i\|}{\|\mathbf{D}^i\|_{\theta=0}} = \frac{D_N^i}{\sqrt{D_N^{i\,2} + D_T^{i\,2}}} \end{aligned}\right\}, \quad (95)$$

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