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Partial Analysis Method Combined with Mode-exciting Method in the Analysis of Lamb Waves Scattering by a Crack in a Plate

部分解析法およびモード励振法を用いた板内の亀裂に対する板波の散乱解析

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The partial analysis method is developed to solve the scattering problem of Lamb waves by a finite crack in a plate. First, the scattering coefficients of Lamb waves due to a semi-infinite crack are obtained by using the mode-exciting method. Then we consider the equivalence between scattering coefficients of Lamb waves by a finite crack and those by a semi-infinite crack to obtain a system of equations depending on the crack length explicitly. The partial analysis method has an advantage in computational time when several plates with single cracks of various lengths are to be analyzed. Comparison between the results of the partial analysis method and the direct mode-exciting method shows a good agreement. The application of the partial analysis method to an inverse analysis is also demonstrated.

Key Words : *Lamb waves, scattering analysis method, crack, inverse analysis*

1. Introduction

Lamb wave ultrasonic method in detecting a defect nondestructively in a thin plate has recently attracted great attention. The properties of Lamb waves such as dispersion relation and wave motion have been studied in detail and summarized in many books[1),2),3)]. However, knowledge about scattering properties of Lamb waves by various types of defects is still insufficient for quantitative evaluation of a plate. This is due to complexity of the scattering mechanism of Lamb waves which is characterized by the mode conversion including a finite number of propagating modes and also an infinite number of nonpropagating modes.

The studies on the scattering analysis of Lamb waves have been carried out by various methods. Rokhlin[4),5),6)] has applied an analytical method of the Wiener-Hopf technique to solve the scattering problem of Lamb waves due to cracks with semi-infinite length and finite length. Liu and Achenbach[7)] applied the strip element method to investigate wave scattering by cracks in anisotropic laminated plates. As numerical methods for the scattering analysis of Lamb waves in the frequency domain, Koshiba *et al.*[8)] and Al-Nassar *et al.*[9)] have applied a finite element method (FEM), whereas Cho and Rose[10),11)] have applied a boundary element method (BEM). Galan[12)] has also developed a hybrid method of finite element and boundary element formulation. Since the scattering problem of Lamb waves is usually formulated for a plate of infinite length, however, FEM or BEM in the frequency domain can not be individually used in solving the scattering problem. They have to be coupled with other technique such as the normal mode expansion technique to express the wave field in the infinity. Recently, the authors[13)] have proposed the mode-exciting method to solve various types of Lamb wave-scattering problems. The mode-exciting method enables us to perform the numerical analysis in a finite domain to solve the scattering problem for the infinite domain. Owing to this benefit, in the mode-exciting method, the numerical method such as FEM and BEM can be used individually without coupling with other technique.

In this paper, we propose an innovative method called *the partial analysis method* combined with the mode-exciting method to solve efficiently the scattering problems of Lamb waves due to finite cracks with various lengths in an infinite plate. In the partial analysis method, the diffraction of Lamb waves at each edge of a finite crack is considered partially in solving the scattering problem by the finite crack. First, the scattering coefficients of Lamb waves due to a semi-infinite crack are obtained by using the mode-exciting method. Next we consider the equivalence between the scattering coefficients of Lamb wave propagating modes in the vicinity of the edges of the finite crack and those by the semi-infinite crack. Then we obtain a system of linear equations depending on the crack length explicitly. The partial analysis method, therefore, has an advantage of computational time when several plates with single cracks of various lengths are to be analyzed because the numerical method such as FEM and BEM, the part of which takes much computational time, need to be carried out only once.

Fundamental relationships of Lamb wave modes and statements of Lamb wave-scattering problems are presented in Section 2 and 3, respectively. Formulation of the partial analysis method is presented in Section 4. Some numerical results and the application of the partial analysis method to estimate the location and the length of an unknown finite crack are shown in Section 5.

2. Fundamental relationships in Lamb wave modes

The dispersive relations of Lamb wave modes propagating in a homogeneous, isotropic and linearly elastic plate with free surfaces are described as[1),2),3)]

$$\frac{\tan(qh)}{\tan(ph)} + \frac{4k^2pq}{(q^2-k^2)^2} = 0, \quad \text{and} \tag{1}$$

$$\frac{\tan(qh)}{\tan(ph)} + \frac{(q^2-k^2)^2}{4k^2pq} = 0, \tag{2}$$

for symmetric modes and antisymmetric modes, respectively, where k and ω denote the wavenumber and the angular frequency, respectively, and h is the half of the thickness of the plate. In Eq. (1),

$$p^2 = \frac{\omega^2}{c_L^2} - k^2, \qquad q^2 = \frac{\omega^2}{c_T^2} - k^2, \tag{3}$$

c_L and c_T are the velocities of the longitudinal and transverse waves, respectively. Fig. 1 shows the dispersion curves for a steel with c_L=5940m/s and c_T=3200m/s. Here S_m and A_m denote the symmetric modes and the antisymmetric modes, respectively, of mth order ($m=0,1,2,\dots$). Detail discussion on dispersion curves of Lamb waves can be found in many books, *e.g.*, see the text book authored by Achenbach.[2)]

The displacement $\boldsymbol{u}$ of the Lamb wave modes can

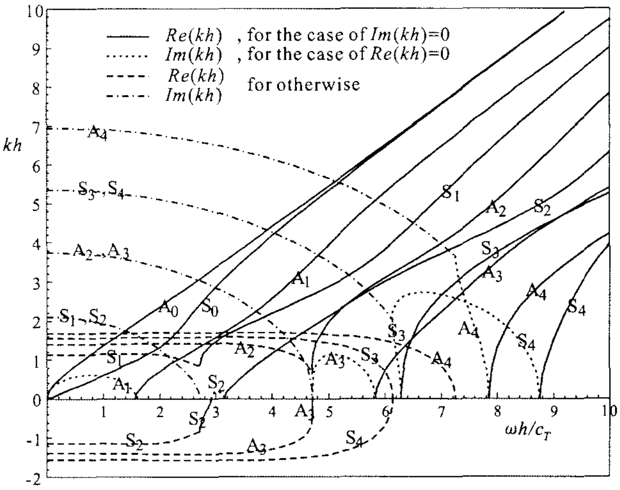

Fig. 1 Dispersion curves of Lamb wave modes in a steel plate. Solid curves and dotted curves denote the propagating waves with pure real wavenumbers and the nonpropagating waves with pure imaginary wavenumbers, respectively. Dashed curves and double dashed curves denote the real parts and the imaginary parts, respectively, of the wavenumbers of the nonpropagating waves with complex wavenumbers.

be expressed as

$$u_1 = IAk\left\{\frac{\cos(px_2)}{\sin(ph)} + \frac{2pq}{k^2-q^2}\frac{\cos(qx_2)}{\sin(qh)}\right\}e^{I(kx_1-\omega t)},$$
$$u_2 = -Ap\left\{\frac{\sin(px_2)}{\sin(ph)} - \frac{2k^2}{k^2-q^2}\frac{\sin(qx_2)}{\sin(qh)}\right\}e^{I(kx_1-\omega t)}, \tag{4}$$

for symmetric modes and

$$u_1 = IAk\left\{\frac{\sin(px_2)}{\cos(ph)} + \frac{2pq}{k^2-q^2}\frac{\sin(qx_2)}{\cos(qh)}\right\}e^{I(kx_1-\omega t)},$$
$$u_2 = Ap\left\{\frac{\cos(px_2)}{\cos(ph)} - \frac{2k^2}{k^2-q^2}\frac{\cos(qx_2)}{\cos(qh)}\right\}e^{I(kx_1-\omega t)}, \tag{5}$$

for antisymmetric modes, where x_1 and x_2 are taken in the longitudinal and thickness' direction, respectively, $I=\sqrt{-1}$, and A is an arbitrary constant. The stress components can be evaluated by substituting Eqs. (4) and (5) into the stress-displacement relation

$$\tau_{ij} = \rho(c_L^2 - 2c_T^2)u_{k,k}\delta_{ij} + \rho c_T^2(u_{i,j} + u_{j,i}), \tag{6}$$

where ρ is the density and δ_{ij} is the Kronecker delta.

3. Scattering problems of Lamb waves

There are two scattering problems of Lamb waves taken into consideration in this study: scattering by a finite crack and scattering by a semi-infinite crack.

The former is called the *main scattering problem*, which is the objective of our analysis in this paper, while the latter is called the *sub-scattering problem*, which will be considered in the application of the partial analysis method to solve the main scattering problem.

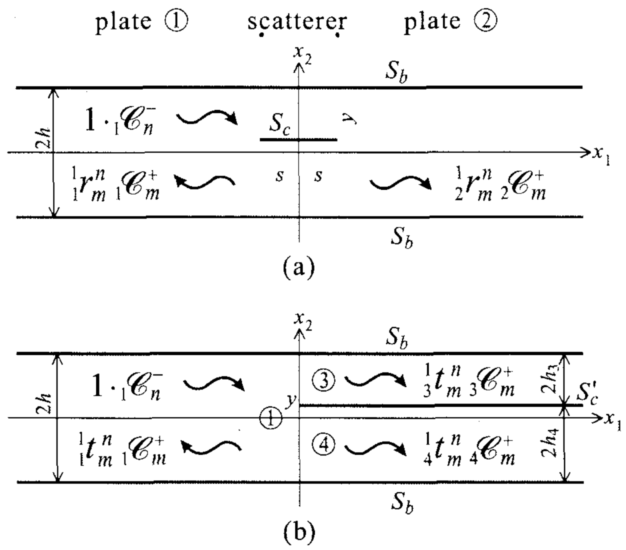

Fig. 2 Scattering of Lamb waves (a) by a finite crack and (b) by a semi-infinite crack in an infinite plate.

3.1 Main scattering problem

The main scattering problem dealt with in this study is the scattering of Lamb waves by a finite crack in an infinite plate as shown in Fig. 2(a). The crack is a horizontal crack with $2s$ in length and lies at the distance y from the center of the plate. The surfaces S_b and S_c of the plate and the crack are assumed to be traction free. Note that the infinite plate can be divided into three parts, *i.e.*, two plain plates ① and ② and the scatterer part containing the finite crack. Let ${}_i\mathscr{C}_n^{\pm}$ denote the propagating Lamb wave modes of nth order in the plain plate ⓘ. Here ${}_i\mathscr{C}_n^{-}$ and ${}_i\mathscr{C}_n^{+}$ denote the incident wave propagating toward the scatterer and the scattered wave traveling away from the scatterer, respectively. It is noted here that ${}_i\mathscr{C}_n^{\pm}$ include both symmetric modes and antisymmetric modes, and are normalized in such a way that the powers of the modes with unit amplitudes are equal to unity.[13)]

Suppose that the Lamb wave ${}_1\mathscr{C}_n^{-}$ of nth propagating mode in the plate ① with unit amplitude is incident to the scatterer and scattered waves of all Lamb wave modes are generated in both plates ① and ② as shown in Fig. 2(a). In the far field where all scattered nonpropagating modes vanish, the scattering process can be written in the form:

$$1 \cdot {}_i\mathscr{C}_n^{-} \longrightarrow \sum_{j=1}^{2} \sum_{m=1}^{\mathcal{N}_j} {}^{i}_{j}r^{n}_{m}\, {}_j\mathscr{C}_m^{+} \qquad (i=1), \quad (7)$$

where ${}^{i}_{j}r^{n}_{m}$ is the amplitude of the scattered Lamb wave ${}_j\mathscr{C}_m^{+}$ of the mth propagating mode in the plate ⓙ due to the incident Lamb wave ${}_i\mathscr{C}_n^{-}$ of the nth propagating mode in the plate ⓘ, and $\mathcal{N}_j$ is the number of propagating modes in the plate ⓙ, which depends on the frequency times the thickness of the plate as shown in Fig. 1. The arrow in Eq. (7) means the scattering process in which the incident wave ${}_i\mathscr{C}_n^{-}$ is transformed into the scattered waves ${}_j\mathscr{C}_m^{+}$ (j=1, 2; m=1, ..., $\mathcal{N}_j$). In the main scattering problem, the scattering coefficients ${}^{i}_{j}r^{n}_{m}$ are the unknowns which are to be determined. Note that the coefficients ${}^{i}_{j}r^{n}_{m}$ are complex variables, and their square absolute values and arguments represent the powers and the phase shifts, respectively.

3.2 Sub-scattering problem

The sub-scattering problem dealt with in this study is the scattering of Lamb waves by a semi-infinite crack located at the distance y from the center of the plate as shown in Fig. 2(b). The sub-scattering problem is necessary for the application of the partial analysis method in solving the main scattering problem. Note that because the crack's surface S_c' is traction free, the regions above and below the crack can be considered as plain plates. Hence, the infinite plate in the sub-scattering problem can be divided into three plain plates ①, ③, and ④ as shown in Fig. 2(b), where the plate ① is perfectly connected with the plates ③ and ④.

Similarly to the main scattering problem, the sub-scattering problem can be described as follows. Assume that when the Lamb wave ${}_i\mathscr{C}_n^{-}$ of the nth propagating mode in the plate ⓘ (i=1,2,3) with unit amplitude is incident to the plates' connection, the scattered waves ${}_j\mathscr{C}_m^{+}$ of the mth propagating modes in the plates ⓙ are generated with the scattering coefficients ${}^{i}_{j}t^{n}_{m}$, as shown in Fig. 2(b) for i=1. The scattering process of the sub-scattering problem can be written in the form:

$$1 \cdot {}_i\mathscr{C}_n^{-} \longrightarrow \sum_{j=1,3,4} \sum_{m=1}^{\mathcal{N}_j} {}^{i}_{j}t^{n}_{m}\, {}_j\mathscr{C}_m^{+} \qquad (i=1,2,3). \quad (8)$$

The scattering coefficients ${}^{i}_{j}r^{n}_{m}$ of the main scattering problem and ${}^{i}_{j}t^{n}_{m}$ of the sub-scattering problem can be found by using the mode-exciting method. Detail explanation about the mode-exciting method can be found in our previously published paper[13)] and will be omitted in this paper. In the next section, the relation between ${}^{i}_{j}r^{n}_{m}$ and ${}^{i}_{j}t^{n}_{m}$ will be discussed.

4. Partial analysis method

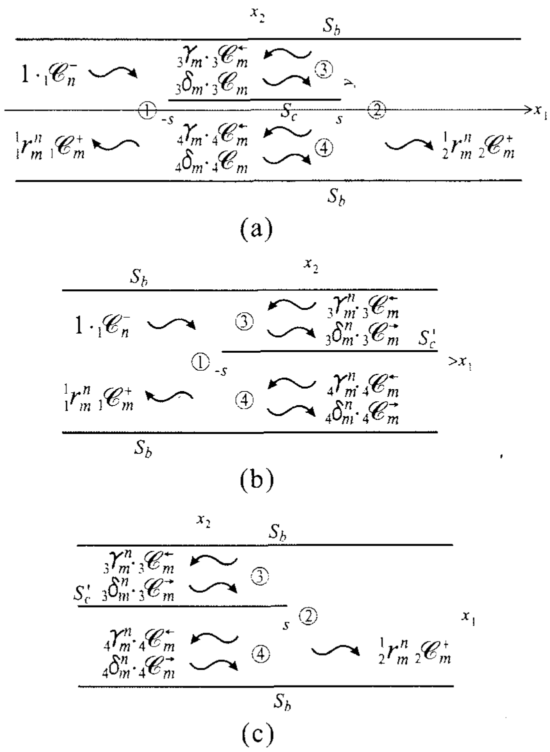

Fig. 3 Basic concept of the partial analysis method, where the diffraction properties of Lamb wave modes in the vicinities of the left and the right edges of the finite crack in the main scattering problem (a) are equivalent to those in the sub-scattering problems (b) and (c), respectively.

Fig. 3(a) shows the scattering process of the main scattering problem, where the Lamb mode ${}_1\mathscr{C}_n^-$ with unit amplitude is incident to the finite crack. Let us first consider the region above the crack ($-s \leq x_1 \leq s$, $y \leq x_2 \leq h$). Since the crack's surface S_c is traction free, it is obvious that the region above the crack can be considered as the finite plain plate ③, where the wave field is decomposed into Lamb wave modes.[3),13)] Similar decomposition can be made for the wave field in the region below the crack, *i.e.*, the plate ④. Let the propagating Lamb wave modes generated in the plates ③ and ④ be ${}_3\gamma_m^n\,{}_3\mathscr{C}_m^{\leftarrow} + {}_3\delta_m^n\,{}_3\mathscr{C}_m^{\rightarrow}$ and ${}_4\gamma_m^n\,{}_4\mathscr{C}_m^{\leftarrow} + {}_4\delta_m^n\,{}_4\mathscr{C}_m^{\rightarrow}$, respectively, as shown in Fig. 3(a). Here, the superscripts $^{\rightarrow}$ and $^{\leftarrow}$ in ${}_i\mathscr{C}_m^{\rightarrow}$ and ${}_i\mathscr{C}_m^{\leftarrow}$ denote the directions of the propagating modes, and ${}_i\gamma_m^n$ and ${}_i\delta_m^n$ are the amplitudes of ${}_i\mathscr{C}_m^{\leftarrow}$ and ${}_i\mathscr{C}_m^{\rightarrow}$, respectively ($i$=3,4). Note that nonpropagating modes in the plates ③ and ④ are neglected here.

Now let us consider the wave field near the left edge of the crack as shown in Fig. 3(b). It is clear that the wave field is expressed by a Lamb wave-scattering process with $1\cdot{}_1\mathscr{C}_n^-$ and ${}_i\gamma_m^n\,{}_i\mathscr{C}_m^{\leftarrow}$ (i=3,4) as incident waves, and ${}_1^1r_m^n\,{}_1\mathscr{C}_m^+$ and ${}_i\delta_m^n\,{}_i\mathscr{C}_m^{\rightarrow}$ (i=3,4) as scattered waves as follows:

$$1\cdot{}_1\mathscr{C}_n^- + \sum_{i=3}^{4}\sum_{m=1}^{\mathcal{N}_i} {}_i\gamma_m^n\,{}_i\mathscr{C}_m^{\leftarrow} \longrightarrow \sum_{l=1}^{\mathcal{N}_1} {}_1^1r_l^n\,{}_1\mathscr{C}_l^+ + \sum_{j=3}^{4}\sum_{l=1}^{\mathcal{N}_j} {}_j\delta_l^n\,{}_j\mathscr{C}_l^{\rightarrow}. \quad (9)$$

It is obvious that the scattering problem shown in Fig. 3(b) is equivalent to the sub-scattering problem shown in Fig. 2(b). As a consequence, there exist relations between the amplitudes of the Lamb wave modes in Fig. 3(b) and the scattering coefficients ${}_j^it_m^n$ of the sub-scattering problem. As a preliminary procedure for deriving these relations, Eq. (8) is modified so as to express the scattering process in Fig. 3(b), in which the location of the edge of the crack differs from that in Fig. 2(b) by s. Then we have

$$e^{I_ik_n^-s}\,{}_i\mathscr{C}_n^- \longrightarrow \sum_{j=1,3,4}\sum_{l=1}^{\mathcal{N}_j} {}_j^it_l^n\, e^{I_jk_l^+s}\,{}_j\mathscr{C}_l^+, \quad (10)$$

or

$${}_i\mathscr{C}_n^- \longrightarrow \sum_{j=1,3,4}\sum_{l=1}^{\mathcal{N}_j} {}_j^it_l^n\, e^{I({}_jk_l^+-{}_ik_n^-)s}\,{}_j\mathscr{C}_l^+$$
$$(i=1,3,4;\ n=1,\ldots,\mathcal{N}_i), \quad (11)$$

where ${}_ik_n^-$ and ${}_jk_l^+$ are the wavenumbers of the modes ${}_i\mathscr{C}_n^-$ and ${}_j\mathscr{C}_l^+$, respectively. Performing some algebraic manipulations to Eq. (11), it can be shown that

$$1\cdot{}_1\mathscr{C}_n^- + \sum_{i=3}^{4}\sum_{m=1}^{\mathcal{N}_i} {}_i\gamma_m^n\,{}_i\mathscr{C}_m^- \longrightarrow \sum_{j=1,3,4}\sum_{l=1}^{\mathcal{N}_j}\Bigg[{}_j^1t_l^n\, e^{I({}_jk_l^+-{}_1k_n^-)s} + \sum_{i=3}^{4}\sum_{m=1}^{\mathcal{N}_i} {}_i\gamma_m^n\,{}_j^it_l^n\, e^{I({}_jk_l^+-{}_ik_n^-)s}\Bigg]{}_j\mathscr{C}_l^+. \quad (12)$$

Considering that ${}_i\mathscr{C}_m^-={}_i\mathscr{C}_m^{\leftarrow}$ and ${}_i\mathscr{C}_m^+={}_i\mathscr{C}_m^{\rightarrow}$ (i=3,4) for the scattering process in Fig. 3(b), comparison between Eq. (9) and Eq. (12) yields the following system of linear equations:

$${}_1^1t_l^n\, e^{I({}_1k_l^+-{}_1k_n^-)s} + \sum_{i=3}^{4}\sum_{m=1}^{\mathcal{N}_i} {}_i\gamma_m^n\,{}_1^it_l^n\, e^{I({}_1k_l^+-{}_ik_n^-)s} = {}_1^1r_l^n \quad (l=1,\ldots,\mathcal{N}_1),$$
$${}_j^1t_l^n\, e^{I({}_jk_l^{\rightarrow}-{}_1k_n^-)s} + \sum_{i=3}^{4}\sum_{m=1}^{\mathcal{N}_i} {}_i\gamma_m^n\,{}_j^it_l^n\, e^{I({}_jk_l^{\rightarrow}-{}_ik_n^{\leftarrow})s} = {}_j\delta_l^n \quad (j=3,4;\ l=1,\ldots,\mathcal{N}_j), \quad (13)$$

where ${}_ik_n^{\leftarrow}$ and ${}_jk_l^{\rightarrow}$ are the wave numbers of ${}_i\mathscr{C}_n^{\leftarrow}$ and ${}_j\mathscr{C}_l^{\rightarrow}$, respectively.

Similar relations can be obtained for the wave field in the vicinity of the right edge of the crack as shown in Fig. 3(c). The Lamb wave-scattering process is expressed as:

$$\sum_{i=3}^{4}\sum_{m=1}^{\mathcal{N}_i} {}_i\delta_m^n\, {}_i\mathscr{C}_m^{\rightarrow} \longrightarrow \sum_{l=1}^{\mathcal{N}_2} {}_2^1r_l^n\, {}_2\mathscr{C}_l^{+} + \sum_{j=3}^{4}\sum_{l=1}^{\mathcal{N}_j} {}_j\gamma_l^n\, {}_j\mathscr{C}_l^{\leftarrow}. \tag{14}$$

For the scattering process in Fig. 3(c), Eq. (8) is modified as

$$e^{-I_ik_n^-s}\, {}_i\mathscr{C}_n^- \longrightarrow \sum_{j=2,3,4}\sum_{l=1}^{\mathcal{N}_j} {}_j^it_l^n\, e^{-I_jk_l^+s}\, {}_j\mathscr{C}_l^+, \tag{15}$$

or

$${}_i\mathscr{C}_n^- \longrightarrow \sum_{j=2,3,4}\sum_{l=1}^{\mathcal{N}_j} {}_j^it_l^n\, e^{I({}_ik_n^- - {}_jk_l^+)s}\, {}_j\mathscr{C}_l^+$$

$$(i = 2,3,4;\ n = 1,\ldots,\mathcal{N}_i), \tag{16}$$

Performing some algebraic manipulations to Eq. (16), it can be shown that

$$\sum_{i=3}^{4}\sum_{m=1}^{\mathcal{N}_i} {}_i\delta_m^n\, {}_i\mathscr{C}_m^- \longrightarrow$$
$$\sum_{j=2,3,4}\sum_{l=1}^{\mathcal{N}_j}\sum_{i=3}^{4}\sum_{m=1}^{\mathcal{N}_i} {}_i\delta_m^n\, {}_j^it_l^n\, e^{I({}_ik_n^- - {}_jk_l^+)s}\, {}_j\mathscr{C}_l^+. \tag{17}$$

Considering that ${}_i\mathscr{C}_m^- = {}_i\mathscr{C}_m^{\rightarrow}$ and ${}_i\mathscr{C}_m^+ = {}_i\mathscr{C}_m^{\leftarrow}$ $(i=3,4)$ for the scattering problem in Fig. 3(c), comparison between Eq. (14) and Eq. (17) yields the following system of linear equations:

$$\sum_{i=3}^{4}\sum_{m=1}^{\mathcal{N}_i} {}_i\delta_m^n\, {}_2^it_l^n\, e^{I({}_ik_n^- - {}_2k_l^+)s} = {}_2^1r_l^n \quad (l = 1,\ldots,\mathcal{N}_1)$$
$$\sum_{i=3}^{4}\sum_{m=1}^{\mathcal{N}_i} {}_i\delta_m^n\, {}_j^it_l^n\, e^{I({}_ik_n^- - {}_jk_l^-)s} = {}_j\gamma_l^n$$
$$(j = 3,4;\ l = 1,\ldots,\mathcal{N}_j) \tag{18}$$

Assuming that the scattering coefficients ${}_j^it_m^n$ of the sub-scattering problem has already been obtained by using the mode-exciting method and the semi-length of the finite crack s is given, the unknown variables in Eqs. (13) and (18) are ${}_1^1r_l^n$ and ${}_2^1r_l^n$ for $l=1,\ldots,\mathcal{N}_1$, and ${}_i\gamma_m^n$ and ${}_i\delta_m^n$ for $i=3,4$ and $m=1,\ldots,\mathcal{N}_i$. The total number of the unknown variables for a fixed n is $2(\mathcal{N}_1+\mathcal{N}_3+\mathcal{N}_4)$. On the other hand, the total number of equations in Eqs. (13) and (18) is $2(\mathcal{N}_1+\mathcal{N}_3+\mathcal{N}_4)$. Since the number of equations is equal to that of the unknown variables, Eqs. (13) and (18) can be solved simultaneously to obtain ${}_1^1r_l^n$, ${}_2^1r_l^n$, which are the solution of the main scattering problem.

It is remarked here that in the above derivation of the partial analysis method, the nonpropagating modes in the plates ③ and ④ are neglected on the assumption that these nonpropagating modes will sufficiently evanesce before they reach the edges of the crack and their contributions to the scattering process in the vicinity of the edges of the crack are very small. In fact, this assumption is acceptable only for a long enough crack. For a relatively small crack, the partial analysis method would give an approximation solution rather than an exact solution of the main scattering problem.

5. Numerical results

5.1 Accuracy of the mode-exciting method

The accuracy of the mode-exciting method is verified by comparing the numerical result obtained by the mode-exciting method to exact solutions found by Rokhlin[14] for the sub-scattering problem with symmetrically located semi-infinite crack (y=0). One of the exact solutions found by Rokhlin is

$$\left|{}_3^3t_n^n \pm {}_3^4t_n^n\right| = \prod_{m=1}^{\mathcal{N}_1} \frac{|{}_1k_m - {}_3k_n|}{|{}_1k_m + {}_3k_n|} \prod_{\substack{m=1\\ m\neq n}}^{\mathcal{N}_3} \frac{|{}_3k_m + {}_3k_n|}{|{}_3k_m - {}_3k_n|}, \tag{19}$$

where the first product includes multipliers with wavenumbers ${}_1k_m$ of only symmetric or only antisymmetric propagating modes depending on the sign (plus or minus) used in the left hand side, and the second product includes all the multipliers for both the symmetric and antisymmetric propagating modes. The values of $\left|{}_3^3t_n^n \pm {}_3^4t_n^n\right|$ obtained by the mode-exciting method and by the exact solution of Eq. (19) are shown in Fig. 4 by symbols and curves, respectively, as a function of the nondimensional frequency $\omega h/c_T$. A fairly good agreement between both values can be observed in this figure and the validity of the mode-exciting method is, therefore, numerically verified.

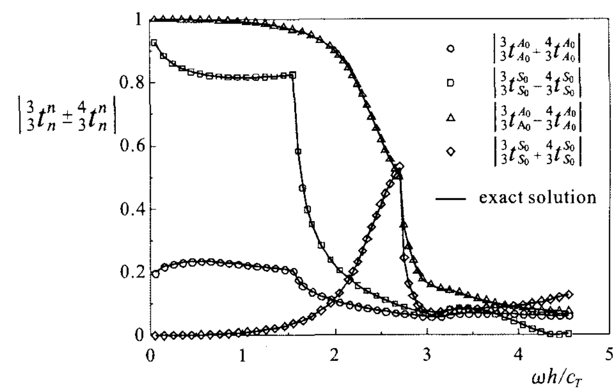

Fig. 4 Comparison between the numerical results obtained by the mode-exciting method and the exact solution[14] of the sub-scattering problem with y=0.

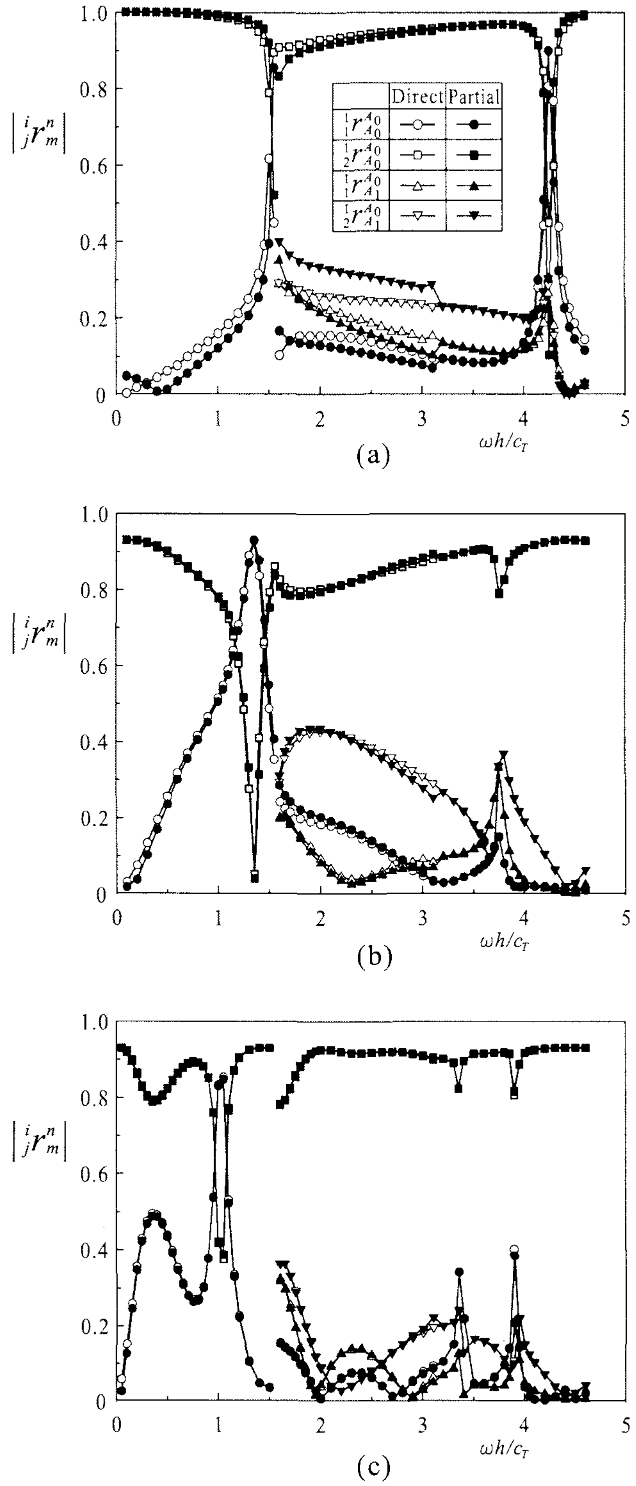

Fig. 5 Scattering coefficients for horizontal cracks with $y=0$ and various lengths of (a) $s/h=0.5$, (b) $s/h=1$, and (c) $s/h=2$.

5.2 Comparison between results of the partial analysis method and the direct mode-exciting method

Here we show two examples for comparison between results obtained by the partial analysis method and by the direct mode-exciting method. First, we consider the scattering problem by a finite horizontal crack located at the center plane of the plate ($y=0$). A_0 mode is considered here as an incident wave. The results are shown in Figs. 5(a), (b), and (c) for the cracks with the lengths $s/h=0.5$, 1, and 2, respectively. The abscissa and ordinate represent the nondimensional frequency $\omega h/c_T$ and the absolute of scattering coefficients ${}_j^i r_m^n$, respectively. The unshaded and the shaded symbols denote the scattering coefficients obtained by the direct mode-exciting method and by the partial analysis method, respectively. Note that these scattering problems are symmetric with respect to the center plane of the plate and, therefore, no symmetric mode appears for the incidence of A_0 mode. Comparison of both results in these figures shows that the results of the partial analysis method are almost in good agreement with those of the direct method. The error in the partial analysis method is, however, relatively large when the crack length is small as shown in Fig. 5(a). The error is due to the neglection of nonpropagating modes in the plates ③ and ④ in the partial analysis method. It can be seen that the error decreases above $\omega h/c_T=\pi$. This frequency is the cut-off frequency of A_1 mode in the plates ③ and ④ (Note that the semi-thickness of these plates is $h/2$). As the frequency increases and becomes close to $\omega h/c_T=\pi$, the contribution of the nonpropagating A_1 mode, which has a quite small imaginary part of wavenumber, to the scattering processes in the plates ③ and ④ becomes significant and, therefore, the neglection of A_1 mode causes large error. The error decreases at the frequency higher than $\omega h/c_T=\pi$, where A_1 mode becomes a propagating mode in the plates ③ and ④ and is taken into account in the partial analysis method. As the crack length s/h increases, the error in the partial analysis method decreases, as shown in Figs. 5(b) and (c). The reason of the decrement of the error is that the contribution of nonpropagating modes to the scattering process in the plates ③ and ④ decreases as the crack becomes longer.

Next, numerical results are shown for the cracks located at $y=0.25h$ subjected to A_0 mode incidence. The results of the partial analysis method and the direct method are shown in Figs. 6(a), (b), and (c) for the cracks with the lengths $s/h=0.5$, 1, and 2, respectively. Since the problem is not symmetric, symmetric modes as well as antisymmetric modes appear as scattered waves even if the antisymmetric A_0 mode is incident to the crack. Although scattered A_1, S_1, and S_2 modes also occur at the frequencies higher than their cut-off frequencies, the results of scattered A_0 and S_0 modes only are shown in these figures. From these figures, we can find that the results of the partial analysis method are in fairly good agreement with the results obtained by the mode-exciting method as the crack becomes longer. Particularly, at the frequency $\omega h/c_T > \pi$, both results show good agreement.

It is also of interest to consider the case when the system of linear equations (13) and (18) becomes nearly ill-posed. Figs. 7(a) and (b) show the values of determinants of Eqs. (13) and (18) for the cracks with y=0 and y=0.25h, respectively. In these figures, the determinants become close to zero at some frequencies which depend on the crack length s/h. The zero determinant means that a free vibration occurs in the plates ③ and ④. At the frequencies where the resonance occurs, large reflection coefficients are often found. In fact, the reflection coefficients in Figs. 5 and 6 become maximum at the frequencies where the determinant are close to zero in Figs. 7(a) and (b). Note that since both results of the direct mode-exciting method results and the partial analysis method show the resonance phenomena at almost the same frequencies, the resonance phenomenon is an intrinsic scattering characteristic of the finite crack which is independent of the analysis methods. Since resonance phenomena are discussed by Rokhlin[15], no further discussion in given here.

5.3 Application of the partial analysis method to the inverse scattering analysis

In the analysis of the main scattering problem by the mode-exciting method as well as other numerical methods, the numerical analysis such as FEM and BEM must be carried out repeatedly as the crack lenght changes. As shown in Section 4, in the partial analysis method the FEM or BEM calculation is, however, carried out only once (when the sub-scattering problem is solved); the scattering coefficients ${}^{i}_{j}r^{n}_{m}$ of the main scattering problem can be found only by solving a system of linear equations with the crack length as a parameter. The partial analysis method can, therefore, solve the scattering problems for several plates with single cracks of various lengths much faster than the mode-exciting method alone. Taking this advantage, we will show the application of the partial analysis method to an inverse Lamb wave-scattering analysis. The problem in the inverse analysis is to determine the location y and the length s of an unknown horizontal crack when the scattering coefficients ${}^{i}_{j}r^{n}_{m}$ are known in advance.

Let ${}^{i}_{j}r^{n}_{m}(\omega_k)$ be the scattering coefficients at several frequencies $\{\omega_k\}$ from which the crack length and location will be estimated. When the scattering coefficients ${}^{i}_{j}t^{n}_{m}$ of the sub-scattering problem have already been obtained for several crack's locations $\{y_q\}$ at the frequencies $\{\omega_k\}$, the approximated scattering coefficients of the main scattering problem for several crack lengths $\{s_p\}$ can be evaluated easily by the partial analysis method. Let ${}^{i}_{j}R^{n}_{m}(\omega_k, s_p, y_q)$ be the approximated scattering coefficients of the main scattering problem evaluated by the partial analysis method. The location y and the length s of the unknown crack can be estimated after evaluating $E(s_p, y_q)$, which is

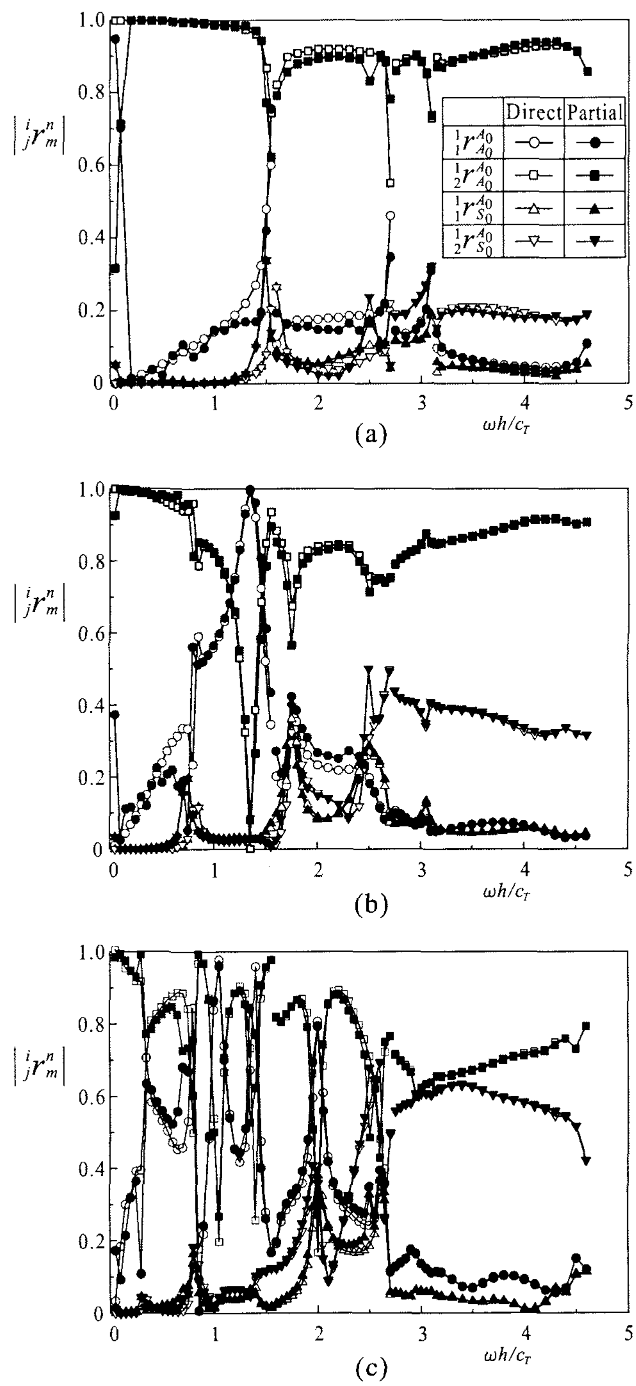

Fig. 6 Scattering coefficients for horizontal cracks with y=0.25h and various lengths of (a) s/h=0.5, (b) s/h=1, and (c) s/h=2.

defined in Eq. (20), for all sets of $\{s_p\}$ and $\{y_q\}$ and finding the values of s_p and y_q for which $E(s_p, y_q)$ becomes minimum.

$$E(s_p, y_q) = \frac{1}{N_\omega N_a} \sum_{k=1}^{N_\omega} \sum_{i,j,n,m} \left| {}^{i}_{j}R^{n}_{m}(\omega_k, s_p, y_q) - {}^{i}_{j}r^{n}_{m}(\omega_k) \right|^2 , \qquad (20)$$

where N_ω is the number of frequencies ω_k at which ${}^{i}_{j}r^{n}_{m}(\omega_k)$ are known and N_a is the number of sets

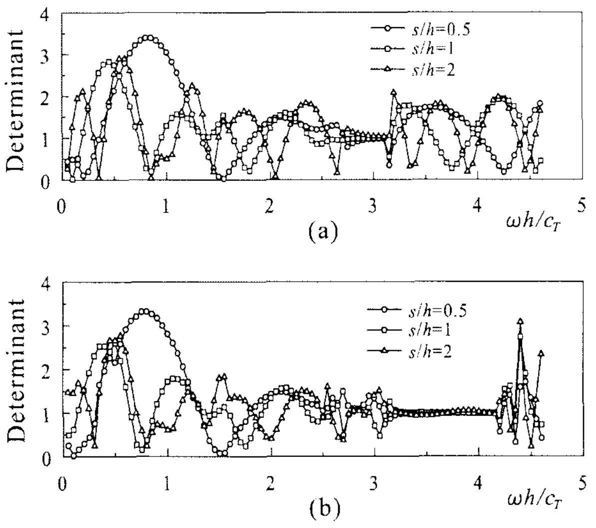

Fig. 7 Values of determinants of matrices when Eqs. (13) and (18) are solved for the cracks which are located (a) at the center plane of the plate and (b) at the distance y=0.25h from the center plate of the plate.

$\{i, j, n, m\}$ used in the summation.

Numerical examples of the inverse analysis are performed for two sets of frequencies: $\omega h/c_T = \{1.6, 1.65, \ldots, 2.6\}$ and $\omega h/c_T = \{3.5, 3.55, \ldots, 4.5\}$. The scattering coefficients obtained by the direct mode-exciting method for the cracks with the lengths s/h=0.5, 1.0, and 2.0 which are located at y/h=0 and 0.25 are used here as the known scattering coefficients ${}_j^i r_m^n(\omega_k)$ in Eq. (20). The values of $E(s, y)$ as functions of s and y are shown in Figs. 8 and 9 for the former and the latter set of frequencies, respectively. Here, $E(s, y)$ are calculated by using all set of $\{i, j, n, m\}$ in Eq. (20) and are shown for $0 \leq s/h \leq 3$ with the interval $\Delta s/h$=0.01 and $0 \leq y/h < 1$ with the interval $\Delta y/h$=0.05. It is noted that to make the locations of the minimums obvious in Figs. 8 and 9, only the values of $E(s, y)$ which are relatively near to zero are shown. It can be seen from Figs. 8 and 9 that the locations of the minimums of $E(s, y)$ are in agreement with the true values of s and y. As an exception, the estimated y obtained from Fig. 8(b) seems to be smaller than the exact value of y. This is due to the large error of the partial analysis method for a short crack, as mentioned in the previous section. By comparison between Figs. 8 and 9, we also find that the values of y can be estimated more accurately by Fig. 9 than by Fig. 8. The reason is that as the frequency becomes higher, the displacement distribution of A_0 mode become more complicated in the thickness direction. In other words, the higher frequency component of the Lamb wave is more sensitive to the resolution in y direction.

In practice, it is not easy to obtain absolute phase

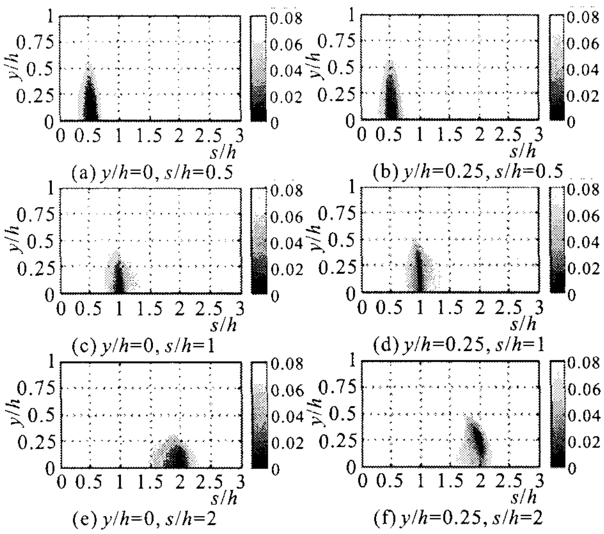

Fig. 8 $E(s, y)$ calculated using the set of frequencies $\omega h/c_T = \{1.6, 1.65, \ldots, 2.6\}$.

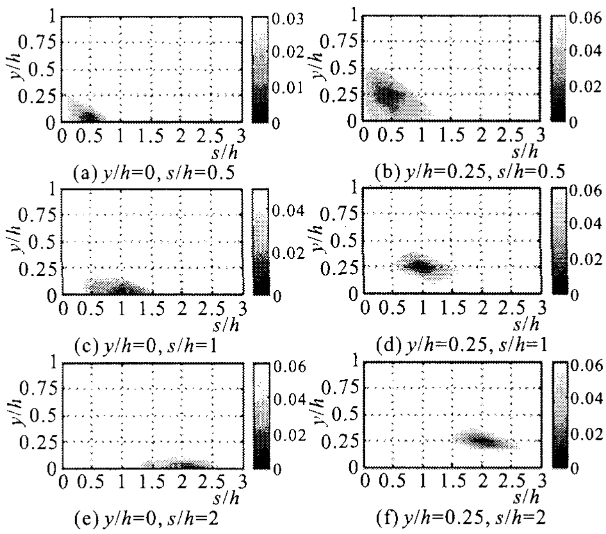

Fig. 9 $E(s, y)$ calculated using the set of frequencies $\omega h/c_T = \{3.5, 3.55, \ldots, 4.5\}$.

information from measurement. For such a case, the inversion scheme based on the absolute values of scattering coefficients is developed as shown below. The indicator E is modified as follows:

$$E'(s, y) = \frac{1}{N_\omega N_a} \sum_{k=1}^{N_\omega} \sum_{i,j,n,m} \left| \left| {}_j^i R_m^n(\omega_k, s, y) \right| - \left| {}_j^i r_m^n(\omega_k) \right| \right|^2. \tag{21}$$

Figs. 10(a)-(h) show the values of $E(s, y)$ and $E'(s, y)$ for several combinations of $\{i, j, n, m\}$ which are evaluated for the same target values s/h=1 and y/h=0.25, and the same set of frequencies $\omega h/c_T =$

$\{3.5, 3.55, \ldots, 4.5\}$. First, we consider the case that all scattering coefficients are available, *i.e.*, all set of $\{i, j, n, m\}$ are used. The values of $E(s, y)$ and $E'(s, y)$ for this case are shown in Figs. 10(a) and (b), respectively. From these figures, it is clear that the lack of phase information makes the location of the minimum ambiguous. In addition to the minimum at *the exact values, a local minimum* is also observed at s/h=2.5 and y/h=0.15, as well as the minimum value of $E'(s, y)$ at s/h=1 and y/h=0.25 in Fig. 10(b).

Secondly, we consider the case that only reflection coefficients of the same mode with the incident mode are used in the inversion. The set of $\{i, j, n, m\}$ is then restricted to i=j and m=n. The values of $E(s, y)$ and $E'(s, y)$ for this case are shown in Figs. 10(c) and (d), respectively. For this case, the length and the position of the crack can be estimated only if the phase information is available, but the location of the minimum is not clear.

Thirdly, we consider the case that only transmission coefficients of the same mode with the incident mode are available. The set of $\{i, j, n, m\}$ is then restricted to $i \neq j$ and m=n. The values of $E(s, y)$ and $E'(s, y)$ for this case are shown in Figs. 10(e) and (f), respectively. The length and position of the crack can be estimated clearly when the phase information is available, but the estimation becomes ambiguous when the phase information is lost.

Lastly, Figs. 10(g) and (h) show the values of $E(s, y)$ for the case that only A_0 mode is observable. The values of $E(s, y)$ estimated from the reflection coefficient ${}_1^1r_{A_0}^{A_0}$ and the transmission coefficient ${}_2^1r_{A_0}^{A_0}$ are shown in Figs. 10(g) and (h), respectively. In Fig. 10(g), it is impossible to specify a minimum point. On the other hand,a minimum value is observed in Fig. 10(h), although the location of the minimum is a little different from the exact location.

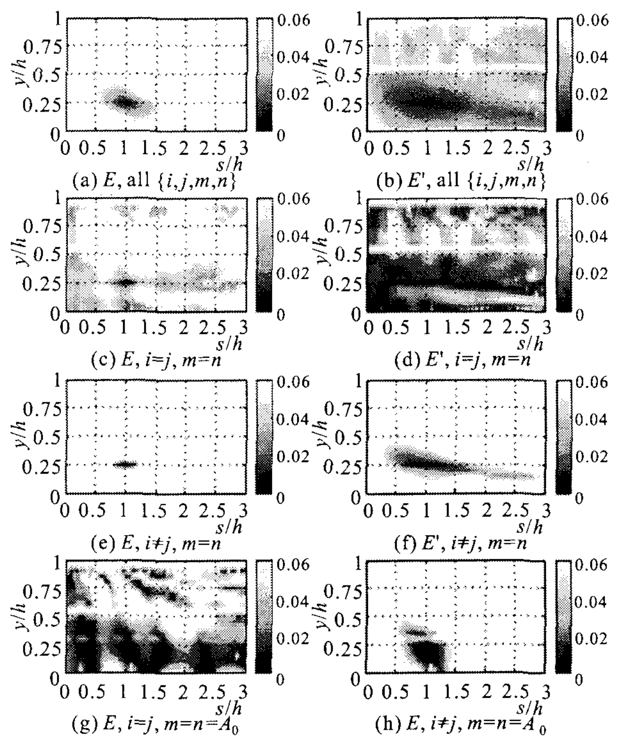

Fig. 10 Comparison of $E(s, y)$ and $E'(s, y)$ for several combinations of $\{i, j, n, m\}$.

6. Conclusions

The partial analysis method has been developed to solve approximately the scattering problem of Lamb waves by a horizontal finite crack in a plate. The partial analysis method is powerful for the scattering analysis of Lamb waves by cracks of various lengths at the same transversal location. The results of the scattering analysis of Lamb waves obtained by the partial analysis method are found to be in good agreement with those obtained by the direct mode-exciting method, especially for a long crack. However, errors of the partial analysis method become relatively large at the frequencies at which a nonpropagating Lamb wave mode in the region around crack has a small imaginary part of wave number.

The application of the partial analysis method to the inverse analysis method has also been proposed. It is shown that the length and the transversal location of finite cracks can be estimated. It is found that the phase information and the number of available scattering coefficients are important to estimate the length and the transversal location of crack clearly.

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