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Proposal of the Concept of the "Tangential Relaxation"

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The novel concept "*tangential stress rate relaxation*", abbreviated as "*tangential relaxation*", is proposed in order to predict rigorously the plastic instability phenomena in which the stress rate has a tangential component deviating severely from the proportional loading. Further, the constitutive equation based on this concept is formulated.

Key Words: Elastoplasticity, Relaxation, Subloading surface model, Tangential-stress rate

1. Introduction

The following facts are generally observed in the elasto-plastic deformation behavior of real materials.

1) The magnitude of the inelastic strain rate depends not only on the component of stress rate normal to the yield surface, called the *normal-stress rate*, but also on the component of stress rate tangential to the yield surface, called the *tangential-stress rate.*

2) The direction of the inelastic strain rate depends not only on the stress but also on the stress rate.

3) Thus, the *non-coaxiality*, i.e., the discordance of principal axes of the plastic strain rate and the stress is exhibited.

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However, the traditional elastoplastic constitutive equation, which has a single smooth yield surface and plastic potential surface and which derives the plastic strain rate based on the consistency condition, is incapable of describing these facts. Then, it has problems in the analysis of the deformation behavior for the loading path that deviates significantly from proportional loading as observed in plastic instability phenomena with localization and/or bifurcation of the deformation; these tend to the prediction of unrealistically stiff mechanical response leading to an excessively high critical load. Then, the extended constitutive equation accounting for these facts has to be formulated. Here, the following facts also have to be considered in the formulation.

4) Rudnicki and Rice (1975) showed that "no vertex can result from hydrostatic stress increments" based on the consideration of the sliding mechanism in a fissure model. For soils, it has been experimentally observed that the contribution of the isotropic part of the tangential stress rate to the inelastic deformation behavior is small compared with the deviatoric part (cf. e.g. Poorooshasb et al., 1966; Lewin and Burland, 1970; Tatsuoka and Ishihara, 1974; Pradel et al, 1990; Gutierrez et al., 1991, 1991). Further, this fact has been verified in numerical experiments based on the discrete element method for granular media (Bardet, 1993, 1994; Kishino and Wu, 1999; Kishino, 2002). Besides, for metals, if inelastic strain rate is induced by the hydrostatic component of the tangential stress rate, it leads to the physically unacceptable prediction that an inelastic volumetric strain rate is induced by that component. Thus, it might be assumed that only the deviatoric part of the tangential-stress rate, called the *deviatoric tangential-stress rate*, influences the inelastic deformation behavior.

5) The direction of the tangential-inelastic strain rate induced by the tangential-stress rate would have the components not only tangential but also outward-normal to the yield surface, as has been found in various experimental and theoretical studies: test data of metals (Ito *et al.*, 1992) and soils (Tatsuoka and Ishihara, 1974; Pradel *et al.*, 1990; Gutierrez *et al.*, 1991, 1993); numerical experiments for metals based on the *KBW model* (Kroner, 1961; Budiansky and Wu, 1962) by Ito (1979) and the *Taylor polycrystalline model* (Taylor, 1938; Asaro and Needleman, 1985; Kuroda and Tvergaard, 1999) by Kuroda and Tvergaard (2001); and numerical experiments for granular media based on the *discrete element method* by Kishino and Wu (1999) Kishino (2002).

In this article, while the relaxation relevant to the normal-stress rate, called the "*normal relaxation*", is induced in the traditional plastic constitutive equation, the novel concept that the relaxation relevant to the deviatoric part of tangential-stress rate, called the *tangential relaxation*, is also induced is proposed in order to take into account the aforementioned fact 4). Then, incorporating this concept into the subloading surface model (Hashiguchi and Ueno, 1977; Hashiguchi, 1980, 1989), the extended constitutive equation is formulated, which is referred to as the *tangential-subloading surface model*. Further, based on it, the concrete constitutive equation of metals is formulated and then it is verified that the equation is capable of describing all of the aforementioned mechanical properties 1)-5).

2. Outline of the Subloading Surface Model

The subloading surface model (Hashiguchi and Ueno, 1977; Hashiguchi, 1980, 1989) is reviewed briefly, which will be later applied to the prediction of soil deformation behavior.

Denoting the current configuration of material particle as $\mathbf{x}$ and the current velocity as $\mathbf{v}$, the velocity gradient is described as $\mathbf{L} = \partial\mathbf{v}/\partial\mathbf{x}$ by which the strain rate and the continuum spin are defined as $\mathbf{D} \equiv (\mathbf{L}+\mathbf{L}^{\mathrm{T}})/2$ and $\mathbf{W} \equiv (\mathbf{L}-\mathbf{L}^{\mathrm{T}})/2$, respectively, $(\ )^{\mathrm{T}}$ standing for the transpose. Let the strain rate be additively decomposed into the elastic strain rate $\mathbf{D}^e$ and the inelastic strain rate $\mathbf{D}^p$, i.e.

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p, \tag{1}$$

where $\mathbf{D}^e$ is given by

$$\mathbf{D}^e = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}}. \tag{2}$$

$\boldsymbol{\sigma}$ is the Cauchy stress and $(^\circ)$ indicates the proper corotational rate and the fourth-order tensor $\mathbf{E}$ is the elastic modulus.

Let the following yield condition be assumed.

$$f(\hat{\boldsymbol{\sigma}},\mathbf{H}) = F(H), \tag{3}$$

where

$$\hat{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \boldsymbol{\alpha}. \tag{4}$$

The scalar H and the second-order tensor $\mathbf{H}$ are the isotropic and the anisotropic hardening variables, respectively, $\boldsymbol{\alpha}$ is the kinematic hardening variable, i.e. the back stress. The function f is assumed to be homogeneous of degree one in the $\hat{\boldsymbol{\sigma}}$.

The subloading surface model falls within the framework of the unconventional plastic constitutive equation (Drucker, 1988) enabling to describe the plastic strain rate due to the rate of stress inside the yield surface by excluding the premise that the interior of the yield surface is a purely elastic domain, while the conventional yield surface is renamed as the *normal-yield surface*. Then, the following *subloading surface* is introduced, which always passes through the current stress point and also keeps a shape similar to the normal-yield surface and the orientation of similarity to the normal-yield surface with respect to the similarity-center $\mathbf{s}$. The degree of approach to the normal-yield state can be described by the ratio $R\ (0 \le R \le 1)$ of the size of the subloading surface to that of the normal-yield surface. Thus, the variable R is called the *normal-yield ratio*. Then, it holds that

$$\boldsymbol{\sigma}_y = \frac{1}{R}\{\boldsymbol{\sigma}-(1-R)\mathbf{s}\} \quad (\boldsymbol{\sigma}-\mathbf{s} = R(\boldsymbol{\sigma}_y-\mathbf{s})), \tag{5}$$

where $\boldsymbol{\sigma}_y$ on the normal-yield surface is the *conjugate stress* of the current stress $\boldsymbol{\sigma}$ on the subloading surface. By substituting Eq. (5) into Eq. (3) (regarding $\boldsymbol{\sigma}$ in Eq. (3) as $\boldsymbol{\sigma}_y$), the subloading surface is described as

$$f(\overline{\boldsymbol{\sigma}},\mathbf{H}) = RF(H), \tag{6}$$

where

$$\overline{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \overline{\boldsymbol{\alpha}}\ \ (= R\hat{\boldsymbol{\sigma}}_y), \tag{7}$$

$$\hat{\boldsymbol{\sigma}}_y \equiv \boldsymbol{\sigma}_y - \boldsymbol{\alpha}, \tag{8}$$

$$\overline{\boldsymbol{\alpha}} \equiv \mathbf{s} - R(\mathbf{s}-\boldsymbol{\alpha}) \quad (\overline{\boldsymbol{\alpha}}-\mathbf{s} = R(\boldsymbol{\alpha}-\mathbf{s})). \tag{9}$$

$\overline{\boldsymbol{\alpha}}$ in the subloading surface is the conjugate point of $\boldsymbol{\alpha}$ in the normal-yield surface. In the calculation, R has to be calculated first by substituting current values of $\boldsymbol{\sigma}, H, \boldsymbol{\alpha}, \mathbf{H}, \mathbf{s}$ into Eq. (6), and thereafter $\overline{\boldsymbol{\alpha}}$ is calculated by Eq. (9).

The material-time derivative of Eq. (6) is given as

$$\mathrm{tr}\Big(\frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\boldsymbol{\sigma}}\Big) - \mathrm{tr}\Big(\frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\overline{\boldsymbol{\alpha}}}\Big) + \mathrm{tr}\Big(\frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial\mathbf{H}}\overset{\circ}{\mathbf{H}}\Big) = \dot{R}F + RF'\dot{H}, \tag{10}$$

where

$$F' \equiv \frac{dF}{dH} \tag{11}$$

and $(\dot{\ })$ stands for the material-time derivative. Here, the material-time derivative can be transformed to the corotational rate $(^\circ)$ (cf. Hashiguchi, 2003). Eq. (10) as it is cannot play the role of the consistency condition for the derivation of plastic strain rate since it contains the variables that are not related to the plastic strain rate. Then, consider below to transform it to the consistency condition.

As observed in experiments, the stress asymptotically approaches the normal-yield surface in the plastic loading process $\mathbf{D}^p \neq \mathbf{0}$. Thus, the following evolution equation of the normal-yield ratio R is assumed.

$$\dot{R} = U(R)\|\mathbf{D}^p\| \ \text{for}\ \mathbf{D}^p \neq \mathbf{0}, \tag{12}$$

where $\|\ \|$ denotes the magnitude and U is a monotonically decreasing function of the normal-yield ratio R, fulfilling the following conditions.

$$U(R) = \begin{cases} \infty & \text{for } R = 0, \\ 0 & \text{for } R = 1, \end{cases} \quad (U(R) < 0 \ \text{for}\ R > 1). \tag{13}$$

Let the function U satisfying Eq. (13) be simply given by

$$U(R) = -u\ln R, \tag{14}$$

where u is a material constant.

The Similarity-center $\mathbf{s}$ has to lie inside the normal-yield surface. Then, it has to hold that

$$f(\hat{\mathbf{s}},\mathbf{H}) \le F(H), \tag{15}$$

$$\hat{\mathbf{s}} \equiv \mathbf{s} - \boldsymbol{\alpha}. \tag{16}$$

The time-differentiation of Eq. (15) in the ultimate state $f(\hat{\mathbf{s}},\mathbf{H}) = F(H)$ where $\mathbf{s}$ lies on normal-yield surface leads to:

$$\mathrm{tr}\Big[\frac{\partial f(\hat{\mathbf{s}},\,\mathbf{H})}{\partial \mathbf{s}}\Big(\mathring{\mathbf{s}}-\mathring{\boldsymbol{\alpha}}+\frac{1}{F}\Big\{\mathrm{tr}\Big(\frac{\partial f(\hat{\mathbf{s}},\,\mathbf{H})}{\partial \mathbf{H}}\mathring{\mathbf{H}}\Big)-\dot{F}\Big\}\hat{\mathbf{s}}\Big)\Big]\leq 0$$

$$\text{for } f(\hat{\mathbf{s}},\,\mathbf{H})=F(H), \qquad (17)$$

while the relation $\mathrm{tr}[\{\partial f(\hat{\mathbf{s}},\,\mathbf{H})/\partial\mathbf{s}\}\hat{\mathbf{s}}]=F$ due to the Euler's theorem for homogeneous function is used for deriving Eq. (17). The inequality (15) or (17) is called the *enclosing condition for the similarity-center*. In the ultimate state $f(\hat{\mathbf{s}},\,\mathbf{H})=F(H)$, the vector $\boldsymbol{\sigma}_y-\mathbf{s}\ (=(\boldsymbol{\sigma}-\mathbf{s})/R)$ makes an obtuse angle with the vector $\partial f(\hat{\mathbf{s}},\,\mathbf{H})/\partial\mathbf{s}$ which is the outward-normal to the similarity-center surface $f(\hat{\mathbf{s}},\,\mathbf{H})=F(H)$ coinciding with the normal-yield surface, provided that the normal-yield surface is convex. Noting this fact and considering the fact that the similarity-center moves only with the plastic deformation, let the following equation be assumed so as to fulfill the inequality (17):

$$\mathring{\mathbf{s}}-\mathring{\boldsymbol{\alpha}}+\frac{1}{F}\Big\{\mathrm{tr}\Big(\frac{\partial f(\hat{\mathbf{s}},\,\mathbf{H})}{\partial \mathbf{H}}\mathring{\mathbf{H}}\Big)-\dot{F}\Big\}\hat{\mathbf{s}}=c\,\|\mathbf{D}^p\|\frac{\tilde{\boldsymbol{\sigma}}}{R} \qquad (18)$$

from which the translation rule of the similarity-center is given as follows:

$$\mathring{\mathbf{s}}=c\|\mathbf{D}^p\|\frac{\tilde{\boldsymbol{\sigma}}}{R}+\mathring{\boldsymbol{\alpha}}+\frac{1}{F}\Big\{F'\dot{H}-\mathrm{tr}\Big(\frac{\partial f(\hat{\mathbf{s}},\,\mathbf{H})}{\partial \mathbf{H}}\mathring{\mathbf{H}}\Big)\Big\}\hat{\mathbf{s}}, \qquad (19)$$

where c is a material constant influencing the translating rate of the similarity-center and

$$\tilde{\boldsymbol{\sigma}}\equiv\boldsymbol{\sigma}-\mathbf{s}. \qquad (20)$$

It is conceivable that the similarity-center $\mathbf{s}$ approaches the current stress $\boldsymbol{\sigma}$ as can be seen from the simple case of the nonhardening state ($\mathring{\boldsymbol{\alpha}}=\mathring{\mathbf{H}}=\mathbf{0}$, $\dot{H}=0$), although the translation rule (19) is assumed to fulfill the requirement (17) in the ultimate state $f(\hat{\mathbf{s}},\,\mathbf{H})=F(H)$.

Substitution of Eqs. (9), (12) and (19) into Eq. (10) leads to the *consistency condition* for the subloading surface:

$$\mathrm{tr}\Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\partial \boldsymbol{\sigma}}\mathring{\boldsymbol{\sigma}}\Big)-\mathrm{tr}\Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\partial \boldsymbol{\sigma}}\mathring{\bar{\boldsymbol{\alpha}}}\Big)+\mathrm{tr}\Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\partial \mathbf{H}}\mathring{\mathbf{H}}\Big)$$

$$=U\|\mathbf{D}^p\|F+RF'\dot{H} \qquad (21)$$

with

$$\mathring{\bar{\boldsymbol{\alpha}}}=R\,\mathring{\boldsymbol{\alpha}}+(1-R)\mathring{\mathbf{s}}-\hat{\mathbf{s}}\,U\|\mathbf{D}^p\|. \qquad (22)$$

Adopt the associated flow rule

$$\mathbf{D}^p=\bar{\lambda}\,\bar{\mathbf{N}}, \qquad (23)$$

where $\bar{\lambda}$ is a positive proportionality factor and the second-order tensor $\bar{\mathbf{N}}$ denotes the normalized outward-normal to the subloading surface, i.e.

$$\bar{\mathbf{N}}\equiv\frac{\partial f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\partial\boldsymbol{\sigma}}\Big/\Big\|\frac{\partial f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\partial\boldsymbol{\sigma}}\Big\| \quad (\|\bar{\mathbf{N}}\|=1). \qquad (24)$$

The proportionality factor $\bar{\lambda}$ is obtained by substituting Eq. (23) into Eq. (21) with Eq. (22)leads to

$$\bar{\lambda}=\frac{\mathrm{tr}(\bar{\mathbf{N}}\mathring{\boldsymbol{\sigma}})}{\bar{M}^p}, \qquad (25)$$

and thus

$$\mathbf{D}^p=\frac{\mathrm{tr}(\bar{\mathbf{N}}\mathring{\boldsymbol{\sigma}})}{\bar{M}^p}\bar{\mathbf{N}}, \qquad (26)$$

where

$$\bar{M}^p\equiv\mathrm{tr}\Big[\bar{\mathbf{N}}\Big(\bar{\mathbf{a}}+\Big\{\frac{F'}{F}h-\frac{1}{RF}\mathrm{tr}\Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\partial\mathbf{H}}\mathbf{h}\Big)+\frac{U}{R}\Big\}\bar{\boldsymbol{\sigma}}\Big)\Big] \qquad (27)$$

The variables h, $\mathbf{h}$ and $\bar{\mathbf{a}}$ are functions of the stress, plastic internal state variables and $\bar{\mathbf{N}}$ of homogeneous degree one, while these functions are related to $\dot{H}$, $\mathring{\mathbf{H}}$ and $\mathring{\boldsymbol{\alpha}}$ as

$$h\equiv\frac{\dot{H}}{\bar{\lambda}},\ \mathbf{h}\equiv\frac{\mathring{\mathbf{H}}}{\bar{\lambda}},\ \mathbf{a}\equiv\frac{\mathring{\boldsymbol{\alpha}}}{\bar{\lambda}}, \qquad (28)$$

$$\bar{\mathbf{a}}\equiv\frac{\mathring{\bar{\boldsymbol{\alpha}}}}{\bar{\lambda}}=R\mathbf{a}+(1-R)\mathbf{z}-U\hat{\mathbf{s}}, \qquad (29)$$

$$\mathbf{z}\equiv\frac{\mathring{\mathbf{s}}}{\bar{\lambda}}=c\frac{\tilde{\boldsymbol{\sigma}}}{R}+\mathbf{a}+\frac{1}{F}\Big\{F'h-\mathrm{tr}\Big(\frac{\partial f(\hat{\mathbf{s}},\,\mathbf{H})}{\partial\mathbf{H}}\mathbf{h}\Big)\Big\}\hat{\mathbf{s}}, \qquad (30)$$

while the following relation due to the Euler's theorem for homogeneous function is used for deriving Eq. (25).

$$\frac{\partial f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\partial\bar{\boldsymbol{\sigma}}}=\frac{\mathrm{tr}\Big(\dfrac{\partial f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\partial\bar{\boldsymbol{\sigma}}}\bar{\boldsymbol{\sigma}}\Big)}{\mathrm{tr}(\bar{\mathbf{N}}\bar{\boldsymbol{\sigma}})}\bar{\mathbf{N}}=\frac{f(\bar{\boldsymbol{\sigma}},\,\mathbf{H})}{\mathrm{tr}(\bar{\mathbf{N}}\bar{\boldsymbol{\sigma}})}\bar{\mathbf{N}}$$

$$=\frac{RF}{\mathrm{tr}(\bar{\mathbf{N}}\bar{\boldsymbol{\sigma}})}\bar{\mathbf{N}}. \qquad (31)$$

The strain rate is given from Eqs. (1), (2) and (26) as

$$\mathbf{D}=\mathbf{E}^{-1}\mathring{\boldsymbol{\sigma}}+\frac{1}{\bar{M}^p}\mathrm{tr}(\bar{\mathbf{N}}\mathring{\boldsymbol{\sigma}})\bar{\mathbf{N}}. \qquad (32)$$

3. Tangential Relaxation

It can be written from Eq. (32) that

$$\mathring{\boldsymbol{\sigma}}=\mathbf{E}\mathbf{D}-\frac{1}{\bar{M}^p}\mathbf{E}\,\mathring{\boldsymbol{\sigma}}_N, \qquad (33)$$

where $\mathring{\boldsymbol{\sigma}}_N$ is the *normal-stress rate*, i.e.

$$\mathring{\boldsymbol{\sigma}}_N\equiv\mathrm{tr}(\bar{\mathbf{N}}\mathring{\boldsymbol{\sigma}})\bar{\mathbf{N}}=(\bar{\mathbf{N}}\otimes\bar{\mathbf{N}})\mathring{\boldsymbol{\sigma}}. \qquad (34)$$

It is observed in Eq. (33) that the relaxation relevant to the normal-stress rate is induced. Let it be called the "*normal relaxation*". Now, in order to take into account the fact 4) described in the introduction, let it be postulated that the relaxation is induced also in the deviatoric-tangential direction to the subloading surface. Let it be referred to as the "*deviatoric-tangential relaxation*", abbreviated as "*tangential relaxation*". In what follows, let the extended subloading surface model, called the *tangential-subloading surface model*, be formulated.

Now, let Eq. (33) be extended as

$$\mathring{\boldsymbol{\sigma}}=\mathbf{E}\Big(\mathbf{D}-\frac{1}{\bar{M}^p}\mathring{\boldsymbol{\sigma}}_N-\frac{1}{\bar{M}^t}\mathring{\boldsymbol{\sigma}}^*_T\Big), \qquad (35)$$

where $\mathring{\boldsymbol{\sigma}}^*_T$ is called the *tangential-deviatoric relaxation stress* *rate* and is given as

$$\overset{\circ}{\boldsymbol{\sigma}}_T^* \equiv \Big(\frac{\overset{\circ}{\boldsymbol{\sigma}}_t^*}{\|\overset{\circ}{\boldsymbol{\sigma}}_t^*\|} + d_n \bar{\mathbf{n}}^*\Big)\|\overset{\circ}{\boldsymbol{\sigma}}_t^*\|, \tag{36}$$

or

$$\overset{\circ}{\boldsymbol{\sigma}}_T^* \equiv \overset{\circ}{\boldsymbol{\sigma}}_t^* + d_n \bar{\mathbf{n}}^* \|\overset{\circ}{\boldsymbol{\sigma}}_t^*\|, \tag{37}$$

where the deviatoric-tangential stress rate $\overset{\circ}{\boldsymbol{\sigma}}_t^*$ is given as follows:

$$\overset{\circ}{\boldsymbol{\sigma}}^* = \overset{\circ}{\boldsymbol{\sigma}}_n^* + \overset{\circ}{\boldsymbol{\sigma}}_t^*, \tag{38}$$

$$\left.\begin{aligned} &\overset{\circ}{\boldsymbol{\sigma}}_n^* = \mathrm{tr}(\bar{\mathbf{n}}^*\overset{\circ}{\boldsymbol{\sigma}})\bar{\mathbf{n}}^* = (\bar{\mathbf{n}}^*\otimes\bar{\mathbf{n}}^*)\overset{\circ}{\boldsymbol{\sigma}} = (\bar{\mathbf{n}}^*\otimes\bar{\mathbf{n}}^*)\overset{\circ}{\boldsymbol{\sigma}}, \\ &\overset{\circ}{\boldsymbol{\sigma}}_t^* = \overset{\circ}{\boldsymbol{\sigma}}^* - \overset{\circ}{\boldsymbol{\sigma}}_n^* = (\breve{\mathbf{I}}^* - \bar{\mathbf{n}}^*\otimes\bar{\mathbf{n}}^*)\overset{\circ}{\boldsymbol{\sigma}} \end{aligned}\right\} \tag{39}$$

$$\bar{\mathbf{n}}^* \equiv \Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial \boldsymbol{\sigma}}\Big)^* \Big/ \Big\|\Big(\frac{\partial f(\bar{\boldsymbol{\sigma}},\mathbf{H})}{\partial \boldsymbol{\sigma}}\Big)^*\Big\| = \frac{\bar{\mathbf{N}}^*}{\|\bar{\mathbf{N}}^*\|} \quad (\|\bar{\mathbf{n}}^*\| = 1). \tag{40}$$

$(\ )^*$ stands for the deviatoric component and $\breve{\mathbf{I}}^*$ is the fourth-order *deviatoric transformation tensor*, i.e.

$$\breve{I}_{ijkl}^* \equiv \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}. \tag{41}$$

The material function $\bar{M}^t$, called the *tangential-relaxation modulus*, is a monotonically decreasing function of R and is simply given by

$$\bar{M}^t = \frac{1}{\xi R^n}, \tag{42}$$

where n is a material constant and ξ is a material parameter which is a function of stress and plastic internal variables in general: a material constant for metals and a function of stress for frictional materials. d_n is a material constant by which the relaxation is induced in the direction not only tangential but also inward-normal to the subloading surface .

The strain rate is expressed in terms of the stress rate from Eqs. (36) and (39) as

$$\mathbf{D} = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} + \frac{1}{\bar{M}^p}\mathrm{tr}(\bar{\mathbf{N}}\overset{\circ}{\boldsymbol{\sigma}})\bar{\mathbf{N}} + \frac{1}{\bar{M}^t}(\overset{\circ}{\boldsymbol{\sigma}}_t^* + d_n\|\overset{\circ}{\boldsymbol{\sigma}}_t^*\|\bar{\mathbf{n}}^*). \tag{43}$$

Then, the strain rate is additively decomposed into the elastic strain rate $\mathbf{D}^e$ and the inelastic strain rate $\mathbf{D}^i$, while the latter is further additively decomposed into the plastic strain rate $\mathbf{D}^p$ and the tangential strain rate $\mathbf{D}^t$, i.e.

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^i, \quad \mathbf{D}^i = \mathbf{D}^p + \mathbf{D}^t, \tag{44}$$

while the tangential strain rate is given for Eq. (43) as follows:

$$\mathbf{D}^t = \frac{1}{\bar{M}^t}(\overset{\circ}{\boldsymbol{\sigma}}_t^* + d_n\|\overset{\circ}{\boldsymbol{\sigma}}_t^*\|\bar{\mathbf{n}}^*). \tag{45}$$

The tangential strain rate equation (45) does not fulfill the exact-differential form, i.e. the complete integrability condition with respect to the stress rate (cf. Hashiguchi, 1980), while the elastic strain rate equation (2) fulfills it.

The positive proportionality factor in the associated flow rule (26) is expressed in terms of strain rate with the tangential stress rate, rewriting $\bar{\lambda}$ by $\bar{\Lambda}$, from Eq. (43) as follows:

$$\bar{\Lambda} = \frac{\mathrm{tr}(\bar{\mathbf{N}}\mathbf{E}\mathbf{D}) - \frac{1}{\bar{M}^t}\mathrm{tr}\{\bar{\mathbf{N}}\mathbf{E}(\overset{\circ}{\boldsymbol{\sigma}}_t^* + d_n\bar{\mathbf{n}}^*\|\overset{\circ}{\boldsymbol{\sigma}}_t^*\|)\}}{\bar{M}^p + \mathrm{tr}(\bar{\mathbf{N}}\mathbf{E}\bar{\mathbf{N}})} \quad \Big(= \frac{\mathrm{tr}(\bar{\mathbf{N}}\overset{\circ}{\boldsymbol{\sigma}})}{\bar{M}^p}\Big) \tag{46}$$

The loading criterion for the plastic strain rate is given as follows (Hashiguchi, 2000):

$$\left.\begin{aligned} &\mathbf{D}^p \neq \mathbf{0}: \ \bar{\Lambda} > 0, \\ &\mathbf{D}^p = \mathbf{0}: \ \text{otherwise} \end{aligned}\right\} \tag{47}$$

or

$$\left.\begin{aligned} &\mathbf{D}^p \neq \mathbf{0}: \ \mathrm{tr}(\bar{\mathbf{N}}\mathbf{E}\mathbf{D}) - \frac{1}{\bar{M}^t}\mathrm{tr}\{\bar{\mathbf{N}}\mathbf{E}(\overset{\circ}{\boldsymbol{\sigma}}_t^* + d_n\bar{\mathbf{n}}^*\|\overset{\circ}{\boldsymbol{\sigma}}_t^*\|)\} > 0, \\ &\mathbf{D}^p = \mathbf{0}: \text{otherwise} \end{aligned}\right\} \tag{48}$$

since it can be assumed that $\bar{M}^p + \mathrm{tr}(\bar{\mathbf{N}}\mathbf{E}\bar{\mathbf{N}}) > 0$, while the tangential strain rate $\mathbf{D}^t$ is always induced for $\overset{\circ}{\boldsymbol{\sigma}}_t^* \neq \mathbf{0}$. Eq. (43) is rate-nonlinear and thus an inverse expression becomes rather complicated form. It should be noted that the loading criterion has to be defined essentially by the sign of the proportionality factor $\bar{\Lambda}$ as has been revealed by Hashiguchi (2000), while it has been defined merely by the quantity $\mathrm{tr}(\bar{\mathbf{N}}\mathbf{E}\mathbf{D})$ in the traditional elastoplastic constitutive equation after Hill (1958, 1967) and even in the past tangential-subloading surface model (Hashiguchi and Tsutsumi, 2001, 2003; Hashiguchi and Protasov, 2004; Khojastehpour and Hashiguchi, 2004a, b).

Hereafter, assume that the elastic modulus tensor E is given by Hooke's type, i.e.

$$E_{ijkl} = \big(K - \tfrac{2}{3}G\big)\delta_{ij}\delta_{kl} + G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{49}$$

where K and G are the elastic bulk modulus and the elastic shear modulus, respectively, which leads to the relations $\mathrm{tr}(\mathbf{SET}^*) = 2G\mathrm{tr}(\mathbf{ST}^*)$ and $\breve{\mathbf{I}}^*\mathbf{ET} = \mathbf{ET}^* = 2G\mathbf{T}^*$ for arbitrary second-order tensors $\mathbf{S}$ and $\mathbf{T}$. Then , it holds from Eq. (35) that

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{ED} - \frac{\mathrm{tr}(\bar{\mathbf{N}}\overset{\circ}{\boldsymbol{\sigma}})}{\bar{M}^p}\mathbf{E}\bar{\mathbf{N}} - \frac{2G}{\bar{M}^t}(\overset{\circ}{\boldsymbol{\sigma}}_t^* + d_n\bar{\mathbf{n}}^*\|\overset{\circ}{\boldsymbol{\sigma}}_t^*\|), \tag{50}$$

It is obtained from Eq. (50) that

$$\overset{\circ}{\boldsymbol{\sigma}}^* = 2G\Big\{\mathbf{D}^* - \frac{\mathrm{tr}(\bar{\mathbf{N}}\overset{\circ}{\boldsymbol{\sigma}})}{\bar{M}^p}\bar{\mathbf{N}}^* - \frac{1}{\bar{M}^t}(\overset{\circ}{\boldsymbol{\sigma}}_t^* + d_n\|\overset{\circ}{\boldsymbol{\sigma}}_t^*\|\bar{\mathbf{n}}^*)\Big\} \tag{51}$$

from which one has

$$\mathrm{tr}(\bar{\mathbf{n}}^*\overset{\circ}{\boldsymbol{\sigma}}^*)\bar{\mathbf{n}}^* = 2G\Big\{\mathrm{tr}(\bar{\mathbf{n}}^*\mathbf{D}^*) - \frac{\mathrm{tr}(\bar{\mathbf{N}}\overset{\circ}{\boldsymbol{\sigma}})}{\bar{M}^p}\mathrm{tr}(\bar{\mathbf{n}}^*\bar{\mathbf{N}}^*) - \frac{d_n}{\bar{M}^t}\|\overset{\circ}{\boldsymbol{\sigma}}_t^*\|\Big\}\bar{\mathbf{n}}^*. \tag{52}$$

Noting $\bar{\mathbf{N}}_t^* \ (= \bar{\mathbf{N}}^* - \mathrm{tr}(\bar{\mathbf{n}}^*\bar{\mathbf{N}}^*)\}\bar{\mathbf{n}}^*) = 0$, the subtraction of Eq. (52) from Eq. (51) leads to

$$\overset{\circ}{\boldsymbol{\sigma}}_t^* = \frac{2G\bar{M}^t}{\bar{M}^t + 2G}\mathbf{D}_t^*, \tag{53}$$

where the deviatoric tangential-stress rate $\overset{\circ}{\boldsymbol{\sigma}}_t^*$ is proportional to the deviatoric tangential-strain rate.

Substituting Eq. (53) and noting

$$\mathrm{tr}(\overline{\mathbf{N}}\mathbf{E}\overset{\circ}{\boldsymbol{\sigma}}_t^*)\ (=2G\mathrm{tr}(\overline{\mathbf{N}}\overset{\circ}{\boldsymbol{\sigma}}_t^*)=0, \qquad (54)$$

Eq. (46) reduces to

$$\bar{\Lambda}=\frac{\mathrm{tr}(\overline{\mathbf{N}}\mathbf{E}\mathbf{D})-d_n\dfrac{2G}{\bar{M}^t+2G}\mathrm{tr}(\overline{\mathbf{N}}\bar{\mathbf{n}}^*)\|\mathbf{D}_t^*\|}{\bar{M}^p+\mathrm{tr}(\overline{\mathbf{N}}\mathbf{E}\mathbf{N})}. \qquad (55)$$

The inverse expression, i.e. the analytical expression of stress rate in terms of strain rate is derived as follows:

$$\overset{\circ}{\boldsymbol{\sigma}}=\mathbf{E}\mathbf{D}-\frac{\mathrm{tr}(\overline{\mathbf{N}}\mathbf{E}\mathbf{D})-d_n\dfrac{2G}{\bar{M}^t+2G}\mathrm{tr}(\overline{\mathbf{N}}\bar{\mathbf{n}}^*)\|\mathbf{D}_t^*\|}{\bar{M}^p+\mathrm{tr}(\overline{\mathbf{N}}\mathbf{E}\mathbf{N})}\mathbf{E}\overline{\mathbf{N}}$$
$$-\frac{(2G)^2}{\bar{M}^t+2G}(\mathbf{D}_t^*+d_n\|\mathbf{D}_t^*\|\bar{\mathbf{n}}^*), \qquad (56)$$

where the stress rate $\overset{\circ}{\boldsymbol{\sigma}}$ is additively decomposed into and the elastic stress rate $\overset{\circ}{\boldsymbol{\sigma}}^e$, the plastic stress rate $\overset{\circ}{\boldsymbol{\sigma}}^p$ and the tangential stress rate $\overset{\circ}{\boldsymbol{\sigma}}^t$ as

$$\overset{\circ}{\boldsymbol{\sigma}}=\overset{\circ}{\boldsymbol{\sigma}}^e+\overset{\circ}{\boldsymbol{\sigma}}^p+\overset{\circ}{\boldsymbol{\sigma}}^t, \qquad (57)$$

setting

$$\left.\begin{aligned}&\overset{\circ}{\boldsymbol{\sigma}}^e\equiv\mathbf{E}\mathbf{D},\\ &\overset{\circ}{\boldsymbol{\sigma}}^p\equiv-\mathbf{E}\mathbf{D}^p=-\frac{\mathrm{tr}(\overline{\mathbf{N}}\mathbf{E}\mathbf{D})-d_n\dfrac{2G}{\bar{M}^t+2G}\mathrm{tr}(\overline{\mathbf{N}}\bar{\mathbf{n}}^*)\|\mathbf{D}_t^*\|}{\bar{M}^p+\mathrm{tr}(\overline{\mathbf{N}}\mathbf{E}\mathbf{N})}\mathbf{E}\overline{\mathbf{N}},\\ &\overset{\circ}{\boldsymbol{\sigma}}^t\equiv-\mathbf{E}\mathbf{D}^t=-\frac{(2G)^2}{\bar{M}^t+2G}(\mathbf{D}_t^*+d_n\|\mathbf{D}_t^*\|\bar{\mathbf{n}}^*)\end{aligned}\right\} \qquad (58)$$

The loading criterion is given as

$$\left.\begin{aligned}&\mathbf{D}^p\neq\mathbf{0}:\mathrm{tr}(\overline{\mathbf{N}}\mathbf{E}\mathbf{D})-d_n\frac{2G}{\bar{M}^t+2G}\mathrm{tr}(\overline{\mathbf{N}}\bar{\mathbf{n}}^*)\|\mathbf{D}_t^*\|>0,\\ &\mathbf{D}^p=\mathbf{0}:\text{otherwise}\end{aligned}\right\} \qquad (59)$$

For $\overline{\mathbf{N}}=\bar{\mathbf{n}}^*$ (pressure-independent yield surface) Eqs. (55) and (56) reduce to

$$\bar{\Lambda}=\frac{2G}{\bar{M}^p+2G}\left\{\mathrm{tr}(\overline{\mathbf{N}}\mathbf{D})-d_n\frac{2G}{\bar{M}^t+2G}\|\mathbf{D}_t^*\|\right\}, \qquad (60)$$

$$\overset{\circ}{\boldsymbol{\sigma}}=\mathbf{E}\mathbf{D}-\frac{(2G)^2}{\bar{M}^p+2G}\left\{\mathrm{tr}(\overline{\mathbf{N}}\mathbf{D})-d_n\frac{2G}{\bar{M}^t+2G}\|\mathbf{D}_t^*\|\right\}\overline{\mathbf{N}}$$
$$-\frac{(2G)^2}{\bar{M}^t+2G}(\mathbf{D}_t^*+d_n\|\mathbf{D}_t^*\|\overline{\mathbf{N}}) \qquad (61)$$

The plastic strain rate (26) is obtained by substituting the associated flow rule (23) into the consistency condition which is obtained by incorporating the evolution rule (12) of the normal-yield ratio R into the time-differentiation (10) of Eq. (6) for the subloading surface. Then, the plastic loading process develops gradually as the stress approaches the yield surface, exhibiting a *smooth elastic-plastic transition.* Thus, the subloading surface model fulfills the *smoothness condition* (Hashiguchi, 1993a, b, 1997, 2000) defined as "the stress rate induced by the identical strain rate changes continuously for a continuous change of stress state". This can be expressed mathematically as follows:

$$\lim_{\delta\boldsymbol{\sigma}\to\mathbf{0}}\overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma}+\delta\boldsymbol{\sigma},\ \mathbf{S}_i,\mathbf{D})=\overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma},\ \mathbf{S}_i,\mathbf{D}), \qquad (62)$$

where $\overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma},\mathbf{S}_i,\mathbf{D})$ designates the stress rate induced by the strain rate $\mathbf{D}$ for the sate of stress $\boldsymbol{\sigma}$ and the identical internal variables $\mathbf{S}_i$ (i=1, 2, 3, ･･･, m) which denotes collectively scalar- or tensor-valued internal state variables describing the alteration of the mechanical response property due to the irreversible deformation, $\delta(\ )$ stands for an infinitesimal variation. The rate-linear constitutive equation can be described as

$$\overset{\circ}{\boldsymbol{\sigma}}=\mathbf{M}^{ep}(\boldsymbol{\sigma},\ \mathbf{S}_i)\mathbf{D}, \qquad (63)$$

where the fourth-order tensor $\mathbf{M}^{ep}$ is the elastoplastic modulus which is the function of stress and internal variables and can be formulated as

$$\mathbf{M}^{ep}=\frac{\partial\overset{\circ}{\boldsymbol{\sigma}}}{\partial\mathbf{D}}. \qquad (64)$$

Therefore, Eq. (62) can be rewritten as

$$\lim_{\delta\boldsymbol{\sigma}\to\mathbf{0}}\mathbf{M}^{ep}(\boldsymbol{\sigma}+\delta\boldsymbol{\sigma},\ \mathbf{S}_i)=\mathbf{M}^{ep}(\boldsymbol{\sigma},\ \mathbf{S}_i). \qquad (65)$$

Thus, the subloading surface model and its extension to the tangential relaxation have the notable advantages as follows:

1) It predicts a smooth response (e.g. a smooth axial stress-axial logarithmic strain relation in the uniaxial loading) for a smooth monotonic loading. By contrast, a nonsmooth response is predicted by constitutive models violating the smoothness condition as in the conventional plasticity with the yield surface enclosing a purely elastic domain.
2) Only the decision for the sign of the proportionality factor Λ is required in the loading criterion of the subloading surface model, since the stress always lies on the subloading surface, which now plays the role of the loading surface, while the determination of whether or not the stress lies on the yield surface is not required. On the other hand, the judgment whether or not the yield condition is also fulfilled is required in conventional plasticity.
3) A stress is automatically drawn back to the normal-yield surface even if it goes out from that surface since it is formulated that $\dot{R}>0$ for $R<1$ (subyield state) and $\dot{R}<0$ for $R>1$ (over the normal-yield state) in Eq. (12) with the condition (13).). Thus, a rough calculation with a large loading step is allowed in the subloading surface model when the explicit method is adopted in numerical calculation.
4) The tangential relaxation is induced gradually as the stress approaches the normal-yield surface, fulfilling not only the smoothness condition but also the *continuity condition* (Ha-

shiguchi, 1993a, b, 1997; 2000) defined as "the stress rate changes continuously for a continuous change of the strain rate" which is expressed mathematically as follows:

$$\lim_{\delta\mathbf{D}\to 0}\overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma},\ \mathbf{S}_i,\ \mathbf{D}+\delta\mathbf{D})=\overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma},\ \mathbf{S}_i,\ \mathbf{D}). \qquad (66)$$

On the other hand, the tangential-strain rate is induced suddenly at the moment when the stress reaches the yield surface in the other tangential-plasticity models violating the smoothness condition, e.g. Rudnicki and Rice's (1975) and Papamichos and Vardoulakis's (1995) models, and thus the continuity condition is also violated in these models. Further, the unconventional models other than the subloading surface model, e.g. the multi surface model and the two surface model also violate both the smoothness and the continuity conditions if the tangential strain rate is incorporated. Therefore, they lead to the serious defect that the uniqueness of solution is violated for the stress path along the yield surface.

4. Examination of mechanical response

Let the concrete constitutive equation of metals be formulated based on the tangential-subloading surface model formulated in the preceding section. Then, let the mechanical response be specified in this section.

4. 1 Constitutive equation of metals

Let the von Mises yield condition with the isotropic-kinematic hardening (Hashiguchi and Yoshimaru, 1995) and the associated flow rule be adopted for the normal-yield/subloading surfaces:

$$f(\bar{\boldsymbol{\sigma}})=\sqrt{\frac{3}{2}}\|\bar{\boldsymbol{\sigma}}^*\|, \qquad (67)$$

$$F(H)=F_0[1+h_1\{1-\exp(-h_2H)\}], \qquad (68)$$

$$\dot{H}=\sqrt{\frac{2}{3}}\mathrm{tr}(\bar{\mathbf{N}}\mathbf{D}^p)\ \left(=\sqrt{\frac{2}{3}}\lambda=\sqrt{\frac{2}{3}}\|\mathbf{D}^p\|\right),\quad h=\sqrt{\frac{2}{3}}, \qquad (69)$$

$$\overset{\circ}{\boldsymbol{\alpha}}=\left(k_1\frac{\bar{\boldsymbol{\sigma}}^*}{\|\bar{\boldsymbol{\sigma}}^*\|}-k_2\boldsymbol{\alpha}\right)\|\mathbf{D}^p\|,\quad \mathbf{a}\equiv k_1\frac{\bar{\boldsymbol{\sigma}}^*}{\|\bar{\boldsymbol{\sigma}}^*\|}-k_2\boldsymbol{\alpha}. \qquad (70)$$

The variables k_1, k_1, h_1 and h_2 are material constants, and F_0 is the initial value of F. The functions in the plastic strain rate (26) with Eqs. (14) and (24) are given from Eqs. (67)-(70) as

$$\bar{\mathbf{N}}=\bar{\mathbf{N}}^*=\bar{\mathbf{n}}^*=\frac{\bar{\boldsymbol{\sigma}}^*}{\|\bar{\boldsymbol{\sigma}}^*\|}, \qquad (71)$$

$$\bar{M}^p=\mathrm{tr}\left[\bar{\mathbf{N}}\left\{\bar{\mathbf{a}}+\left(\sqrt{\frac{2}{3}}\frac{F'}{F}-\frac{u\ln R}{R}\right)\bar{\boldsymbol{\sigma}}\right\}\right], \qquad (72)$$

$$F'=F_0h_1h_2\exp(-h_2H). \qquad (73)$$

In what follows, in order to exhibit concisely the mechanical properties, let the simple constitutive equation fulfilling the isotropy with $\boldsymbol{\alpha}=\mathbf{0}$ ($k_1=k_2=0$) and $\mathbf{s}=\mathbf{0}$ ($c=0$) resulting in $\bar{\boldsymbol{\alpha}}=\mathbf{0}$ be examined, leading to

$$\mathrm{tr}(\bar{\mathbf{N}}\bar{\boldsymbol{\sigma}})=\sqrt{\frac{2}{3}}RF, \qquad (74)$$

$$\bar{M}^p=\sqrt{\frac{2}{3}}\left\{\sqrt{\frac{2}{3}}RF'-(u\ln R)F\right\}. \qquad (75)$$

For the elastic modulus, Hooke's type in Eq. (49) is adopted.

4. 2 Mechanical response

The mechanical features of the constitutive equation of metals formulated in the preceding sub-section is examined by analyzing the response in the π-plane. In what follows, only the principal components are shown since all the variables of the inputs and outputs have only the principal components because of the mechanical isotropy.

1) Strain rate response

Consider the response of the strain rate $\mathbf{D}$ to the input of the deviatoric stress rate $\overset{\circ}{\boldsymbol{\sigma}}^*\ (=\overset{\circ}{\boldsymbol{\sigma}})$ in the π-plane, which is given as

$$\begin{Bmatrix}\overset{\circ}{\sigma}_1\\ \overset{\circ}{\sigma}_2\\ \overset{\circ}{\sigma}_3\end{Bmatrix}=\sqrt{\frac{2}{3}}\|\overset{\circ}{\boldsymbol{\sigma}}\|\begin{Bmatrix}\cos\theta\\ \cos\{\theta-(2/3)\pi)\}\\ \cos\{\theta+(2/3)\pi)\}\end{Bmatrix}, \qquad (76)$$

keeping the magnitude of stress rate to be constant, i.e. $\|\overset{\circ}{\boldsymbol{\sigma}}\|=$ const. from the state of stress

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{Bmatrix}=\frac{1}{3}RF\begin{Bmatrix}2\\ -1\\ -1\end{Bmatrix}, \qquad (77)$$

resulting in

$$\begin{Bmatrix}\bar{N}_1\\ \bar{N}_2\\ \bar{N}_3\end{Bmatrix}=\frac{1}{\sqrt{6}}\begin{Bmatrix}2\\ -1\\ -1\end{Bmatrix}, \qquad (78)$$

$$\mathrm{tr}(\bar{\mathbf{N}}\overset{\circ}{\boldsymbol{\sigma}})=\|\overset{\circ}{\boldsymbol{\sigma}}\|\cos\theta, \qquad (79)$$

$$\begin{Bmatrix}(\overset{\circ}{\boldsymbol{\sigma}}_t^*)_1\\ (\overset{\circ}{\boldsymbol{\sigma}}_t^*)_2\\ (\overset{\circ}{\boldsymbol{\sigma}}_t^*)_3\end{Bmatrix}=\frac{1}{\sqrt{2}}\|\overset{\circ}{\boldsymbol{\sigma}}\|\sin\theta\begin{Bmatrix}0\\ 1\\ -1\end{Bmatrix}, \qquad (80)$$

where

$$\cos 3\theta\equiv\sqrt{6}\,\mathrm{tr}\left(\frac{\overset{\circ}{\boldsymbol{\sigma}}^*}{\|\overset{\circ}{\boldsymbol{\sigma}}^*\|}\right)^3, \qquad (81)$$

θ standing for the angle measured in the clock-wise direction from the σ_1^*-direction to the direction of stress rate $\overset{\circ}{\boldsymbol{\sigma}}$ on the π-plane.

Now, let the following unit vector $\bar{\mathbf{T}}$ be introduced.

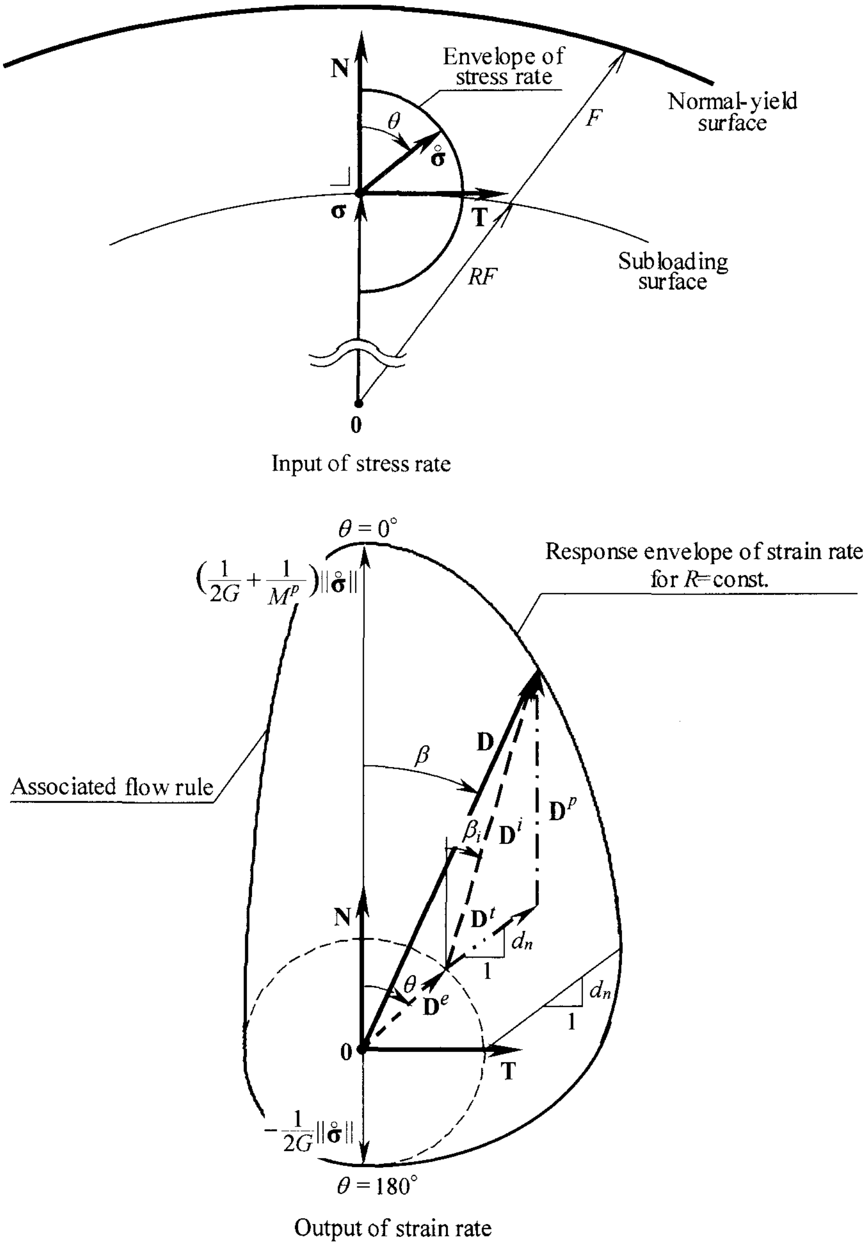

Fig. 1. Illustration of response envelope of $\mathbf{D}$ (= $\mathbf{D}^*$) to the input of the stress rate $\overset{\circ}{\boldsymbol{\sigma}}(=\overset{\circ}{\boldsymbol{\sigma}}{}^*)$ with a constant magnitude, i.e. $\|\overset{\circ}{\boldsymbol{\sigma}}\|$ $(=\|\overset{\circ}{\boldsymbol{\sigma}}{}^*\|)$=const. for metals.

$$\overline{\mathbf{T}} = \begin{Bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{Bmatrix} \tag{82}$$

which has the direction rotated $\pi/2$ in clock-wise direction from $\overline{\mathbf{N}}$ and thus is tangential to the subloading surface. Consider the two-dimensional orthogonal coordinate system with the unit base vectors $\overline{\mathbf{N}}$ and $\overline{\mathbf{T}}$. Further, let the directions of the coordinate axes of strain rate coincide with those of stress rate, and let components in the directions $\overline{\mathbf{N}}$ and $\overline{\mathbf{T}}$ be denoted as $(\ )_n$ and $(\ )_t$, respectively. Then, it holds from Eqs. (76) -(82) that

$$\left.\begin{aligned} D_n &= \mathrm{tr}(\overline{\mathbf{N}}\mathbf{D}) = \left\{\left(\frac{1}{2G}+\frac{1}{\overline{M}^p}\right)\cos\theta + \frac{d_n}{\overline{M}^t}|\sin\theta|\right\}\|\overset{\circ}{\boldsymbol{\sigma}}\|, \\ D_t &= \mathrm{tr}(\overline{\mathbf{T}}\mathbf{D}) = \left(\frac{1}{2G}+\frac{1}{\overline{M}^t}\right)\|\overset{\circ}{\boldsymbol{\sigma}}\|\sin\theta \end{aligned}\right\} \tag{83}$$

$$\left.\begin{aligned} D_n^e &= \frac{1}{2G}\|\overset{\circ}{\boldsymbol{\sigma}}\|\cos\theta\,, \\ D_t^e &= \frac{1}{2G}\|\overset{\circ}{\boldsymbol{\sigma}}\|\sin\theta \end{aligned}\right\} \tag{84}$$

$$\left.\begin{aligned} D_n^p &= \mathrm{tr}(\overline{\mathbf{N}}\mathbf{D}^p) = \frac{\mathrm{tr}(\overline{\mathbf{N}}\overset{\circ}{\boldsymbol{\sigma}})}{\overline{M}^p} = \frac{1}{\overline{M}^p}\|\overset{\circ}{\boldsymbol{\sigma}}\|\cos\theta, \\ D_t^p &= \mathrm{tr}(\overline{\mathbf{T}}\mathbf{D}^p) = 0 \end{aligned}\right\} \tag{85}$$

$$D_n^t = \mathrm{tr}(\bar{\mathbf{N}}\mathbf{D}^t) = \frac{d_n}{\bar{M}^t}\|\mathring{\boldsymbol{\sigma}}\||\sin\theta|,\quad D_t^t = \mathrm{tr}(\bar{\mathbf{T}}\mathbf{D}^t) = \frac{1}{\bar{M}^t}\|\mathring{\boldsymbol{\sigma}}\|\sin\theta \tag{86}$$

It has to be put in Eqs. (83) and (85) that $1/\bar{M}^p = 0$ for $\pi/2 < \theta \le \pi$ leading to $\bar{\Lambda} \le 0$.

2) Relaxation response (stress rate response)

Consider the response of the stress rate $\mathring{\boldsymbol{\sigma}}$ ($= \mathring{\boldsymbol{\sigma}}^*$) to the input of the deviatoric strain rate $\mathbf{D}(=\mathbf{D}^*)$ in the π-plane, keeping the magnitude of stress rate to be constant, i.e. $\|\mathbf{D}\| =$ const. from the stress state of Eq. (77) resulting in Eq. (78) and (82), while $\mathbf{D}$ is given as

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \|\mathbf{D}\| \begin{Bmatrix} \cos\beta \\ \cos\{\beta-(2/3)\pi\} \\ \cos\{\beta+(2/3)\pi\} \end{Bmatrix}, \tag{87}$$

where

$$\cos 3\beta \equiv \sqrt{6}\,\mathrm{tr}\left(\frac{\mathbf{D}^*}{\|\mathbf{D}^*\|}\right)^3. \tag{88}$$

Here, it holds from Eqs. (78), (82) and (87) that

$$\mathrm{tr}(\bar{\mathbf{N}}\mathbf{D}) = \|\mathbf{D}\|\cos\beta, \tag{89}$$

$$\mathbf{D}_t^* = \frac{1}{\sqrt{2}}\|\mathbf{D}\| \begin{Bmatrix} 0 \\ \sin\beta \\ -\sin\beta \end{Bmatrix},\quad \|\mathbf{D}_t^*\| = \|\mathbf{D}\||\sin\beta|. \tag{90}$$

The stress rates are described in the coordinate system with the bases ($(\bar{\mathbf{N}}, \bar{\mathbf{T}})$ as follows:

$$\begin{aligned} \mathring{\sigma}_n &= 2G\|\mathbf{D}\|\Big[\cos\beta - \frac{2G/\bar{M}^p}{1+2G/\bar{M}^p}\Big\{\cos\beta - d_n\frac{2G/\bar{M}^t}{1+2G/\bar{M}^t}|\sin\beta|\Big\} - d_n\frac{2G/\bar{M}^t}{1+2G/\bar{M}^t}|\sin\beta|\Big] \\ \mathring{\sigma}_t &= \frac{2G}{1+2G/\bar{M}^t}\|\mathbf{D}\|\sin\beta \end{aligned} \tag{91}$$

$$\mathring{\sigma}_n^e = 2G\|\mathbf{D}\|\cos\beta,\quad \mathring{\sigma}_t^e = 2G\|\mathbf{D}\|\sin\beta \tag{92}$$

$$\mathring{\sigma}_n^p = -2G\frac{2G/\bar{M}^p}{1+2G/\bar{M}^p}\|\mathbf{D}\|\Big\{\cos\beta - d_n\frac{2G/\bar{M}^t}{1+2G/\bar{M}^t}|\sin\beta|\Big\},\quad \mathring{\sigma}_t^p = 0 \tag{93}$$

$$\mathring{\sigma}_n^t = -2Gd_n\frac{2G/\bar{M}^t}{1+2G/\bar{M}^t}\|\mathbf{D}\||\sin\beta|,\quad \mathring{\sigma}_t^t = -2G\frac{2G/\bar{M}^t}{1+2G/\bar{M}^t}\|\mathbf{D}\|\sin\beta \tag{94}$$

provided that $1/\bar{M}^p = 0$ for $\bar{\Lambda} \le 0$ due to the loading criterion (47), while $\bar{\Lambda}$ is given as follows:

$$\bar{\Lambda} = \frac{2G/\bar{M}^p}{1+2G/\bar{M}^p}\|\mathbf{D}\|\Big(\cos\beta - d_n\frac{2G/\bar{M}^t}{1+2G/\bar{M}^t}|\sin\beta|\Big). \tag{95}$$

The relaxation response is illustrated in Fig. 2.

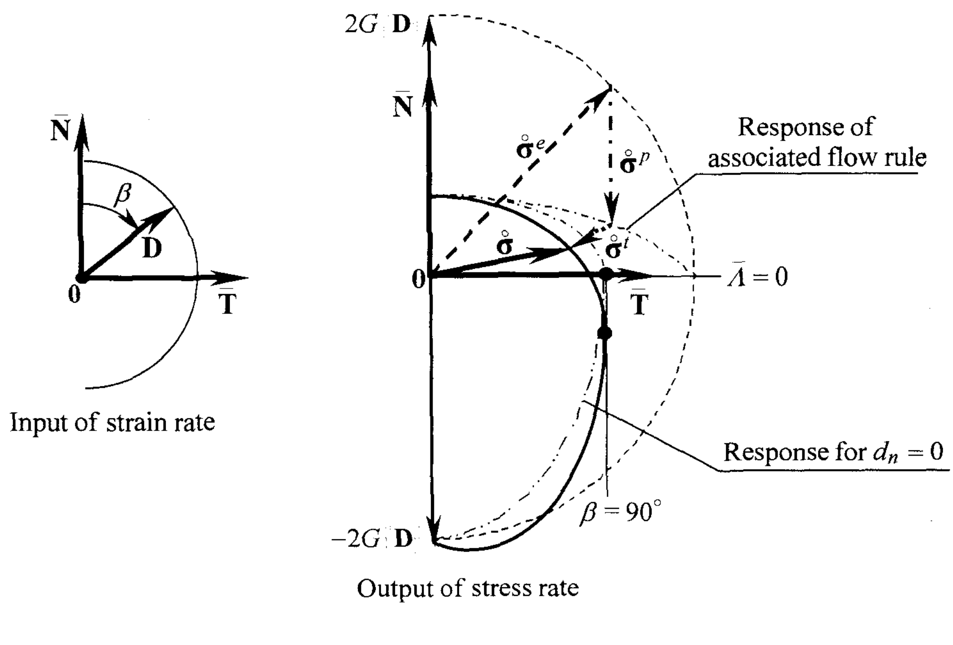

Fig. 2. Response envelopes of stress rates to the input of strain rate, i.e, the relaxation for three levels of the material parameter for metals.

5. Concluding Remarks

Numerous models have been proposed up to the present in order to extend the elastoplastic constitutive equations so as to realistically describe the dependence of the inelastic strain rate on the tangential-stress rate. The rigorous formulation could be provided in this by incorporating the novel concept of the tangential relaxation. The fundamental features of the present model are as follows:

i) Single smooth surface is adopted for the normal-yield surface.

ii) Both the plastic and the tangential strain rates are gradually induced as the stress approaches the normal-yield surface, where the smoothness condition is always fulfilled. Then, the present model keeps the fundamental features of the subloading surface model described in the foregoing, i.e. the loading criterion without the judgment whether or not the stress lies on the normal-yield surface, the controlling function that the stress automatically approaches the normal-yield surface during a plastic loading process and the expression of smooth mechanical response for a smooth loading path.

iii) Both the magnitude and the direction of the plastic strain rate depend on the tangential-stress rate.

iv) The tangential strain rate is oriented between the outward-normal and tangential directions to the subloading surface; the direction can be controlled by the material parameter d_n . Here, note that $d_n \neq 0$ causing the outward-normal component of the tangential strain rate $\mathbf{D}^t$ to the subloading surface makes the constitutive equation rate-nonlinear.

v) The constitutive relation fulfills the continuity and the smoothness conditions (Hashiguchi, 1993a, b, 1997, 2000).

vi) The novel loading criterion described in terms of the strain rate, which is applicable to not only the hardening but also the perfectly-plastic and the softening processes, is formulated.

vii) The reciprocal expression, i.e. the expression of the strain rate in terms of the stress rate and vice versa is derived in spite of the rate-nonlinearity of the constitutive equation, while the stiffness matrix tensor cannot be expressed analytically because of the nonlinearity.

viii) The constitutive relation is applicable to the analysis of deformation in the general loading process including unloading, reloading and reverse loading for a wide class of materials including metals and geomaterials. On the other hand, the existing models with the tangential-stress rate effect except the model of Hashiguchi (1998) and Hashiguchi and Tsutsumi (2001) are applicable only to the monotonic loading process. Since the interior of the yield surface is assumed to be the purely elastic domain, they violate both the smoothness condition (62) and the continuity condition (66): their tangential strain rate is induced suddenly when the stress reaches the normal-yield state.

The original subloading surface (Hashiguchi and Ueno, 1977; Hashiguchi, 1980; Hashiguchi and Chen, 1998; Hashiguchi et al., 2002) has been extended already to describe the cyclic loading behavior (Hashiguchi, 1989), the time-dependent behavior (Hashiguchi, 2000b; Hashiguchi and Okayasu, 2000; Hashiguchi et al., 2004) and the friction phenomenon (Hashiguchi et al., 2004). It is extended here so as to describe the non-proportional loading behavior in the simple and natural way.

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