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Spectral Stochastic Finite Element Method for Log-Normal Uncertainty

対数正規分布を有する不確定性のためのスペクトル確率有限要素法

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Log-normal distribution is widely used to approximate probability density distributions of uncertain parameters in stochastic problems. This paper presents an analytical expression of the expectation of the product of log-normal stochastic variable and polynomial chaos and it enables efficient computation of spectral stochastic finite element method in the context of log-normal uncertainties. Accuracy of the presented method is validated through numerical simulations and comparison with Monte Carlo Simulation.

Key Words : *Spectral Stochastic FEM, Log-normal distribution, uncertainty*

1. Introduction

Stochastic finite element methods are powerful tools and widely used for the analysis of stochastic problems, as an efficient alternative of Monte Carlo Simulation (MCS) which is usually quite expensive in terms of computational effort. Various types of formulations have been proposed for stochastic finite element methods, such as those based on perturbation methods or Neumann expansion etc.[1,2] In addition to those methodologies, Ghanem and Spanos[3] proposed a spectral stochastic finite element method (SSFEM). SSFEM has a variety of advantages such as that it can estimate probability density function as well as mean values and variances.

Applicability of SSFEM has been explored in wide range of problems.[4-9] Most of those applications assume that uncertain parameters have fluctuations with a Gaussian distribution. SSFEM, however, is also applicable to problems with non-Gaussian uncertainty.

A log-normal distribution is suitable for approximation of many uncertain parameters and is used in many stochastic problems. Gaussian distribution ranges from $-\infty$ to ∞ and therefore it is not necessarily suitable to represent uncertain parameters that never takes negative values such as stiffness coefficients. For both practical and theoretical reasons, a log-normal distribution is frequently employed. Consequently, SSFEM to consider a log-normal distribution is studied in some research works.[10,11] They expand a stochastic parameter with a log-normal distribution in terms of polynomial chaos. This paper will present a simpler and more efficient scheme.

In the following, first we summarize the basic formulation of SSFEM, which is followed by a proposed scheme to use SSFEM with log-normal distributions. Then we present numerical examples to illustrate accuracy and efficiency of the proposed scheme, using a two-dimensional elastostatic problem where the stiffness is assumed to have uncertainty. Influences of Gaussian uncertainty and log-normal uncertainty are compared based on the computation results obtained by the presented scheme.

[1] Dedicated to the memory of Prof. Michihiro KITAHARA

2. Spectral Stochastic Finite Element Method

Here we briefly derive the formulation of SSFEM. Further detail description of SSFEM formulation can be found in the preceding works.[3,5,12] First we introduce two stochastic representations that play important roles in SSFEM; Karhunen-Loève expansion and polynomial chaos expansion. It is followed by a description of SSFEM formulation.

2.1 Karhunen-Loève Expansion

SSFEM regards uncertain parameters as stochastic processes and their spatial distributions are expressed in the form of truncated Karhunen-Loève expansion. The Karhunen-Loève expansion is a representation of a stochastic process in terms of uncorrelated random variables. When applied to a stochastic process whose covariance function is known, Karhunen-Loève expansion can provide the optimal representation of the target stochastic process in the mean-square sense.

Let us consider the domain S and a stochastic process $E(x, \theta)$ defined in S where $x \in S$ denotes the spatial coordinate and θ denotes an event in the probability space. Let us also assume that covariance function of the value at arbitrary points $x_1, x_2 \in S$ is given as $C(x_1, x_2)$. Then Karhunen-Loève expansion of a stochastic process $E(x, \theta)$ is obtained as

$$E(x,\theta) = E_0(x) + \sum_{i=1}^{\infty} \xi_i(\theta)\sqrt{\lambda_i} f_i(x) \qquad (1)$$

where $E_0(x)$ is a mean value of $E(x, \theta)$ at x; $\xi_i(\theta)$'s are orthonormal random variables. Scalars λ_i's and functions $f_i(x)$'s are respectively given as eigenvalues and normalized eigenfunctions of the integral equation

$$\int_S C(x_1, x_2) f_i(x_2) \mathrm{d}x_2 = \lambda_i f_i(x_1). \qquad (2)$$

Since it is impossible to expand Eq.(1) to an infinite order in practice, summation of Eq.(1) is truncated at a finite order, which is denoted by N_{KL}.

The truncation order N_{KL} should be determined depending on desired accuracy and the eigenvalues λ_i's. As can be seen in the right-hand side of Eq.(1), terms with small eigenvalues make small contribution. Those terms can be truncated for the purpose of computational efficiency.

2.2 Polynomial Chaos Expansion

Since it is impossible to obtain the information of covariance function of the solution in advance, solution process can not be represented by Karhunen-Loève expansion. Therefore, in SSFEM, the solution process is approximated by a stochastic process belonging to the homogeneous chaos[13] which is truncated after a specified finite order. Finitely truncated homogeneous chaos is spanned by a finite number of polynomial chaos $\Psi_n[\boldsymbol{\xi}]$, where its argument $\boldsymbol{\xi}$ is a vector of orthonormal Gaussian stochastic variables

$$\boldsymbol{\xi} = (\xi_1(\theta), \xi_2(\theta), \ldots,)^\top. \qquad (3)$$

Argument θ will be omitted hereafter for brevity of expression. Polynomial chaos are given as multivariate Hermite polynomials. They are orthogonal to each other with respect to a Gaussian measure and they consist an orthogonal basis of homogeneous chaos.[13,14]

In SSFEM, solution $\boldsymbol{u}(\theta)$ is represented by a square integrable stochastic process, which is expressed as

$$\boldsymbol{u} = \sum_{i=0}^{N_{PC}} \boldsymbol{u}_i \Psi_i^{(m)}[\boldsymbol{\xi}] \qquad (4)$$

where $\Psi_0^{(m)}[\boldsymbol{\xi}] = 1$ and the expansion is truncated after the N_{PC}-th term of polynomial chaos. m denotes the number of Gaussian random variables that appear in arguments and in SSFEM usually $m = N_{KL}$.

It is not easy to find the best truncation order N_{PC} of the summation of Eq.(4) for desired accuracy. High truncation order promises high accuracy but requires large computation efforts. Appropriate order is usually determined empirically considering the trade-off of accuracy and computation resources.

A mean value, variance and probability density function (pdf) of the solution expressed in the form of polynomial chaos expansion can be easily evaluated. Let us take $\boldsymbol{u}$ in Eq.(4) as an example. The 0-th term $\boldsymbol{u}_0$ gives a mean value. Denote n-th component of $\boldsymbol{u}$ by u^n, and n-th component of $\boldsymbol{u}_i$ by u_i^n. Then the variance of n-th component of $\boldsymbol{u}$ is given as

$$\langle \{u^n - \langle u^n \rangle\}^2 \rangle = \sum_{i=1}^{N_{PC}} \langle \Psi_i^{(m)}[\boldsymbol{\xi}] \Psi_i^{(m)}[\boldsymbol{\xi}] \rangle (u_i^n)^2 \qquad (5)$$

where $\langle \cdot \rangle$ denotes the expectation operator. The orthogonality of polynomial chaos ($\langle \Psi_i[\boldsymbol{\xi}] \Psi_j[\boldsymbol{\xi}] \rangle = 0$ for $i \neq j$) is taken into consideration. Pdf can be evaluated by a Monte Carlo Simulation-like manner.

2.3 SSFEM formulation

Now we derive an SSFEM formulation by taking a simple elastostatic problem as an example.

Based on the weak form expression of the equilibrium equation, considering the boundary conditions, an elastostatic problem can be discretized using Galerkin's method. We obtain the FEM expression of the problem as,

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{p} \qquad (6)$$

where $\boldsymbol{K}$ denotes a stiffness matrix, $\boldsymbol{u}$ and $\boldsymbol{p}$ denote a nodal displacement vector and a nodal external force vector, respectively. In the stochastic problem, we assume that Young's modulus has uncertainty and its covariance function is given. Then the stiffness matrix $\boldsymbol{K}$ has uncertainty associated with the uncertainty of Young's modulus. Assume that Young's modulus is given in the form of Karhunen-Loève expansion as in Eq.(1), then correspondingly stiffness matrix $\boldsymbol{K}$ is expressed in the form as

$$\boldsymbol{K} = \sum_{i=0}^{N_{KL}} \xi_i \boldsymbol{K}_i. \qquad (7)$$

Here $\boldsymbol{K}_i$ denotes a stiffness matrix to be obtained by using Galerkin's method when Young's modulus is given as $\sqrt{\lambda_i} f_i(x)$ in Eq.(1). When $i = 0$, $\xi_0 = 1$ and $\boldsymbol{K}_0$ is built from the mean value of Young's modulus $E_0(x)$. Since Eq.(1) gives the stiffness as a linear summation of kernel functions, Eq.(7) is also given as a linear summation of corresponding stiffness matrices. The external force $\boldsymbol{p}$ can be also assumed to have uncertainty and it will be similarly treated. The solution vector $\boldsymbol{u}$ is written in the form of polynomial chaos expansion as Eq.(4).

We substitute Eqs.(4) and (7) in Eq.(6) and aim to obtain the optimal approximation in the finitely truncated homogeneous chaos on which displacement $\boldsymbol{u}$ is expanded. This can be achieved by projecting the equation to each of the basis of the homogeneous chaos, that is, polynomial chaos. This procedure yields

$$\left\langle \Psi_k^{(N_{KL})}[\boldsymbol{\xi}] \left(\sum_{i=0}^{N_{KL}} \xi_i \boldsymbol{K}_i \right) \left(\sum_{j=0}^{N_{PC}} \boldsymbol{u}_j \Psi_j^{(N_{KL})}[\boldsymbol{\xi}] \right) \right\rangle = \left\langle \Psi_k^{(N_{KL})}[\boldsymbol{\xi}] \boldsymbol{p}(\boldsymbol{\xi}) \right\rangle \qquad (8)$$

for $k = 0, 1, \ldots, N_{PC}$. By changing the order of summation, Eq.(8) can be simplified as

$$\sum_{i=0}^{N_{KL}} \sum_{j=0}^{N_{PC}} \left\langle \xi_i \Psi_j^{(N_{KL})}[\xi]\, \Psi_k^{(N_{KL})}[\xi] \right\rangle \boldsymbol{K}_i \boldsymbol{u}_j = \boldsymbol{p}_k \tag{9}$$

where

$$\boldsymbol{p}_k = \left\langle \boldsymbol{p}(\xi)\, \Psi_k^{(N_{KL})}[\xi] \right\rangle. \tag{10}$$

Solving a set of Eq.(9) for $k = 0, 1, \ldots, N_{PC}$ with respect to $\boldsymbol{u}_j$ ($j = 0, 1, \ldots, N_{PC}$), gives the solution of our problem.

3. SSFEM for Log-Normal Uncertainty

Let us introduce the formulation of SSFEM analysis for a stochastic problem with a log-normal uncertainty.

3.1 Karhunen-Loève Expansion with Log-Normal Random Variables

In ordinary SSFEM, Karhunen-Loève expansion plays an important role in consideration of spatial correlation of uncertain parameters in SSFEM. It is also the case with SSFEM for a log-normal uncertainty.

We can avail the same kernel functions of Karhunen-Loève expansion to consider the log-normal stochastic process. Taking into consideration two equalities

$$\langle e^{\xi} \rangle = e^{1/2} \tag{11}$$

$$\langle (e^{\xi})^2 \rangle - \langle e^{\xi} \rangle^2 = e^2 - e, \tag{12}$$

we can introduce a stochastic variable η_i ($i = 0, 1, 2, \ldots$) as

$$\eta_i = \frac{1}{\sqrt{e(e-1)}} \left(e^{\xi_i} - e^{1/2} \right) \tag{13}$$

where each ξ_i denotes an independent normal stochastic variable. Then η_i's exhibit orthogonality as

$$\langle \eta_i \rangle = 0, \tag{14}$$

$$\langle \eta_i\, \eta_j \rangle = \delta_{ij}. \tag{15}$$

It allows us to replace the normalized independent Gaussian stochastic variables ξ_i's in Karhunen-Loève expansion by η_i's. Such substitution in Eq.(1) leads to Karhunen-Loève representation of the stiffness as

$$E(x, \theta) = E_0(x) + \sum_{i=1}^{\infty} \eta_i \sqrt{\lambda_i} f_i(x) \tag{16}$$

This representation gives the optimal approximation of the stochastic process in the sense of the second moment. Note that, in order to generate a stochastic process which provides the approximation in terms of higher order moments or probability density function, we need to adopt appropriate random variables in Eq.(16). This issue is not discussed here.

3.2 Formulation with Log-Normal Random Variables

We assume Young's modulus E as Eq.(16). Discretization by the Galerkin's method yields a stiffness matrix as

$$\boldsymbol{K}(\xi_1, \xi_2, \ldots) = \sum_{i=0}^{\infty} \eta_i \boldsymbol{K}_i \tag{17}$$

where $\eta_0 = 1$. We truncate the summation in Eq.(17) by N_{KL} and also truncate the polynomial chaos expansion of the displacement vector at N_{PC}-th term as in Eq.(4). Substitute them in Eq.(6) and project the equation to each of the orthogonal basis of homogeneous chaos under consideration, $\Psi_\ell[\xi]$. This gives equations corresponding to Eq.(9) as

$$\sum_{i=0}^{N_{KL}} \sum_{j=0}^{N_{PC}} e_{ijk}^{N_{KL}} \boldsymbol{K}_i \boldsymbol{u}_j = \boldsymbol{p}_k \tag{18}$$

where $\boldsymbol{p}_k$'s are identical with those given in Eq.(10) and

$$e_{ijk}^{N_{KL}} = \left\langle \eta_i \Psi_j^{(N_{KL})}[\xi]\, \Psi_k^{(N_{KL})}[\xi] \right\rangle \tag{19}$$

Now we estimate the value of Eq.(19). Let us consider a case that involves a single stochastic variable ξ, which means $N_{KL} = 1$. Then Eq.(19) can be rewritten as

$$e_{ijk}^{1} = \frac{1}{\sqrt{e(e-1)}} \left\langle e^{\xi} \Psi_j^{(1)}[\xi] \Psi_k^{(1)}[\xi] \right\rangle - \frac{e^{1/2}}{\sqrt{e(e-1)}} \left\langle \Psi_j^{(1)}[\xi] \Psi_k^{(1)}[\xi] \right\rangle. \tag{20}$$

This can be evaluated by using an equality

$$\langle e^{s\xi} \Psi_m^{(1)}[\xi] \Psi_n^{(1)}[\xi] \rangle = m!\, s^{n-m} e^{\frac{1}{2}s^2} L_m^{n-m}(-s^2) \tag{21}$$

where $L_m^n(x)$ denotes Laguerre polynomials and it can be evaluated as

$$L_n^{\alpha}(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}) = \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!} \tag{22}$$

More description about the derivation of Eq.(21) is provided in Appendix A.

The formulation presented above is equivalent to the case where a stochastic parameter with a log-normal distribution is expanded by polynomial chaos to infinity and therefore it ensures the higher accuracy than that presented in the reference.[10)]

Extension of the above presented scheme to a multivariate case (two or more stochastic variables are involved) is straightforward, but such cases are not considered in this paper.

4. Numerical Examples

For the purpose of validation of usability, we apply the proposed scheme to a two-dimensional elastostatic problem where Young's modulus is assumed to have uncertainty. Firstly, accuracy of the proposed SSFEM is discussed based on the comparison of computation results obtained by the proposed method and by Monte Carlo Simulation (MCS). Then, influence of the distribution of the

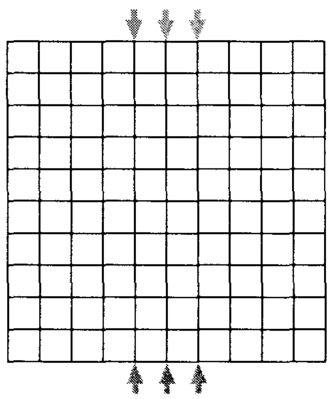


Fig. 1 Mesh configuration of SSFEM model. Arrows on the nodes on top and bottom sides denote external forces applied in the problem.

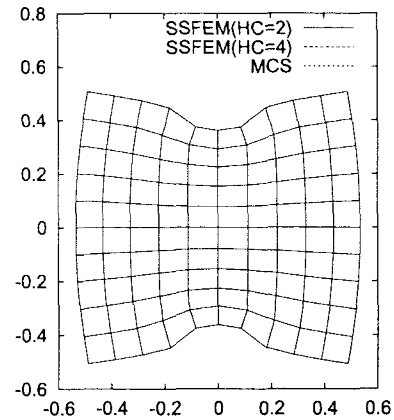


Fig. 2 Deformation representing mean values of the displacements obtained by SSFEM and MCS. Karhunen-Loève expansion is truncated at the first order.

fluctuation is discussed considering the results evaluated by assuming a log-normal uncertainty and a Gaussian uncertainty in Young's modulus of the media.

4.1 Comparison with MCS

A two-dimensional plane strain elastostatic problem in the square domain of 1.0×1.0 is considered. Young's modulus is assumed to have uncertainty and its covariance function for two points in the domain (x_1, y_1) and (x_2, y_2) is given as

$$C(x_1, y_1, x_2, y_2) = (\gamma\bar{G})^2 \exp\left\{-\frac{(|x_1 - x_2| + |y_1 - y_2|)}{b}\right\} \quad (23)$$

where b is a correlation length and $b = 1.0$; γ is a parameter to determine the magnitude of uncertainty. Fig. 1 illustrates a mesh configuration and external nodal forces applied to the model. External force has a magnitude of 1.0×10^{-3} and it is applied to three nodes of top and bottom sides. All other nodes on the boundary are set free.

We truncate the Karhunen-Loève expansion at the first order ($N_{KL} = 1$). γ=0.02 is set in Eq.(23), which implies the standard deviation of fluctuation is equal to 2% of the mean value. Contribution from the higher order homogeneous chaos is studied by comparing the results obtained by considering the homogeneous chaos up to the second order (HC=2) and the forth order (HC=4). They are referred to as SSFEM(HC=2) and SSFEM(HC=4), respectively. Since $N_{KL} = 1$, we have $N_{PC} = 2$ in SSFEM(HC=2) and $N_{PC} = 4$ in SSFEM(HC=4).

Table 1 Property of the model

Parameters	Values
Unit weight	1.0
Young's Modulus (mean value)	0.025
Poisson Ratio	0.25
Correlation length (b) *1	1.0
Standard deviation (γ) *1	2 %

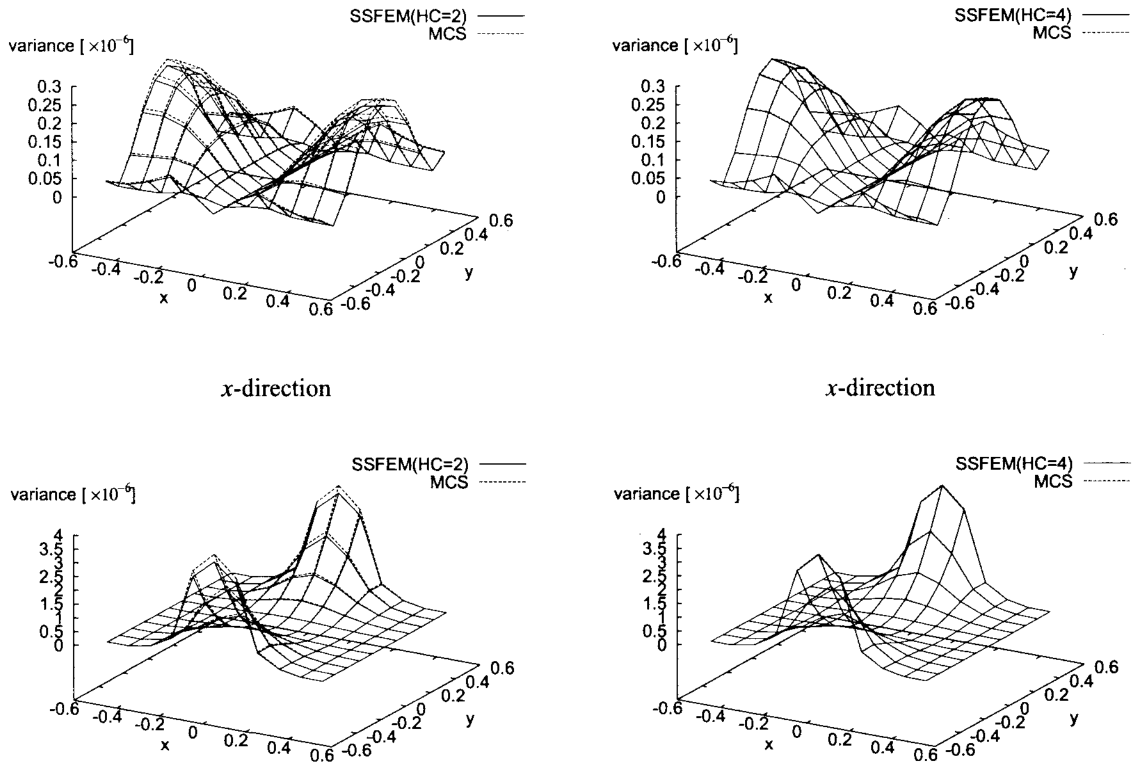
*1 in Eq.(23) for fluctuation of Young's modulus.

MCS was conducted with 50,000 realizations, which we verified is enough to capture the stochastic characteristics of the solution with an accuracy sufficient for our current purpose. In MCS, realization of the stiffness distribution is generated by substituting the random variables in Eq.(16).

Fig. 2 shows deformations representing mean values of displacements obtained by MCS, SSFEM(HC=2) and SSFEM(HC=4). Results of two SSFEM analyses both show good agreement with that obtained by MCS. Distribution of variances of x- and y-directional displacements obtained by SSFEM are plotted with the results by MCS in Fig. 3. SSFEM(HC=2) gives a good approximation of the MCS results. Better agreement is observed in the comparison of the results by MCS and SSFEM(HC=4), where computation results of SSFEM and MCS are virtually identical.

Pdf of displacements at $(-0.5, -0.1)$ and $(-0.1, -0.2)$ are plotted in Fig. 4. The results by SSFEM(HC=2) present a good approximation of the results by MCS, but the displacement at which pdf has its peak is different for SSFEM(HC=2) and MCS. Approximation accuracy is improved in the results by SSFEM(HC=4), which show good agreement with those by MCS. These results illustrate the performance of SSFEM in the problems with log-normal uncertainties.

Let us also mention to the comparison of CPU time required for these computations. MCS with 50,000 realizations takes more than 50,000 seconds in CPU time on Athlon 1700+, while SSFEM takes no more than a few seconds on the same CPU. It clearly illustrates superiority of the proposed scheme over MCS in terms of the computational efficiency.



x-direction x-direction

y-direction y-direction

(a) SSFEM(HC=2) (b) SSFEM(HC=4)

Fig. 3 Distribution of variance of the displacements obtained by SSFEM and MCS. Karhunen-Loève expansion is truncated at the first order.

4.2 Comparison of Gaussian and Log-normal Uncertainties

Let us show the difference of the stochastic characteristics of the solutions we obtain when we assume uncertain parameters have Gaussian distributions and those we get when we assume log-normal distributions.

We make a FEM model by discretizing the same area ($[-0.50.5]\times[-0.5\times0.5]$) with finer ($100\times100$) grid mesh. External load is assumed to have a uniform magnitude of 0.1 per unit length and it is applied to the area $[-0.1, 0.1]$ on the top and bottom sides. Fluctuation of Young's modulus is assumed to have a Gaussian distribution in one case and a log-normal distribution in the other case. Same mean value and same covariance function (Eq.(23) with $b = 1.0$ and $\gamma = 0.1$) are assumed in both cases. Karhunen-Loève expansion is truncated at the first order ($N_{KL} = 1$) and the solution is approximated in the homogeneous chaos truncated at the second order. (HC=2)

Fig. 5 plots distributions of variances of x-direction displacements. They exhibit similarity in terms of geometry, but magnitude of variance in the Gaussian stochastic problem is considerably larger than that in the log-normal uncertainty problem, although the fluctuation of Young's modulus is the same in both two problems.

Influence of the distribution of random parameters are more clearly observed in the comparison of pdf of displacements. Since a log-normal distribution is bounded, stiffness of this problem is always larger than a certain value. Therefore in the log-normal uncertainty problem, displacement is also bounded and it never exceeds a value obtained for the smallest value of stiffness. In the Gaussian uncertainty problem, on the other hand, displacement should distribute over a wide range. Such difference is clearly observed in the pdf of displacements shown in Fig. 6 which plots pdf of x- and y-directional displacements at (-0.1, -0.2) in these two problems.

These comparisons intelligibly illustrate that difference of stochastic characteristics of the problem can lead to significant difference of the stochastic behavior of the solutions. Poor modeling of stochastic problems can lead to a considerably inaccurate interpretation of results. The difference is clearly observable in the comparison of pdf, but may not be obvious in the comparison of mean values or variances, which are of main interest in numerous conventional stochastic finite element methods. These results would stress the importance of methodologies such as a spectral stochastic finite element method.

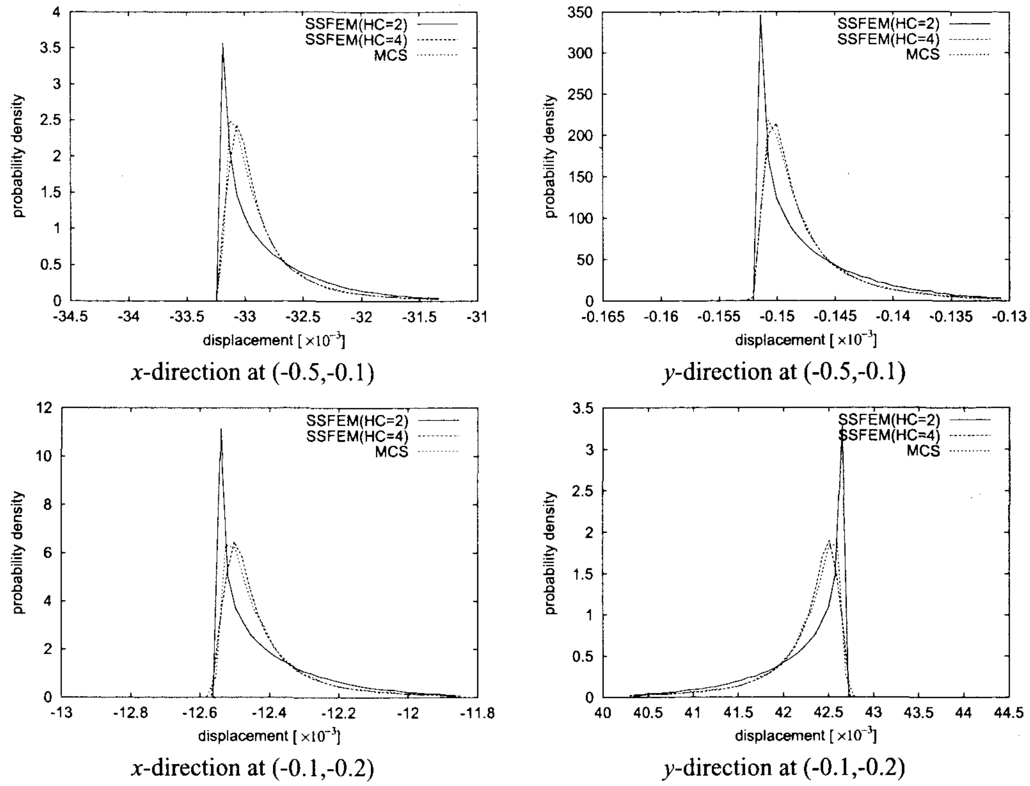


Fig. 4 Pdf of the displacements obtained by SSFEM and MCS. Karhunen-Loève expansion is truncated at the first order.

5. Conclusion

We have presented a practical scheme to apply a spectral stochastic finite element method (SSFEM) to stochastic problems in which uncertain parameters have log-normal distributions.

The presented scheme does not require substantial modification of the program code of the original SSFEM where Gaussian uncertainty is of main interest. Required are (a) replacement of Gaussian stochastic variables ξ_i's in Karhunen-Loève expansion by log-normal stochastic variables η_i's, and (b) replacement of an expectation of the product of a Gaussian stochastic variable ξ_i and polynomial chaos ($\langle \xi_i \Psi_j \Psi_k \rangle$) by an expectation of the product of a log-normal stochastic variable η_i and polynomial chaos ($\langle \eta_i \Psi_j \Psi_k \rangle$). The expectation values can be easily evaluated by an analytical expression and they are provided in this paper (See Table 2).

Validation of the proposed scheme is conducted through the numerical examples using a two-dimensional elastostatic problem where stiffness has a log-normal uncertainty. Computation results obtained by the proposed scheme are compared with the results obtained by Monte Carlo Simulation (MCS) with 50,000 realizations. Resultant good agreement illustrates the usability of the proposed scheme. Efficiency of the proposed scheme is also portrayed by the fact that its computation time is much smaller than that by MCS.

The scheme is also utilized to study the effect of the difference of probability distribution of the fluctuation. Two problems are considered assuming different distributions for the fluctuation of Young's modulus. A Gaussian distribution is assumed in one case and it was analyzed by a conventional SSFEM. In the other case, a log-normal distribution is adopted and this case was treated by the proposed formulation.

Although the same standard deviation is assumed for the fluctuation of Young's modulus in the two cases, probability characteristics of the solutions exhibit significant difference in the variances and the probability density functions. This indicates importance of appropriate modeling of stochastic problems. Now with a help of the proposed scheme, we can consider a log-normal distribution as an available option. It would be of great help when considered are more realistic and practical problems than those treated in this paper.

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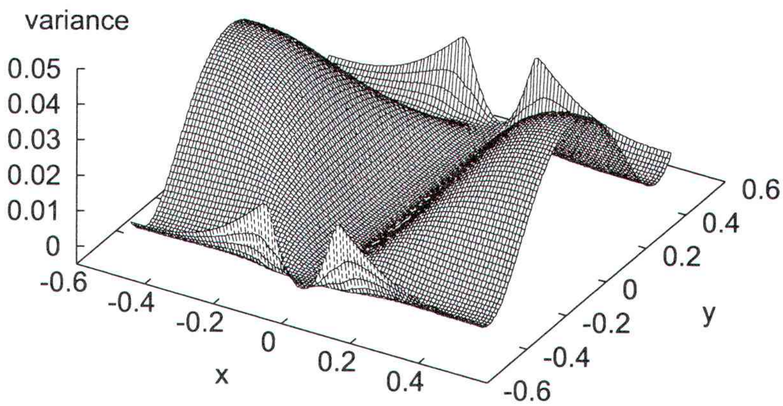
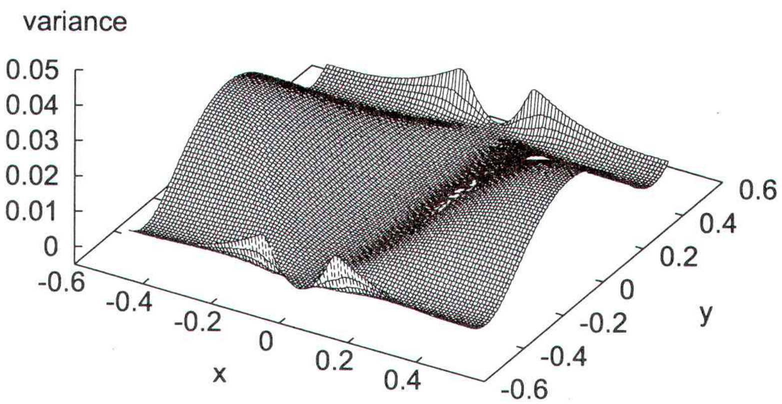
Appendix A Derivation of Eq.(21)

Here we derive the equality Eq.(21).

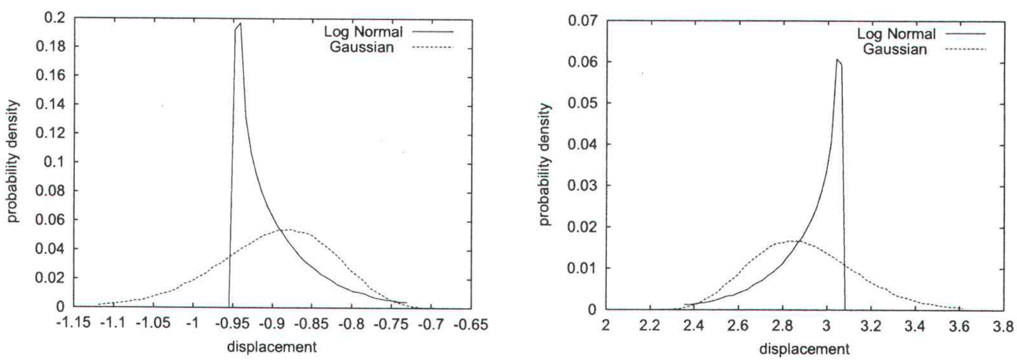


(a) Gaussian



(b) Log Normal

Fig. 5 Distribution of variance of the x-directional displacements obtained by assuming the fluctuation of Young's modulus has a Gaussian uncertainty and a log-normal uncertainty.



(a) x-direction (b) y-direction

Fig. 6 Pdf of the displacements at (-0.1, -0.2) in the problem obtained by assuming the fluctuation of Young's modulus has a Gaussian uncertainty and a log-normal uncertainty.

Integration tables (*e.g.* **7.374.7** in the reference[15]) shows us an equality

$$\int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(x) H_n(x) \mathrm{d}x = 2^n \pi^{1/2} m! y^{n-m} L_m^{n-m}(-2y^2) \tag{A.1}$$

where $H_n(x)$ denotes Hermite polynomial.

The left-hand side of Eq.(21) can be written as

$$\langle e^{s\xi} \Psi_m[\xi] \Psi_n[\xi] \rangle = \int e^{s\xi} \Psi_m[\xi] \Psi_n[\xi] \mathrm{d}\tilde{P} \tag{A.2}$$

where

$$\mathrm{d}\tilde{P} = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\xi^2\right\} \mathrm{d}\xi \tag{A.3}$$

Table 2 Values of $e^{(1)}_{ijk}$

i	j	k	$e^{(1)}_{ijk}$
0	0	0	1.0
0	1	1	1.0
0	2	2	2.0
0	3	3	6.0
0	4	4	24.0
1	0	1	0.7629
1	0	2	0.7629
1	0	3	0.7629
1	0	4	0.7629
1	1	1	0.7629
1	1	2	2.2886
1	1	3	3.0515
1	1	4	3.8144
1	2	2	3.8144
1	2	3	9.9174
1	2	4	16.0204
1	3	3	21.3605
1	3	4	55.6898
1	4	4	141.1317

Substitute in Eq.(A.1)

$$x = \xi / \sqrt{2} \quad \text{(A.4)}$$

$$y = s / \sqrt{2}, \quad \text{(A.5)}$$

and multiply

$$\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2^m}} \frac{1}{\sqrt{2^n}} \cdot e^{\frac{1}{2}s^2}. \quad \text{(A.6)}$$

Then we have

$$\int_{-\infty}^{\infty} e^{s\xi} \frac{1}{\sqrt{2^m}} H_m\left(\frac{\xi}{\sqrt{2}}\right) \frac{1}{\sqrt{2^n}} H_n\left(\frac{\xi}{\sqrt{2}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi^2} \mathrm{d}\xi \quad \text{(A.7)}$$

Using the relationship of polynomial chaos function $\Psi^{(1)}_m[\xi]$ and Hermite Polynomial $H_m(\xi)$

$$\frac{1}{\sqrt{2^m}} H_m\left(\frac{\xi}{\sqrt{2}}\right) = \Psi^{(1)}_m[\xi] \quad \text{(A.8)}$$

we obtain Eq.(21)

$$\langle e^{s\xi} \Psi^{(1)}_m[\xi] \Psi^{(1)}_n[\xi] \rangle = m! s^{n-m} e^{\frac{1}{2}s^2} L^{n-m}_m(-s^2) \quad (21)$$

Values of e_{ijk} with a single stochastic variable are listed in Table 2. Values of terms whose subscripts satisfies $j < k$ are omitted in the table because they have the identical values with e_{ikj} ($j < k$).

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