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Free Vibration Analysis of Square Plates Resting on Non-homogeneous Elastic Foundations

不均一弾性地盤上の正方形板の自由振動の一解析法

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A discrete method is developed for analyzing the free vibration problem of square plates resting on non-homogeneous elastic foundations. The fundamental differential equations are established for the bending problem of the plate on elastic foundations. The Green function, which is obtained by transforming these differential equations into integral equations and using numerical integration, is used to get the characteristic equation of the free vibration. The effects of the modulus of the foundation and the stepped thickness on the frequency parameters are considered. By comparing the present numerical results with those previously published, the efficiency and accuracy of the present method are investigated.

Key Words : *discrete method, elastic foundation, stepped thickness, Green function, vibration*

1. Introduction

Plates on the elastic foundations can be extensively used in engineering and the free vibration problems of these plates have been studied for many years. The fundamental frequency of vibration of circular and regular polygonal plates on a non-homogeneous foundation was obtained by Laura and Guriérrez [1]. The Rayleigh-Ritz method was used. By using the same method, they [2] analyzed the transverse vibration of rectangular plates on non-homogeneous foundations. The boundary conditions were elastically restrained. The fundamental frequency coefficients were given for various aspect ratios and the moduli of the foundations. Based on Gâteaux differential, a mixed finite element formulation was derived and used to analyze the static and dynamic problems of thin plate on elastic foundation by Omurtag [3] *et al.* The numerical results were obtained for clamped and simply supported plates with variable thickness on Winkler or Pasternak foundations. By the same method, Omurtag and Kadioğlu [4] studied the free vibration of orthotropic plates resting on Pasternak foundation and presented some numerical results for plates with simply supported boundary conditions. Matsunaga [5] used the method of power series expansion of the displacement components to investigate the vibration and stability of thick plates on elastic foundation. Based on the higher-order theory of thick plate, the natural frequency and the buckling stress were given for a simply supported square plate on a two-parameter elastic foundation and subjected to in-plane stress. Huang and Thambiratnam [6] used the finite strip method to analyze the static and dynamic responses of plates resting on elastic supports or elastic foundations. A spring system was used to simulate these elastic supports and foundations. Ju, Lee and Lee [7] analyzed the free vibration of rectangular and circular plates with stepped thickness resting on non-homogeneous elastic foundations by using the finite element method. Natural frequency parameters and mode shapes of these plates were presented.

In this paper, early work [8] is extended for analyzing the free vibration of rectangular plates resting

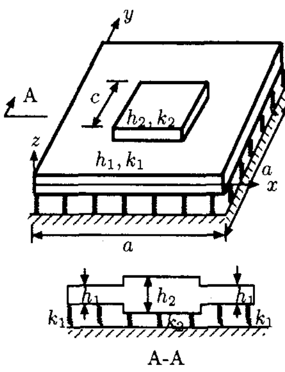

Fig. 1 A square plate with stepped thickness resting on elastic foundations.

on non-homogeneous elastic foundations. A discrete method proposed by some of the authors is used. The fundamental differential equations of a plate on non-homogeneous foundations are established and satisfied exactly throughout the whole plate. By transforming these equations into integral equations and using numerical integration, the solutions are obtained at the discrete points. The Green function, which is the solution for deflection, is used to obtain the characteristic equation of the free vibration. By applying the characteristic equation , the behaviour of the free vibration of the plates on foundations can be analyzed efficiently without a calculation by a trial and error method. The efficiency and accuracy of the present method for the free vibration of square plates on Winkler foundation are investigated. The effect of the foundation modulus on the frequency parameter is considered. Numerical results are obtained for the plates on homogeneous foundations, local uniformly distributed supports and non-homogeneous foundations. As an application of the proposed method, some numerical results are also given for plates with stepped thickness in central part resting on local uniformly distributed supports and the non-homogeneous foundations.

2. Fundamental Differential Equations

Figure 1 shows a square plate of length a, density ρ and stepped thickness h resting on non-homogeneous foundations of foundation modulus k. The thickness and the foundation modulus in the central square part are h_2 and k_2, and those for the other part are h_1 and k_1, respectively. An xyz coordinate system is used in the present study with its $x-y$ plane contained in middle plane of the square plate, the $z-$axis perpendicular to the middle plane of the plate and the origin at one of the corners of the plate.

In this paper, the elastic foundation is modelled as a spring system and the intensity of the reaction of the foundation is assumed to be proportional to the deflection w of the plate. By considering the reaction of the foundation as a kind of lateral load, the fundamental differential equations of the plate having a concentrated load $\overline{P}$ at a point (x_q, y_r) and resting on a Winkler foundation of the foundation modulus k are as follows:

$$
\begin{aligned}
&\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+\overline{P}\delta(x-x_q)\delta(y-y_r)-kw=0,\\
&\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y=0,\\
&\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=0,\\
&\frac{\partial \theta_x}{\partial x}+\nu\frac{\partial \theta_y}{\partial y}=\frac{M_x}{D},\\
&\frac{\partial \theta_y}{\partial y}+\nu\frac{\partial \theta_x}{\partial x}=\frac{M_y}{D},\\
&\frac{\partial \theta_x}{\partial y}+\frac{\partial \theta_y}{\partial x}=\frac{2}{(1-\nu)}\frac{M_{xy}}{D},\\
&\frac{\partial w}{\partial x}+\theta_x=\frac{Q_x}{\kappa Gh},\\
&\frac{\partial w}{\partial y}+\theta_y=\frac{Q_y}{\kappa Gh},
\end{aligned}
\tag{1}
$$

where Q_x, Q_y are the shearing forces, M_{xy} the twisting moment, M_x, M_y the bending moments, θ_x, θ_y the rotations of the x- and y-axes, w the deflection, $D=Eh^3/(12(1-\nu^2))$ the bending rigidity, E, G modulus, shear modulus of elasticity, respectively, ν Poisson's ratio, h the thickness of plate, $\kappa=5/6$ is the shear correction factor, $\delta(x-x_q)$, $\delta(x-x_r)$ Dirac's delta functions.

By introducing the non-dimensional expressions,

$$[X_1, X_2]=\frac{a^2}{D_0(1-\nu^2)}[Q_y, Q_x],$$

$$[X_3, X_4, X_5]=\frac{a}{D_0(1-\nu^2)}[M_{xy}, M_y, M_x],$$

$$[X_6, X_7, X_8]=[\theta_y, \theta_x, w/a],$$

the equation (1) is rewritten as the following non-dimensional forms:

$$
\begin{aligned}
&\mu\frac{\partial X_2}{\partial \eta}+\frac{\partial X_1}{\partial \zeta}+P\delta(\eta-\eta_q)\delta(\zeta-\zeta_r)-\overline{k}X_8=0,\\
&\mu\frac{\partial X_3}{\partial \eta}+\frac{\partial X_4}{\partial \zeta}-\mu X_1=0,\\
&\mu\frac{\partial X_5}{\partial \eta}+\frac{\partial X_3}{\partial \zeta}-\mu X_2=0,
\end{aligned}
$$

$$\mu\frac{\partial X_7}{\partial\eta}+\nu\frac{\partial X_6}{\partial\zeta}-\overline{D}X_5=0,$$
$$\nu\mu\frac{\partial X_7}{\partial\eta}+\frac{\partial X_6}{\partial\zeta}-\overline{D}X_4=0,$$
$$\mu\frac{\partial X_6}{\partial\eta}+\frac{\partial X_7}{\partial\zeta}-\frac{2}{1-\nu}\overline{D}X_3=0,$$
$$\frac{\partial X_8}{\partial\eta}+X_7-\overline{H}X_2=0,$$
$$\frac{\partial X_8}{\partial\zeta}+\mu X_6-\mu\overline{H}X_1=0, \qquad (2)$$

where $\mu = b/a$, $\overline{D} = \mu(1-\nu^2)(h_0/h)^3$, $\overline{H} = ((1+\nu)/5)(h_0/a)^2(h_0/h)$, $P = \overline{P}a/(D_0(1-\nu^2))$, $D_0 = Eh_0^3/(12(1-\nu^2))$ is the standard bending rigidity, h_0 is the standard thickness of the plate, $\delta(\eta-\eta_q)$ and $\delta(\zeta-\zeta_r)$ are Dirac's delta functions, $\overline{k} = \mu K/(1-\nu^2)$, K is the dimensionless modulus of the foundation, it is defined as follows:

$$K = ka^4/D_0,$$

In the above equation, the variable quantity h_0/h has been separated and expressed only in the quantities $\overline{D}$ and $\overline{H}$ so that the equation can be used for the plate with stepped thickness.

The equation (2) can also be expressed as the following simple form.

$$\sum_{s=1}^{8}\{F_{1ts}\frac{\partial X_s}{\partial\zeta}+F_{2ts}\frac{\partial X_s}{\partial\eta}+F_{3ts}X_s\}$$
$$+P\delta(\eta-\eta_q)\delta(\zeta-\zeta_r)\delta_{1t}=0 \quad (t=1\sim 8), \quad (3)$$

where δ_{1t} is Kronecker's delta, $F_{111}=F_{124}=F_{133}=F_{156}=F_{167}=F_{188}=1$, $F_{146}=\nu$, $F_{212}=F_{223}=F_{235}=F_{247}=F_{266}=\mu$, $F_{257}=\mu\nu$, $F_{278}=1$, $F_{318}=-\overline{k}$, $F_{321}=F_{332}=-\mu$, $F_{345}=F_{354}=-\overline{D}$, $F_{363}=-2\overline{D}/(1-\nu)$, $F_{372}=-\overline{H}$, $F_{377}=1$, $F_{381}=-\mu\overline{H}$, $F_{386}=\mu$, other$F_{kts}=0$.

3. Discrete Green Function

As given in Ref [8], by dividing a square plate vertically into m equal-length parts and horizontally into n equal-length parts as shown in Figure 2, the plate can be considered as a group of discrete points which are the intersections of the $(m+1)$-vertical and $(n+1)$-horizontal dividing lines. To describe the present method conveniently, the rectangular area, $0\le\eta\le\eta_i$, $0\le\zeta\le\zeta_j$, corresponding to the arbitrary intersection (i,j) as shown in Figure 2 is denoted as the area $[i,j]$, the intersection (i,j) denoted by ○ is called the main point of the area $[i,j]$, the intersections denoted by ∘ are called the inner dependent points of the area, and the intersections denoted by • are called the boundary dependent points of the area.

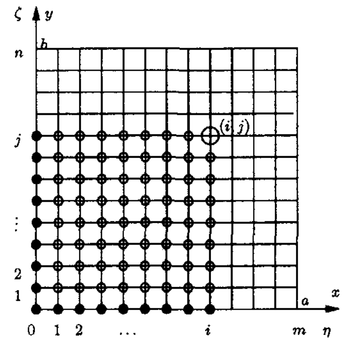

Fig. 2 Discrete points on a rectangular plate.

By integrating the equation (3) over the area $[i,j]$, the following integral equation is obtained:

$$\sum_{s=1}^{8}\left\{F_{1ts}\int_0^{\eta_i}[X_s(\eta,\zeta_j)-X_s(\eta,0)]d\eta\right.$$
$$+F_{2ts}\int_0^{\zeta_j}[X_s(\eta_i,\zeta)-X_s(0,\zeta)]d\zeta$$
$$\left.+F_{3ts}\int_0^{\eta_i}\int_0^{\zeta_j}X_s(\eta,\zeta)d\eta d\zeta\right\}$$
$$+Pu(\eta-\eta_q)u(\zeta-\zeta_r)\delta_{1t}=0, \qquad (4)$$

where $u(\eta-\eta_q)$ and $u(\zeta-\zeta_r)$ are the unit step functions.

Next, by applying the numerical integration method, the simultaneous equation for the unknown quantities $X_{sij}=X_s(\eta_i,\zeta_j)$ at the main point (i,j) of the area $[i,j]$ is obtained as follows:

$$\sum_{s=1}^{8}\left\{F_{1ts}\sum_{k=0}^{i}\beta_{ik}(X_{skj}-X_{sk0})\right.$$
$$+F_{2ts}\sum_{l=0}^{j}\beta_{jl}(X_{sil}-X_{s0l})$$
$$\left.+F_{3ts}\sum_{k=0}^{i}\sum_{l=0}^{j}\beta_{ik}\beta_{jl}X_{skl}\right\}$$
$$+Pu_{iq}u_{jr}\delta_{1t}=0, \qquad (5)$$

where $\beta_{ik}=\alpha_{ik}/m, \beta_{jl}=\alpha_{jl}/n$, $\alpha_{ik}=1-(\delta_{0k}+\delta_{ik})/2$, $\alpha_{jl}=1-(\delta_{0l}+\delta_{jl})/2$, $t=1\sim 8$, $i=1\sim m$, $j=1\sim n$, $u_{iq}=u(\eta_i-\eta_q)$, $u_{jr}=u(\zeta_j-\zeta_r)$.

By retaining the quantities at main point (i,j) on the left hand side of the equation and putting other quantities on the right hand side, and using the matrix transition, the solution X_{pij} of the above equation (5) is obtained as follows:

$$X_{pij}=\sum_{t=1}^{8}\left\{\sum_{k=0}^{i}\beta_{ik}A_{pt}[X_{tk0}-X_{tkj}(1-\delta_{ik})]\right.$$

$$+\sum_{l=0}^{j} \beta_{jl} B_{pt}[X_{t0l} - X_{til}(1-\delta_{jl})]$$

$$+\sum_{k=0}^{i}\sum_{l=0}^{j} \beta_{ik}\beta_{jl} C_{ptkl} X_{tkl}(1-\delta_{ik}\delta_{jl})\Big\}$$

$$-A_{p1} P u_{iq} u_{jr}, \tag{6}$$

where $p = 1 \sim 8$, A_{pt}, B_{pt} and C_{ptkl} are given in Ref [8].

In the equation (6), the quantity X_{pij} is not only related to the quantities X_{tk0} and X_{t0l} at the boundary dependent points but also the quantities X_{tkj}, X_{til} and X_{tkl} at the inner dependent points. The maximal number of the unknown quantities is $6(m-1)(n-1)+3(m+n+1)$. In order to reduce the unknown quantities, the area $[i,j]$ is spread according to the regular order as $[1,1]$, $[1,2]$, $\cdots$, $[1,n]$, $[2,1]$, $[2,2]$, $\cdots$, $[2,n]$, $\cdots$, $[m,1]$, $[m,2]$, $\cdots$, $[m,n]$. With the spread of the area according to the above mentioned order, the quantities X_{tkj}, X_{til} and X_{tkl} at the inner dependent points can be eliminated by substituting the obtained results into the corresponding terms of the right hand side of equation (6). By repeating this process, the quantity X_{pij} at the main point is only related to the quantities X_{rk0} (r=1,3,4,6,7,8) and X_{s0l} (s=2,3,5,6,7,8) at the boundary dependent points. The maximal number of the unknown quantities is reduced to $3(m+n+1)$. It can be noted the number of the unknown quantities of the present method is fewer than that of the finite element method for the same divisional number $m(\geq 3)$ and $n(\geq 3)$. Based on the above consideration, the equation (6) is rewritten as follows.

$$X_{pij} = \sum_{d=1}^{6}\left\{\sum_{f=0}^{i} a_{pijfd} X_{rf0} + \sum_{g=0}^{j} b_{pijgd} X_{s0g}\right\} + \overline{q}_{pij} P, \tag{7}$$

where a_{pijfd}, b_{pijgd} and $\overline{q}_{pij}$ are given in Appendix A.

The equation (7) gives the discrete solution of the fundamental differential equation (3) of the bending problem of a plate resting on an elastic foundation and having a concentrated load, and the discrete Green function is chosen as $X_{8ij}a^2/[PD_0(1-\nu^2)]$, that is $w(x_0,y_0,x,y)/\overline{P}$.

The integral constants X_{rfo} and X_{sog} involved in the discrete solution (7) are all quantities at the discrete points along the edges $\zeta = 0$ ($y = 0$) and $\eta = 0$ ($x = 0$) of the rectangular plate. There are six integral constants at each discrete point. Half of them are self-evident according to the boundary conditions along the edges $\zeta = 0$ and $\eta = 0$ and half of them are needed to determine by the boundary conditions along the edges $\zeta = 1$ and $\eta = 1$.

The simply supported boundary conditions are as follows.

$M_y = \theta_x = w = 0$ for the edges $\zeta = 0$ and $\zeta = 1$

$M_x = \theta_y = w = 0$ for the edges $\eta = 0$ and $\eta = 1$

4. Characteristic equation

In this paper, the analysis is carried out for the thin plates and the effect of the rotary inertia is not taken into account.

By applying the Green function $w(x_0,y_0,x,y)/\overline{P}$ which is the displacement at a point (x_0,y_0) of a plate with a concentrated load $\overline{P}$ at a point (x,y), the displacement amplitude $\hat{w}(x_0,y_0)$ at a point (x_0,y_0) of the square plate during the free vibration is given as follows:

$$\hat{w}(x_0,y_0) = \int_0^b\int_0^a \rho h\omega^2 \hat{w}(x,y)[w(x_0,y_0,x,y)/\overline{P}]dxdy, \tag{8}$$

where ρ is the mass density of the plate material and ω is the circular frequency.

By using the numerical integration method and the following non-dimensional expressions,

$$\lambda^4 = \frac{\rho_0 h_0 \omega^2 a^4}{D_0(1-\nu^2)}, \quad \Lambda = 1/(\mu\lambda^4),$$

$$H(\eta,\zeta) = \frac{\rho(x,y)}{\rho_0}\frac{h(x,y)}{h_0}, W(\eta,\zeta) = \frac{\hat{w}(x,y)}{a},$$

$$G(\eta_0,\zeta_0,\eta,\zeta) = \frac{w(x_0,y_0,x,y)}{a}\frac{D_0(1-\nu^2)}{\overline{P}a},$$

where ρ_0 is the standard mass density , the characteristic equation is obtained from the equation (8) as

$$\begin{pmatrix} \mathbf{S_{00}} & \mathbf{S_{01}} & \mathbf{S_{02}} & \cdots & \mathbf{S_{0m}} \\ \mathbf{S_{10}} & \mathbf{S_{11}} & \mathbf{S_{12}} & \cdots & \mathbf{S_{1m}} \\ \mathbf{S_{20}} & \mathbf{S_{21}} & \mathbf{S_{22}} & \cdots & \mathbf{S_{2m}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{S_{m0}} & \mathbf{S_{m1}} & \mathbf{S_{m2}} & \cdots & \mathbf{S_{mm}} \end{pmatrix} = 0 \tag{9}$$

where

$$\mathbf{S_{ij}} = \beta_{mj}\begin{bmatrix} \beta_{n0}H_{j0}G_{i0j0} - \Lambda\delta_{ij} & \cdots & \beta_{nn}H_{jn}G_{i0jn} \\ \beta_{n0}H_{j0}G_{i1j0} & \cdots & \beta_{nn}H_{jn}G_{i1jn} \\ \beta_{n0}H_{j0}G_{i2j0} & \cdots & \beta_{nn}H_{jn}G_{i2jn} \\ \vdots & \vdots & \vdots \\ \beta_{n0}H_{j0}G_{inj0} & \cdots & \beta_{nn}H_{jn}G_{injn} - \Lambda\delta_{ij} \end{bmatrix}.$$

5. Numerical results

To investigate the validity of the proposed method, the frequency parameters are given for the plate

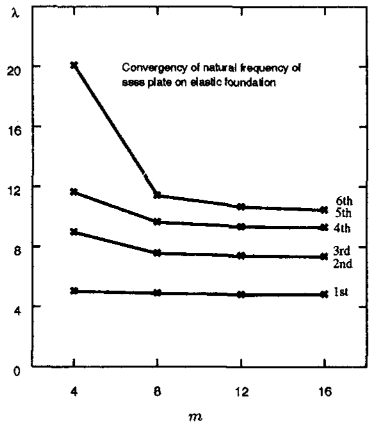

Fig. 3 The natural frequency parameter λ versus the divisional number $m(=n)$ for the SSSS square plate with uniform thickness on the elastic foundation ($K = 100$).

shown in Figure 1. The standard thickness h_0 is chosen as h_1 and $h_0/a = 1/1000$ is used. Simply supported boundary conditions are considered and denoted by four symbols SSSS. All the convergent values of the frequency parameters are obtained for simply supported square plates ($\mu = b/a = 1$) by using Richardson's extrapolation formula for two cases of divisional numbers m (=n). Some of the results are compared with those reported previously.

5.1 A square plate on homogeneous foundations

In order to examine the convergency, numerical calculation is carried out by varying the number of divisions m and n for a square plate with uniform thickness on homogeneous foundations. The plate is special case of the plate shown in Figure 1 with $c/a = 0.0$. The lowest 6 natural frequency parameters of this plate with the dimensionless modulus of the foundation $K = K_1 = 100$ are shown in Figure 3. It shows a good convergency of the numerical results obtained by the present method. After studying the figure, it is decided to obtain the convergent results of frequency parameter by using Richardson's extrapolation formula for two cases of divisional numbers m (=n) of 12 and 16. By the same method, the suitable number of divisions $m(=n)$ can be determined for the other plates.

Table 1 shows the numerical values for the lowest 4 natural frequency parameter λ of square plates on homogeneous foundation with $K = 0, 10, 100, 1000, 10000$. The results obtained by Matsunaga [5] and the exact values of the plate with $K = 0$ [9] are also shown in the table. It can be seen that the numerical results of the present method have satisfactory accuracy. From this table, it can be also seen that the effect of the constant K on the fundamental frequency parameter is much more significant than that on higher frequency parameters, the frequency parameters increase with increase of the constant K, and they increase quickly when K is larger than 100.

Table 1 Natural frequency parameter λ for a SSSS square plate on homogeneous foundations

		Mode sequence number			
K	References	1st	2nd	3rd	4th
0	Present				
	12 × 12	4.575	7.336	7.336	9.311
	16 × 16	4.564	7.272	7.272	9.216
	20 × 20	4.553	7.261	7.261	9.126
	Ex.	4.549	7.190	7.190	9.094
	Ref. [5]	4.549	–	–	–
	Exact [9]	4.549	7.192	7.192	9.098
10	Present				
	12 × 12	4.603	7.343	7.343	9.315
	16 × 16	4.592	7.279	7.279	9.220
	20 × 20	4.582	7.266	7.267	9.133
	Ex.	4.578	7.198	7.198	9.098
	Ref. [5]	4.578	–	–	–
10^2	Present				
	12 × 12	4.838	7.405	7.405	9.345
	16 × 16	4.829	7.343	7.343	9.251
	20 × 20	4.824	7.314	7.314	9.210
	Ex.	4.816	7.263	7.263	9.131
	Ref. [5]	4.816	–	–	–
10^3	Present				
	12 × 12	6.261	7.950	7.950	9.635
	16 × 16	6.257	7.900	7.900	9.549
	20 × 20	6.254	7.879	7.880	9.506
	Ex.	6.251	7.836	7.836	9.439
	Ref. [5]	6.251	–	–	–
10^4	Present				
	12 × 12	10.339	10.855	10.855	11.664
	16 × 16	10.338	10.836	10.836	11.616
	20 × 20	10.338	10.827	10.827	11.595
	Ex.	10.337	10.811	10.811	11.554
	Ref. [5]	10.337	–	–	–

Ex.:The values obtained by using Richardson's extrapolation formula.

Table 2 Natural frequency parameter λ for a SSSS square plate with the central part on local uniform supports ($c/a = 0.6$)

K_1	K_2	References	Mode sequence number		
			1st	2nd (3rd)	4th
0	320	Present			
		10 × 10	5.156	7.552	9.465
		15 × 15	5.157	7.434	9.290
		Ex.	5.158	7.341	9.151
		Ref. [2]	5.168	–	–
		Ref. [7]	5.123	–	–
0	800	Present			
		10 × 10	5.766	7.736	9.542
		15 × 15	5.782	7.627	9.371
		Ex.	5.794	7.540	9.235
		Ref. [2]	5.813	–	–
		Ref. [7]	5.774	–	–
0	1600	Present			
		10 × 10	6.491	8.000	9.662
		15 × 15	6.514	7.907	9.498
		Ex.	6.533	7.833	9.367
		Ref. [2]	6.563	–	–
		Ref. [7]	6.517	–	–
320	0	Present			
		15 × 15	4.757	7.346	9.286
		20 × 20	4.747	7.296	9.275
		Ex.	4.733	7.232	9.262
		Ref. [2]	4.715	–	–
		Ref. [7]	4.656	–	–
800	0	Present			
		15 × 15	4.998	7.447	9.364
		20 × 20	4.971	7.425	9.292
		Ex.	4.936	7.397	9.200
		Ref. [2]	4.937	–	–
		Ref. [7]	4.871	–	–
1600	0	Present			
		15 × 15	5.316	7.618	9.491
		20 × 20	5.281	7.595	9.433
		Ex.	5.235	7.566	9.357
		Ref. [2]	5.250	–	–
		Ref. [7]	5.161	–	–

5.2 A square plate with uniform thickness on non-homogenous foundations

Table 2 shows the numerical values for the lowest 4 natural frequency parameter λ of the plate shown in Figure 1 with $K_1 = 0$ or $K_2 = 0$, which is the case of the local uniformly distributed support. The side ratio of the local square part and the plate $c/a = 0.6$ and

Table 3 Natural frequency parameter λ for a SSSS square plate with the central part on non-homogeneous foundations ($c/a = 0.6$)

K_1	K_2	References	Mode sequence number		
			1st	2nd (3rd)	4th
320	800	Present			
		10 × 10	5.878	7.798	9.591
		15 × 15	5.883	7.695	9.424
		Ex.	5.887	7.613	9.290
		Ref. [2]	5.895	–	–
		Ref. [7]	5.862	–	–
320	1600	Present			
		10 × 10	6.571	8.063	9.710
		15 × 15	6.588	7.974	9.549
		Ex.	6.602	7.902	9.421
		Ref. [2]	6.620	–	–
		Ref. [7]	6.584	–	–
800	320	Present			
		15 × 15	5.476	7.602	9.420
		20 × 20	5.456	7.596	9.349
		Ex.	5.430	7.589	9.257
		Ref. [2]	5.446	–	–
		Ref. [7]	5.402	–	–
1600	320	Present			
		15 × 15	5.729	7.767	9.547
		20 × 20	5.710	7.731	9.518
		Ex.	5.685	7.684	9.481
		Ref. [2]	5.685	–	–
		Ref. [7]	5.627	–	–

the thickness ratio $h_1/h_2 = 1.0$ are adopted. The convergent results of frequency parameter are obtained by using Richardson's extrapolation formula for two cases of divisional numbers m (=n) pointed in Table 2. The present results are compared with those obtained by Laura and Gutiérrez [2] and Ju, Lee and Lee [7]. They are in good agreement.

Table 3 shows the numerical values for the lowest 4 natural frequency parameter λ of the plate on non-homogeneous foundations with $h_1/h_2 = 1.0, c/a = 0.6$ and four kinds of combination of K_1 and K_2. The convergent results of frequency parameter are obtained by using Richardson's extrapolation formula for two cases of divisional numbers m (=n) pointed in Table 3. The present results are also in good agreement with those obtained by Laura and Gutiérrez [2] and Ju, Lee and Lee [7]. From Tables 1 ~ 3, it can be seen the present method can be used to solve the problem of plates on homogeneous foundations, local

Table 4 Natural frequency parameter λ for SSSS square plates with stepped thickness in the central part

h_1/h_2	c/a	References	Mode sequence number	
			1st	2nd (3rd)
0.7	0.5	Ex.	4.829	7.498
		Ref. [7]	4.831	–
0.8	0.5	Ex.	4.726	7.404
		Ref. [7]	4.721	7.439
1.5	0.5	Ex.	4.308	6.708
		Ref. [7]	4.280	6.757

Table 5 Natural frequency parameter λ for a SSSS square plate with stepped thickness in the central square part on local uniformly distributed supports ($h_1/h_2 = 0.8, c/a = 0.5$)

K_1	K_2	Reference	Mode sequence number	
			1st	2nd (3rd)
100	0	Ex.	4.805	7.443
1000	0	Ex.	5.364	7.771
0	100	Ex.	4.863	7.426
0	1000	Ex.	5.753	7.613

uniformly distributed supports and non-homogeneous foundations.

5.3 Square plates with stepped thickness in central square part

The numerical calculation is carried out for the plate shown in Figure 1 with $K_1 = K_2 = 0$, which is the case without foundations. The numerical values for the lowest 3 natural frequency parameter λ of the square plate with $c/a = 0.5$ and $h_1/h_2 = 0.7, 0.8, 1.5$ are presented in Table 4. The convergent results of frequency parameter are obtained by using Richardson's extrapolation formula for two cases of divisional numbers m ($=n$) of 12 and 16. The present results are compared with those obtained by Ju, Lee and Lee [7]. It shows the present results have satisfactory accuracy. From Table 4, it can be noted that the frequency parameters decrease with the increase of the ratio h_1/h_2.

5.4 Square plates with stepped thickness on non-homogeneous elastic foundations

As an application of the present method, some numerical results are presented for the plate with stepped thickness in the central square part resting on local uniformly distributed supports or non-homogeneous foundations. The ratios of $h_1/h_2 = 0.8$ and $c/a = 0.5$ are considered. The convergent results of frequency parameter of these plates are obtained by using Richardson's extrapolation formula for two cases of divisional numbers 12 and 16 in Tables 5 and 6. From these two tables, it can be noted for the specific modulus of the foundation K and the ratios of c/a and h_1/h_2, the fundamental frequency parameters of the plate with the central part having higher foundation modulus are higher than those of the plate with the central part having lower foundation modulus and it can also be seen that the modulus of the foundation affects the frequency parameters greatly.

Figure 4 shows the effects of the thickness ratio and the dimensionless modulus of the foundation on the

Table 6 Natural frequency parameter λ for a SSSS square plate with stepped thickness in the central square part on non-homogeneous foundations ($h_1/h_2 = 0.8, c/a = 0.5$)

K_1	K_2	Reference	Mode sequence number	
			1st	2nd (3rd)
10	100	Ex.	4.870	7.430
10	1000	Ex.	5.758	7.617
100	10	Ex.	4.818	7.445
100	1000	Ex.	5.799	7.653
1000	10	Ex.	5.374	7.773
1000	100	Ex.	5.462	7.791

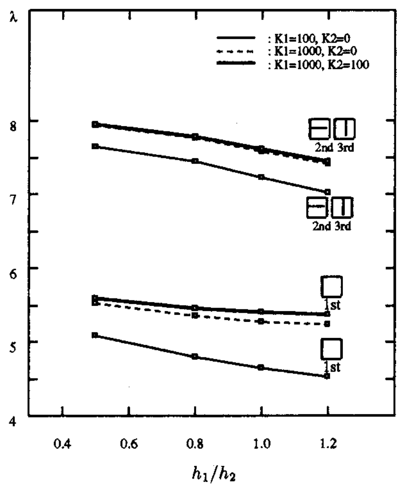

Fig. 4 The natural frequency parameter λ versus the thickness ratio $m(=n)$ for the SSSS square plate on the elastic foundation .

frequency parameter. It can be noted with increase of the thickness ratio, the first and the second frequency parameters decrease for the plates with the specific modulus of the foundation. With increase of the value of the modulus of the foundation, the frequency parameters increase. The nodal lines of the first, second and third mode shapes of the plate with $h_1/h_2 = 1.2$ are also shown in the figure.

6. Conclusions

A discrete method is extended for analyzing the free vibration problem of square plates with stepped thickness on the elastic foundations. No prior assumption of shape of deflection, such as shape functions used in the Finite Element Method, is employed in this method. The spring system is used to simulate the foundations. The characteristic equation of the free vibration is gotten by using the Green function. The effects of the elastic constant of the foundations and the stepped thickness on the frequencies are considered. The results by the present method have been compared with those previously reported. It shows that the present results have a good convergence and satisfactory accuracy.

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Appendix A

$$a_{1i0i1} = a_{3i0i2} = a_{4i0i3} = 1, a_{6i0i4} = a_{7i0i5} = a_{8i0i6} = 1$$

$$b_{20jj1} = b_{30jj2} = b_{50jj3} = 1, b_{60jj4} = b_{70jj5} = b_{80jj6} = 1,$$

$$b_{30002} = 0$$

$$a_{pijfd} = \sum_{t=1}^{8}\left\{\sum_{k=0}^{i} \beta_{ik} A_{pt}[a_{tk0fd} - a_{tkjfd}(1-\delta_{ki})] + \sum_{l=0}^{j} \beta_{jl} B_{pt}[a_{t0lfd} - a_{tilfd}(1-\delta_{lj})] + \sum_{k=0}^{i}\sum_{l=0}^{j} \beta_{ik}\beta_{jl} C_{ptkl} a_{tklfd}(1-\delta_{ki}\delta_{lj})\right\}$$

$$b_{pijfd} = \sum_{t=1}^{8}\left\{\sum_{k=0}^{i} \beta_{ik} A_{pt}[b_{tk0gd} - b_{tkjgd}(1-\delta_{ki})] + \sum_{l=0}^{j} \beta_{jl} B_{pt}[b_{t0lgd} - b_{tilgd}(1-\delta_{lj})] + \sum_{k=0}^{i}\sum_{l=0}^{j} \beta_{ik}\beta_{jl} C_{ptkl} b_{tklgd}(1-\delta_{ki}\delta_{lj})\right\}$$

$$\overline{q}_{pij} = \sum_{t=1}^{8}\left\{\sum_{k=0}^{i} \beta_{ik} A_{pt}[\overline{q}_{tk0} - \overline{q}_{tkj}(1-\delta_{ki})] + \sum_{l=0}^{j} \beta_{jl} B_{pt}[\overline{q}_{t0l} - \overline{q}_{til}(1-\delta_{lj})] + \sum_{k=0}^{i}\sum_{l=0}^{j} \beta_{ik}\beta_{jl} C_{ptkl}\right\} - A_{p1} u_{iq} u_{jr}$$

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