

A Discrete Method for Bending and Free Vibration Analysis of Orthotropic Plates with Non-uniform Thickness

離散的近似解法による変厚直交異方性板の曲げおよび自由振動解析

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A discrete method is proposed for analyzing the bending and free vibration problems of isotropic and orthotropic plates with uniform or non-uniform thickness. The fundamental differential equations involving Dirac's delta functions are established and satisfied exactly throughout the whole plate. By transforming these fundamental equations into integral equations and using numerical integration, the solutions are obtained at the discrete points. The Green function, which is the solution for deflection at the discrete points, is used to obtain the characteristic equation of the free vibration. Some numerical analyses are carried out for the bending problem of isotropic plate with uniform thickness, the free vibration problem of isotropic with uniform thickness and orthotropic plates with variable thickness in one or two directions. The efficiency and accuracy of the numerical solutions by the present method are investigated.

Key Words : *discrete method, orthotropic plate, bending, Green function, vibration*

1. Introduction

The bending and free vibration problems of plates have been studied for many years. Except for the special cases, it is difficult to get the analytical solution of these problems. So a variety of numerical methods, such as the finite element method, the Rayleigh-Ritz method, the boundary element method and the finite strip method, has been developed. Fan [1] studied the bending of the plate with variable thickness in one direction. The concentrated load and the uniform load were considered. Malhotra, Ganesan and Veluswami [2] investigated the vibrations of orthotropic square plates with parabolic thickness variation in one direction. Rayleigh-Ritz method was used to obtain fundamental frequencies for four kinds of boundary conditions. Many papers on the subject of free vibration of plates were well compiled in Ref. [3].

In this paper, a discrete method [4, 5] is used to analyze the bending and free vibration problems of plates. No prior assumption of shape of deflection,

such as shape function used in Rayleigh-Ritz method, is employed in the present method. The fundamental differential equations involving Dirac's delta functions are established and satisfied exactly throughout the whole plate. By transforming these equations into integral equations and using numerical integration, the solutions are obtained at the discrete points. The solutions are only related to the unknown quantities along the boundary. That makes the number of the unknown quantities decrease greatly. With the increase of the divisional numbers, the discrete solutions can converge to the exact solution to any desired degree theoretically. The Green function, which is the solution for deflection, is used to obtain the characteristic equation of the free vibration. The purpose of the paper is to 1) compare the results of the present method with those of the finite element method for the bending problem of the isotropic plate with uniform load or concentrated load, 2) present the numerical results for the free vibration problem of the

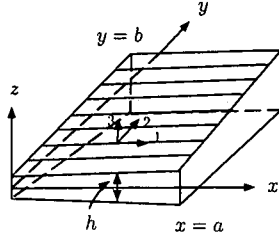


Fig. 1 An orthotropic rectangular plate and coordinate system.

isotropic plate to show the accuracy of the present method, and 3) extend the discrete method for the free vibration of orthotropic plate and give some new data and mode shapes for the plates with general boundary condition and variable thickness in one or two directions.

2. Fundamental Differential Equations

An xyz coordinate system (shown in Figure 1) is used in the present study with its x - y plane contained in middle plane of an orthotropic rectangular plate and the z -axis perpendicular to the middle plane of the plate. The thickness, the length and the width of the orthotropic rectangular plate are h , a and b , respectively. The principle material axes of the plate in the longitudinal, transverse and normal directions are designated as 1, 2 and 3.

The displacements u , v and w in the x , y and z directions are assumed to be

$$u = z\theta_x(x, y), \quad v = z\theta_y(x, y), \quad w = w(x, y), \quad (1)$$

where $\theta_x(x, y)$ and $\theta_y(x, y)$ are the rotations in the xz and yz planes.

For small displacements, the strain-displacement relations of elasticity yield

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} z \frac{\partial \theta_x}{\partial x} \\ z \frac{\partial \theta_y}{\partial y} \\ z \left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right) \\ \frac{\partial w}{\partial y} + \theta_y \\ \frac{\partial w}{\partial x} + \theta_x \end{bmatrix}. \quad (2)$$

For orthotropic plates, the stress-strain relations can be expressed as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 & 0 & 0 \\ 0 & 0 & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & 0 \\ 0 & 0 & 0 & 0 & \bar{Q}_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}, \quad (3)$$

where $\bar{Q}_{11} = E_1/(1 - \nu_{12}\nu_{21})$, $\bar{Q}_{12} = \nu_{12}E_2/(1 - \nu_{12}\nu_{21})$, $\bar{Q}_{22} = E_2/(1 - \nu_{12}\nu_{21})$, $\bar{Q}_{66} = G_{12}$, $\bar{Q}_{44} = G_{23}$, $\bar{Q}_{55} = G_{13}$, E_1 is the axial modulus in the 1-direction, E_2 is the axial modulus in the 2-direction, ν_{12} is the Poisson's ratio associated with loading in the 1-direction and strain in the 2-direction, ν_{21} is the Poisson's ratio associated with loading in the 2-direction and strain in the 1-direction, G_{23} , G_{13} and G_{12} are the shear moduli in 2-3, 1-3 and 1-2 planes.

The moments and the shear forces can be given by

$$\begin{aligned} M_x &= \int_{-h/2}^{h/2} \sigma_x z dz, \quad M_y = \int_{-h/2}^{h/2} \sigma_y z dz, \\ M_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} z dz, \quad Q_y = \int_{-h/2}^{h/2} \tau_{yz} dz, \\ Q_x &= \int_{-h/2}^{h/2} \tau_{xz} dz \end{aligned} \quad (4)$$

By using the equations (2) ~ (4), the relations of the moment-displacement and the shear force-displacement can be obtained as

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_y \\ Q_x \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 & 0 & 0 \\ D_{12} & D_{22} & 0 & 0 & 0 \\ 0 & 0 & D_{66} & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 \\ 0 & 0 & 0 & 0 & A_{55} \end{bmatrix} \begin{bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \\ \frac{\partial w}{\partial y} + \theta_y \\ \frac{\partial w}{\partial x} + \theta_x \end{bmatrix}, \quad (5)$$

where the extensional stiffness $A_{ij} = \bar{Q}_{ij}h$ ($i, j = 4, 5$) and the bending stiffness $D_{ij} = \bar{Q}_{ij}h^3/12$ ($i, j = 1, 2, 6$).

By using the non-dimensional expressions,

$$\begin{aligned} [X_1, X_2] &= \frac{a^2}{D_0(1 - \nu_{12}\nu_{21})} [Q_y, Q_x], \\ [X_3, X_4, X_5] &= \frac{a}{D_0(1 - \nu_{12}\nu_{21})} [M_{xy}, M_y, M_x], \\ [X_6, X_7, X_8] &= \left[\theta_y, \theta_x, \frac{w}{a} \right], \quad [\eta, \zeta, \xi] = \left[\frac{x}{a}, \frac{y}{a}, \frac{z}{h} \right], \end{aligned}$$

the differential equations of the plate with uniform load $\bar{q}$ and a concentrated load $\bar{P}$ at point (x_q, y_r) are established as follows:

$$\sum_{s=1}^8 \left\{ F_{1ts} \frac{\partial X_s}{\partial \zeta} + F_{2ts} \frac{\partial X_s}{\partial \eta} + F_{3ts} X_s \right\} q \delta_{1t} + P \delta(\eta - \eta_q) \delta(\zeta - \zeta_r) \delta_{1t} = 0, \quad (6)$$

where δ_{1t} is Kronecker's delta, $F_{111} = F_{123} = F_{134} = 1$, $F_{146} = \bar{D}_{12}$, $F_{156} = \bar{D}_{22}$, $F_{167} = \bar{D}_{66}$, $F_{188} = \mu k \bar{A}_{45}$, $F_{212} = F_{225} = F_{233} = \mu$, $F_{247} = \mu \bar{D}_{11}$, $F_{257} = \mu \bar{D}_{12}$, $F_{266} = \mu \bar{D}_{66}$, $F_{278} = k \bar{A}_{44}$, $F_{322} = F_{331} = -\mu$, $F_{345} = F_{354} = F_{363} = -\mu \bar{D}$, $F_{371} = F_{382} = -\mu \bar{D}T$, $F_{376} = \mu k \bar{A}_{44}$, $F_{387} = \mu k \bar{A}_{55}$,

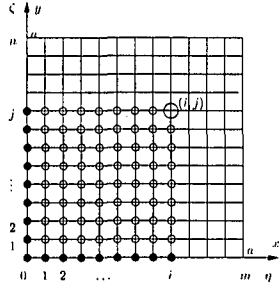


Fig. 2 Discrete points on a rectangular plate.

other $F_{kts} = 0$, $q = \mu(\bar{q}/q_0)q_0a^3/(D_0(1 - \nu_{12}\nu_{21}))$, $P = \bar{P}a/(D_0(1 - \nu_{12}\nu_{21}))$, $\bar{D}_{ij} = \bar{Q}_{ij}/(2E_2)$, $\bar{D} = (h_0/h)^3$, $\bar{A}_{ij} = 12(a/h_0)^2(\bar{Q}_{ij}/E_2)$, $\bar{DT} = h_0/h$, $D_0 = E_2h_0^3/(12(1 - \nu_{12}\nu_{21}))$ is the standard bending rigidity, h_0 is the standard thickness of the plate, $k = 5/6$ is the shear correction factor, $\delta(\eta - \eta_q)$ and $\delta(\zeta - \zeta_r)$ are Dirac's delta functions.

3. Discrete Green Function

By dividing a rectangular plate vertically into m equal-length parts and horizontally into n equal-length parts as shown in Figure 2, the plate can be considered as a group of discrete points which are the intersections of the $(m+1)$ -vertical and $(n+1)$ -horizontal dividing lines. To describe the present method conveniently, the rectangular area, $0 \leq \eta \leq \eta_i$, $0 \leq \zeta \leq \zeta_j$, corresponding to the arbitrary intersection (i, j) as shown in Figure 2 is denoted as the area $[i, j]$, the intersection (i, j) denoted by $\bigcirc$ is called the main point of the area $[i, j]$, the intersections denoted by $\circ$ are called the inner dependent points of the area, and the intersections denoted by $\bullet$ are called the boundary dependent points of the area. By integrating the equation (6) over the area $[i, j]$, the following integral equation is obtained:

$$\begin{aligned} \sum_{s=1}^8 \left\{ F_{1ts} \int_0^{\eta_i} [X_s(\eta, \zeta_j) - X_s(\eta, 0)] d\eta \right. \\ + F_{2ts} \int_0^{\zeta_j} [X_s(\eta_i, \zeta) - X_s(0, \zeta)] d\zeta \\ + F_{3ts} \int_0^{\eta_i} \int_0^{\zeta_j} X_s(\eta, \zeta) d\eta d\zeta \left. \right\} \\ + \int_0^{\eta_i} \int_0^{\zeta_j} q(\eta, \zeta) \delta_{1t} d\eta d\zeta \\ + Pu(\eta - \eta_q)u(\zeta - \zeta_r)\delta_{1t} = 0, \end{aligned} \quad (7)$$

where $u(\eta - \eta_q)$ and $u(\zeta - \zeta_r)$ are the unit step functions.

Next, by applying the numerical integration method, the simultaneous equation for the unknown quantities $X_{sij} = X_s(\eta_i, \zeta_j)$ at the main point (i, j) of the area $[i, j]$ is obtained as follows:

$$\begin{aligned} \sum_{s=1}^8 \left\{ F_{1ts} \sum_{k=0}^i \beta_{ik}(X_{skj} - X_{sk0}) \right. \\ + F_{2ts} \sum_{l=0}^j \beta_{jl}(X_{sil} - X_{s0l}) \\ + F_{3ts} \sum_{k=0}^i \sum_{l=0}^j \beta_{ik}\beta_{jl}X_{skl} \left. \right\} \\ + \sum_{k=0}^i \sum_{l=0}^j \beta_{ik}\beta_{jl}q_{kl}\delta_{1t} \left. \right\} \\ + Pu_{iq}u_{jr}\delta_{1t} = 0, \end{aligned} \quad (8)$$

where $\beta_{ik} = \alpha_{ik}/m$, $\beta_{jl} = \alpha_{jl}/n$, $\alpha_{ik} = 1 - (\delta_{0k} + \delta_{ik})/2$, $\alpha_{jl} = 1 - (\delta_{0l} + \delta_{jl})/2$, $t = 1 \sim 8$, $i = 1 \sim m$, $j = 1 \sim n$, $u_{iq} = u(\eta_i - \eta_q)$, $u_{jr} = u(\zeta_j - \zeta_r)$.

By retaining the quantities at main point (i, j) on the left hand side of the equation and putting other quantities on the right hand side, and using the matrix transition, the solution X_{pij} of the above equation (8) is obtained as follows.

$$\begin{aligned} X_{pij} = \sum_{t=1}^8 \left\{ \sum_{k=0}^i \beta_{ik}A_{pt}[X_{tk0} - X_{tkj}(1 - \delta_{ik})] \right. \\ + \sum_{l=0}^j \beta_{jl}B_{pt}[X_{t0l} - X_{til}(1 - \delta_{jl})] \\ + \sum_{k=0}^i \sum_{l=0}^j \beta_{ik}\beta_{jl}C_{ptkl}X_{tkl}(1 - \delta_{ik}\delta_{jl}) \left. \right\} \\ - \sum_{k=0}^i \sum_{l=0}^j \beta_{ik}\beta_{jl}A_{p1}q_{kl} \\ - A_{p1}Pu_{iq}u_{jr}, \end{aligned} \quad (9)$$

where $p = 1 \sim 8$, A_{pt} , B_{pt} and C_{ptkl} are given in Appendix A.

In the equation (9), the quantity X_{pij} is not only related to the quantities X_{tk0} and X_{t0l} at the boundary dependent points but also the quantities X_{tkj} , X_{til} and X_{tkl} at the inner dependent points. The maximal number of the unknown quantities is $6(m - 1)(n - 1) + 3(m + n + 1)$. In order to reduce the unknown quantities, the area $[i, j]$ is spread according to the regular order as $[1, 1]$, $[1, 2]$, $\dots$, $[1, n]$, $[2, 1]$, $[2, 2]$, $\dots$, $[2, n]$, $\dots$, $[m, 1]$, $[m, 2]$, $\dots$, $[m, n]$. With the spread of the area $[i, j]$, a main point of smaller area becomes one of the inner dependent points of the following larger areas. Whenever the quantity X_{pij} at the main point (i, j) is obtained using the equation

(9) in the above-mentioned order, the quantities X_{tkj} , X_{tli} and X_{tkl} at the inner dependent points of the following larger areas can be eliminated by substituting the obtained results into the corresponding terms of the right hand side of equation (9). By repeating this process, the quantity X_{pij} at the main point is only related to the quantities X_{rk0} ($r=1,3,4,6,7,8$) and X_{s0l} ($s=2,3,5,6,7,8$) at the boundary dependent points. There are six quantities at each boundary dependent points. But half of them are self-evident according to the boundary conditions along the edges $y = 0$ and $x = 0$. Therefore the maximal number of the unknown quantities is reduced to $3(m + n + 1)$. Based on the above consideration, the equation (9) is rewritten as follows.

$$X_{pij} = \sum_{d=1}^6 \left\{ \sum_{f=0}^i a_{pijfd} X_{rf0} + \sum_{g=0}^j b_{pijgd} X_{s0g} \right\} + \bar{q}_{pij} + \bar{q}_{pij} P, \quad (10)$$

where a_{pijfd} , b_{pijgd} , $\bar{q}_{pij}$ and $\bar{q}_{pij}$ are given in Appendix B.

The equation (10) gives the discrete solution of the fundamental differential equation (6) of the bending problem of a plate under uniform load and a concentrated load.

4. Characteristic Equation of Rectangular Plate

By applying the Green function $w(x_0, y_0, x, y)/\bar{P}$ which is the displacement at a point (x_0, y_0) of a plate with a concentrated load $\bar{P}$ at a point (x, y) , the displacement amplitude $\hat{w}(x_0, y_0)$ at a point (x_0, y_0) of the rectangular plate during the free vibration is given as follows:

$$\hat{w}(x_0, y_0) = \int_0^a \int_0^a \rho h \omega^2 \hat{w}(x, y) [w(x_0, y_0, x, y)/\bar{P}] dx dy, \quad (11)$$

where ρ is the mass density of the plate material.

By using the numerical integration method and the following non-dimensional expressions,

$$\begin{aligned} \lambda^4 &= \frac{\rho_0 h_0 \omega^2 a^4}{D_0(1 - \nu_{12}\nu_{21})}, H(\eta, \zeta) = \frac{\rho(x, y)}{\rho_0} \frac{h(x, y)}{h_0}, \\ G(\eta_0, \zeta_0, \eta, \zeta) &= \frac{w(x_0, y_0, x, y)}{a} \frac{D_0(1 - \nu_{12}\nu_{21})}{\bar{P}a}, \\ W(\eta, \zeta) &= \frac{\hat{w}(x, y)}{a}, \quad \mu = b/a, \quad k_f = 1/(\mu\lambda^4), \end{aligned}$$

where ρ_0 is the standard mass density, the characteristic equation is obtained from the equation (11)

Table 1 Material properties of isotropic and orthotropic square plates

Material	$\frac{E_1}{E_2}$	$\frac{G_{12}}{E_2}$	$\frac{G_{13}}{E_2}$	$\frac{G_{23}}{E_2}$	ν_{12}
isotropic	1	0.385	0.385	0.385	0.3
orthotropic	2.448	0.484	0.484	0.484	0.23

as

$$\begin{vmatrix} K_{00} & K_{01} & K_{02} & \dots & K_{0m} \\ K_{10} & K_{11} & K_{12} & \dots & K_{1m} \\ K_{20} & K_{21} & K_{22} & \dots & K_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{m0} & K_{m1} & K_{m2} & \dots & K_{mm} \end{vmatrix} = 0, \quad (12)$$

where

$$K_{ij} = \beta_{mj} \begin{bmatrix} \beta_{n0} H_{j0} G_{i0j0} - k_f \delta_{ij} & \dots & \beta_{nn} H_{jn} G_{i0jn} \\ \beta_{n0} H_{j0} G_{i1j0} & \dots & \beta_{nn} H_{jn} G_{i1jn} \\ \beta_{n0} H_{j0} G_{i2j0} & \dots & \beta_{nn} H_{jn} G_{i2jn} \\ \vdots & \ddots & \vdots \\ \beta_{n0} H_{j0} G_{inj0} & \dots & \beta_{nn} H_{jn} G_{injn} - k_f \delta_{ij} \end{bmatrix}.$$

Using the QR double-step method and the inverse iteration method, the eigenvalues and eigenvectors of the equation (12) can be calculated, respectively.

5. Numerical Results

The described method is used to analyze the bending and free vibration problems of plate with uniform or non-uniform thickness. Isotropic and orthotropic materials are used. In order to compare the present results with those previously reported, the orthotropic material is chosen as E-glass/epoxy material. Its properties are given as $E_1 = 60.7$ GPa, $E_2 = 24.8$ GPa, $G_{12} = G_{13} = G_{23} = 12.0$ GPa, $\nu_{12} = 0.23$. The ratios of E_1/E_2 , G_{12}/E_2 , G_{13}/E_2 and G_{23}/E_2 are shown in Table 1. The thickness functions are chosen as $h = h_0(1 + \alpha x/a)$ and $h = h_0(1 + \alpha x/a)(1 + \beta y/b)$ for variable thickness in one and two directions, respectively. In all tables and figures, the symbols F, S, and C denote free, simply supported and clamped edges. Four symbols such as SCFC delegate the boundary conditions of the plate, the first indicating the conditions at $x = 0$, the second at $y = 0$, the third at $x = a$ and the fourth at $y = b$.

5.1 Bending Analysis of SSSS and CCCC Square Isotropic Plates

Due to the symmetry, only the $1/4$ plate is used to obtain the deflections at the central point $C(a/2, b/2)$ of the whole plate for SSSS and CCCC plates ($\mu=1$). Uniform load and concentrated load are considered.

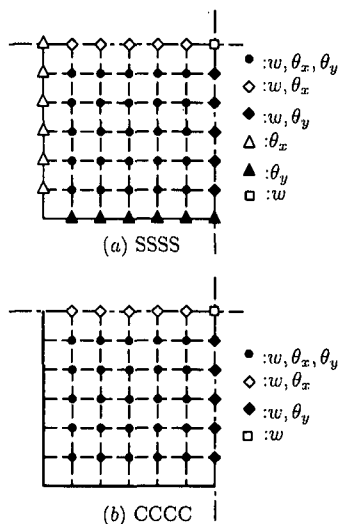


Fig. 3 Unknown quantities at every discrete point for the finite element method.

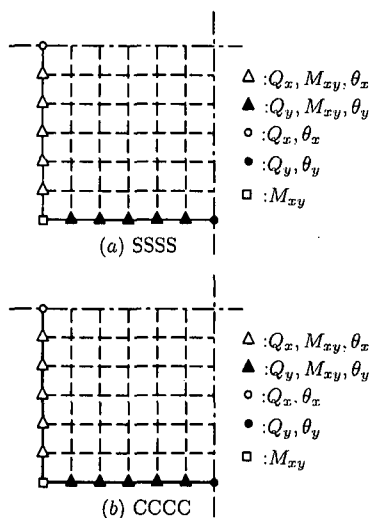


Fig. 4 Unknown quantities at every discrete point for the present method.

Figures 3 and 4 show the unknown quantities at every discrete point for the FEM and the present method, respectively. It can be concluded if the divisional number is m and the FEM is used, the numbers of the unknown quantities are $3m^2$ for SSSS plate and $3m^2 - 2m$ for CCCC plate. If the present method is used, the number of the unknown quantities is $6m - 1$ for both the SSSS and CCCC plates. So for the same divisional number m , the numbers of the unknown quantities of the FEM and the present method are different. The unknown quantities of the present method are fewer than those of the FEM when $m \geq 3$. Ta-

Table 2 The central deflection w_c of SSSS isotropic plate under uniform and concentrated loads

Source	Element	Num.*	Uni. L. ($w_c/qa^4/D \times 10^3$)	Con. L. ($w_c/Pa^2/D \times 10^2$)
Present	2×2	11	4.133	0.836
	4×4	23	4.077	1.038
	6×6	35	4.069	1.093
	8×8	47	4.066	1.119
	14×14	83	4.063	1.144
	16×16	95	4.063	1.147
FEM [6]	2×2	12	3.939	1.233
	4×4	48	4.033	1.183
	6×6	108	4.050	1.172
	8×8	192	4.056	1.167
Exact [7]			4.062	1.160

Num.*: the number of the unknown quantities.

Table 3 The central deflection w_c of CCCC isotropic plate under uniform and concentrated loads

Source	Element	Num.	Uni. L. ($w_c/qa^4/D \times 10^3$)	Con. L. ($w_c/Pa^2/D \times 10^2$)
Present	2×2	11	1.304	0.2936
	4×4	23	1.274	0.4563
	6×6	35	1.271	0.5046
	8×8	47	1.269	0.5256
	14×14	83	1.268	0.5479
	16×16	95	1.268	0.5509
FEM [6]	2×2	8	1.403	0.6134
	4×4	40	1.304	0.5803
	6×6	96	1.283	0.5710
	8×8	176	1.275	0.5672
Exact [7]			1.265	0.5600

bles 2 ~ 3 show the number of elements, the number of the unknown quantities, and the numerical results. From these tables, it can be seen that the tendency of the change of the deflection with the divisional number m is different for the uniform load and the concentrated load in the present method. The uniform load can be estimated accurately even for the small divisional number m , but the concentrated load can't. For the small number m , the concentrated load is underestimated as a uniform load in a definite area. Therefore, the deflection obtained by the present method is smaller than the exact solution. With the increase of the number m , the concentrated load can be satisfied more and more accurately and the present result is more and more close to the exact solution. Com-

Table 4 Natural frequency parameter λ for SSSS isotropic square plate with uniform thickness

Source	Mode sequence number				
	1st	2nd	3rd	4th	5th
Present					
12 × 12	4.574	7.333	7.333	9.306	10.672
16 × 16	4.563	7.270	7.270	9.211	10.442
Ex.*	4.548	7.188	7.188	9.011	10.146
Ref. [9]	4.558	7.246	7.246	9.365	10.163
Ref. [10]	4.529	7.110	7.110	9.001	10.034
Ref. [11]	4.549	7.192	7.192	9.098	10.172

Ex.*:the values obtained by using Richardson's extrapolation formula.

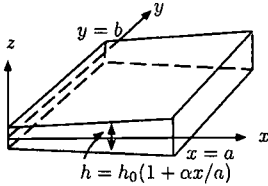


Fig. 5 An orthotropic rectangular plate with variable thickness in one direction.

pared with the FEM, the present method has the advantage of using less unknown quantities to obtain satisfactory accuracy. Both the present method and the FEM agree well with the exact solution [7] for the bending plate with a concentrated load and a uniform load.

5.2 Free Vibration of Isotropic and Orthotropic Plates with Uniform or Variable Thickness

Table 4 presents the numerical values for the lowest 5 natural frequency parameter λ of SSSS isotropic square plates with uniform thickness. The convergent values of frequency parameter are obtained by using Richardson's extrapolation formula for two cases of divisional numbers $m = n = 12$ and $m = n = 16$. The FEM values obtained by Ali and Atwal [9], Kaushal and Bhat [10] and the exact solutions [11] are also shown. By comparing the present results and FEM results with the exact solutions, it can be found the present results are more accuracy than the FEM results.

Table 5 presents numerical values for the lowest 3 natural frequency parameter λ of CSSS, SSSS, SSFS

Table 5 Natural frequency parameter λ for CSSS, SSSS and SSFS orthotropic plates with variable thickness in one direction

B.C.	b/a	α	Source	Mode number		
				1st	2nd	3rd
CSSS	1.0	0.0	Ex.	5.572	7.548	9.249
			Ref. [12]	5.574	7.553	9.263
		0.4	Ex.	6.030	8.273	10.050
			Ref. [12]	6.034	8.281	10.068
		0.8	Ex.	6.439	8.905	10.745
	2.0	0.0	Ex.	5.084	5.572	6.415
			Ref. [12]	5.086	5.574	6.425
		0.4	Ex.	5.445	6.030	7.003
			Ref. [12]	5.448	6.034	7.015
		0.8	Ex.	5.761	6.436	7.524
SSSS	1.0	0.0	Ex.	4.902	7.253	8.374
			Ref. [12]	4.902	7.256	8.382
		0.4	Ex.	5.360	7.928	9.150
			Ref. [12]	5.362	7.932	9.159
		0.8	Ex.	5.770	8.525	9.831
	2.0	0.0	Ex.	4.190	4.901	5.966
			Ref. [12]	4.191	4.902	5.975
		0.4	Ex.	4.575	5.360	6.526
			Ref. [12]	4.575	5.362	6.637
		0.8	Ex.	4.909	5.771	7.029
SSFS	1.0	0.0	Ex.	3.533	5.945	6.509
			Ref. [12]	3.528	5.956	6.520
		0.4	Ex.	3.916	6.485	7.310
			Ref. [12]	3.921	6.492	7.316
		0.8	Ex.	4.280	6.967	7.994
	2.0	0.0	Ex.	2.126	3.523	4.994
			Ref. [12]	2.125	3.528	5.002
		0.4	Ex.	2.323	3.918	5.620
			Ref. [12]	2.325	3.921	5.600
		0.8	Ex.	2.519	4.281	5.982

orthotropic plates (shown in Figure 5). Taper ratios $\alpha = 0.0, 0.4, 0.8$ and aspect ratios $b/a = 1.0, 2.0$ are considered. It can be seen that the frequency parameters increase with increase of the taper ratio α for the plate with specific boundary condition and aspect ratio b/a , and decrease with the increase of aspect ratio b/a for the plate with the same boundary condition and taper ratio α . The effect of boundary condition on the frequency parameters can be observed by comparing the corresponding results presented in Table 5. In the table, the highest frequency parameters were obtained for CSSS plates, then successively for SSSS and SSFS. It shows that with decrease of boundary

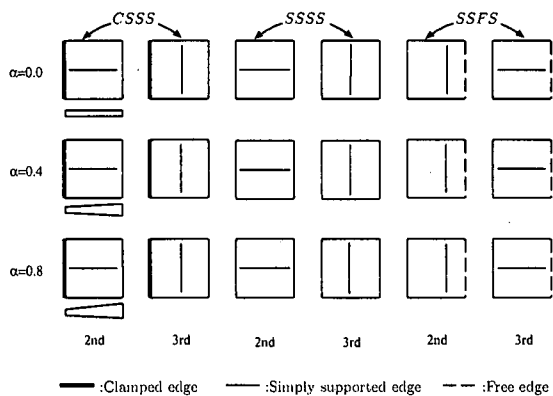


Fig. 6 Nodal patterns for CSSS, SSSS and SSFS orthotropic square plates with variable thickness in one direction ($h = h_0(1 + \alpha \times x/a)$).

constraints, which results in decrease of stiffness, frequency parameters decrease significantly. The results obtained by Bert and Malik [12] are also shown in above table. It can be seen that the numerical results of the present method have satisfactory accuracy. The results presented in Table 5 are limited to plates with two opposite edges having the same thickness and simply supported boundary conditions.

The nodal patterns of the lowest 3 modes of the above plates with $b/a = 1$ are shown in Figure 6. From Figure 6, it is noticed that the vertical nodal lines tend to be close to simply supported edges for CSSS plates and to be close to the free edges for SSFS plates. These show the vertical nodal lines have the tendency to be close to the edge with less boundary constraint. It can be also noted that with increase of taper ratio, the vertical nodal lines move to the thinner part of the plates gradually.

Table 6 presents the numerical results for the lowest 3 natural frequency parameter λ of the CCCC, CCSC and FCCC plates with taper ratios $\alpha = 0.0, 0.4, 0.8$ and aspect ratios $b/a = 0.5, 1.0$. The results obtained by the present method are compared with those of Ashour [13]. It can be noticed that the present results again agree well with Ashour's results for CCCC plates. For CCSC plates with $b/a = 1$, they still agree each other, but for plates with $b/a = 0.5$, difference can be found. The difference can also be found in the results of FCCC plates. Due to the lack of the published information on orthotropic plates with variable thickness, no other suitable references can be used for comparison. But comparing the results of CCCC and FCCC, it can be inferred Ashour may have lost some

Table 6 Natural frequency parameter λ for CCCC, CCSC and FCCC orthotropic plates with variable thickness in one direction

B.C.	b/a	α	Source	Mode number		
				1st	2nd	3rd
CCCC	0.5	0.0	Ex.	10.194	12.289	15.301
			Ref. [13]	10.206	12.318	15.393
		0.4	Ex.	11.113	13.422	16.695
			Ref. [13]	11.132	13.463	16.813
		0.8	Ex.	11.894	14.419	17.908
	1.0	0.0	Ex.	6.780	8.953	10.293
			Ref. [13]	6.785	8.967	10.317
		0.4	Ex.	7.402	9.770	11.232
			Ref. [13]	7.410	9.787	11.265
		0.8	Ex.	7.945	10.475	12.046
CCSC	0.5	0.0	Ex.	10.013	11.783	14.597
			Ref. [13]	11.805	14.662	16.124
			Ref. [14]	10.035	11.830	14.710
		0.4	Ex.	10.982	12.896	15.928
			Ref. [13]	12.928	16.012	17.535
		0.8	Ex.	11.796	13.892	17.096
	1.0	0.0	Ex.	6.156	8.683	9.435
			Ref. [13]	6.159	8.695	9.450
		0.4	Ex.	6.707	9.528	10.260
			Ref. [13]	6.711	9.544	10.280
		0.8	Ex.	7.196	10.256	10.982
FCCC	0.5	0.0	Ex.	9.601	10.557	12.642
			Ref. [13]	15.632	17.828	22.150
		0.4	Ex.	10.025	11.454	13.788
			Ref. [13]	16.429	17.088	22.831
		0.8	Ex.	10.379	12.264	14.815
	1.0	0.0	Ex.	5.017	7.082	8.050
			Ref. [13]	7.068	9.436	10.457
		0.4	Ex.	5.330	7.793	8.458
			Ref. [13]	7.782	10.258	11.495
		0.8	Ex.	5.636	8.434	8.820

of the lower frequency parameters for FCCC plates. According to the conclusions obtained earlier, the frequencies of FCCC plates should be lower than those corresponding results of CCCC plates. So the fundamental frequency of FCCC plate with $\alpha = 0.4$ and $b/a = 0.5$ is expected to be lower than 11.132 but his result is 16.429. In order to confirm the accuracy of the present method further, the lower 3 frequency parameters of CCSC plate are calculated by using the solution obtained by Hearmon [14] for E-glass/epoxy plates with uniform thickness and $b/a = 0.5$. These results are also shown in Table 6. Although the re-

Table 7 Natural frequency parameter λ for CCCS, SSSC and SSCC orthotropic plates with variable thickness in one direction

B.C.	b/a	α	Mode sequence number		
			1st	2nd	3rd
CCCS	0.5	0.0	8.958	11.616	14.681
		0.4	9.776	12.681	15.944
		0.8	10.482	13.612	16.950
	1.0	0.0	6.533	8.417	10.211
		0.4	7.133	9.189	11.143
		0.8	7.656	9.857	11.949
SSSC	0.5	0.0	8.493	10.475	13.454
		0.4	9.269	11.454	14.693
		0.8	9.938	12.330	15.780
	1.0	0.0	5.238	7.855	8.483
		0.4	5.824	8.463	9.195
		0.8	6.164	9.219	9.966
SSCC	0.5	0.0	8.689	11.009	14.183
		0.4	9.432	12.027	15.658
		0.8	10.079	12.925	16.650
	1.0	0.0	5.818	8.090	9.330
		0.4	6.399	8.795	10.232
		0.8	6.912	9.414	11.017

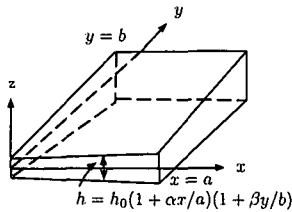


Fig. 7 The orthotropic rectangular plate with variable thickness in two directions

sults presented in Table 6 are not limited to plates with two opposite edges simply supported, they are still limited to plates with two opposite edges having the same boundary conditions and the same thickness.

As an application of the present method, the numerical results are presented for the plates with general boundary conditions. The number of the combination of the boundary conditions is too large to be considered completely, so the numerical results of the lowest 3 natural frequency parameter λ are given only for CCCS, SSSC and SSCC plates with taper ratios $\alpha = 0.0, 0.4, 0.8$ and aspect ratios $b/a = 0.5, 1.0$. These results are shown in Table 7.

Table 8 Natural frequency parameter λ for orthotropic plates with variable thickness in two directions

B.C.	α	β	Mode sequence number		
			1st	2nd	3rd
CCCC	-0.5	-0.5	4.955	6.548	7.440
	-0.5	0.5	6.453	8.510	9.748
	0.5	-0.5	6.447	8.525	9.671
	0.5	0.5	8.390	11.076	12.666
SSSC	-0.5	-0.5	3.872	5.716	6.252
	-0.5	0.5	5.038	7.499	8.015
	0.5	-0.5	5.016	7.442	8.115
	0.5	0.5	6.536	9.729	10.398
SSSS	-0.5	-0.5	3.635	5.335	6.086
	-0.5	0.5	4.704	6.937	7.966
	0.5	-0.5	4.708	6.933	7.904
	0.5	0.5	6.086	9.022	10.350
SCFC	-0.5	-0.5	3.431	4.815	5.475
	-0.5	0.5	4.467	6.303	6.989
	0.5	-0.5	4.831	6.231	7.879
	0.5	0.5	6.301	8.084	10.235
CCCS	-0.5	-0.5	4.734	6.184	7.258
	-0.5	0.5	6.259	8.009	9.720
	0.5	-0.5	6.158	8.052	9.444
	0.5	0.5	8.137	10.417	12.630
SSCC	-0.5	-0.5	4.248	5.959	6.779
	-0.5	0.5	5.481	7.795	8.715
	0.5	-0.5	5.637	7.653	8.922
	0.5	0.5	7.248	9.991	11.475

As another application of the present method, the numerical results are given for the plates with linearly variable thickness in two directions (shown in Figure 7). Table 8 presents the results for the plates with six kinds of boundary conditions and four kinds of thickness variation. It can be noted the boundary conditions and the taper ratio affect the frequency parameters greatly. With the increase of the boundary constraint and the taper ratio, the frequency parameters will increase.

6. Conclusions

A discrete method is described for analyzing the bending and the free vibration problems of isotropic and orthotropic rectangular plates with variable thickness. The characteristic equation of the free vibration is obtained by using the Green function. A comparison of the present method and the FEM is made. The effects of the boundary conditions, aspect ratios and variable thickness in one and two directions on the frequencies are considered. The results by the present method have been compared with those previously reported. It shows that the present method has a good convergence and satisfactory accuracy.

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Appendix A

$$\begin{aligned}
 A_{p1} &= \gamma_{p1}, A_{p2} = 0, A_{p3} = \gamma_{p2}, \\
 A_{p4} &= \gamma_{p3}, A_{p5} = 0, \\
 A_{p6} &= \bar{D}_{12}\gamma_{p4} + \bar{D}_{22}\gamma_{p5} + \bar{D}_{26}\gamma_{p6}, \\
 A_{p7} &= \bar{D}_{16}\gamma_{p6} + \bar{D}_{26}\gamma_{p7} + \bar{D}_{66}\gamma_{p8}, \\
 A_{p8} &= k(\bar{A}_{44}\gamma_{p7} + \bar{A}_{45}\gamma_{p8}), \\
 B_{p1} &= 0, B_{p2} = \mu\gamma_{p1}, B_{p3} = \mu\gamma_{p3}, \\
 B_{p4} &= 0, B_{p5} = \mu\gamma_{p2}, \\
 B_{p6} &= \mu(\bar{D}_{16}\gamma_{p4} + \bar{D}_{26}\gamma_{p5} + \bar{D}_{66}\gamma_{p6}), \\
 B_{p7} &= \mu(\bar{D}_{11}\gamma_{p4} + \bar{D}_{12}\gamma_{p5} + \bar{D}_{16}\gamma_{p6}), \\
 B_{p8} &= \mu k(\bar{A}_{45}\gamma_{p7} + \bar{A}_{55}\gamma_{p8}), \\
 C_{p1kl} &= \mu\gamma_{p3} + \mu\bar{D}T_{kl}\gamma_{p7}, \\
 C_{p2kl} &= \mu\gamma_{p2} + \mu\bar{D}T_{kl}\gamma_{p8}, \\
 C_{p3kl} &= \mu\bar{D}_{kl}\gamma_{p6}, C_{p4kl} = \mu\bar{D}_{kl}\gamma_{p7}, \\
 C_{p5kl} &= \mu\bar{D}_{kl}\gamma_{p4}, \\
 C_{p6kl} &= -\mu k(\bar{A}_{44}\gamma_{p7} + \bar{A}_{45}\gamma_{p8}), \\
 C_{p7kl} &= -\mu k(\bar{A}_{45}\gamma_{p7} + \bar{A}_{55}\gamma_{p8}), \\
 C_{p8kl} &= 0, [\gamma_{pt}] = [\rho_{tp}]^{-1}, \\
 \rho_{11} &= \beta_{ii}, \rho_{12} = \mu\beta_{jj}, \rho_{22} = -\mu\beta_{ij}, \\
 \rho_{23} &= \beta_{ii}, \rho_{25} = \mu\beta_{jj}, \rho_{31} = -\mu\beta_{ij}, \\
 \rho_{33} &= \mu\beta_{jj}, \rho_{34} = \beta_{ii}, \rho_{45} = -\mu\beta_{ij}\bar{D}_{ij}, \\
 \rho_{46} &= \bar{D}_{12}\beta_{ii} + \mu\bar{D}_{16}\beta_{jj}, \rho_{47} = \bar{D}_{16}\beta_{ii} + \mu\bar{D}_{11}\beta_{jj}, \\
 \rho_{54} &= -\mu\beta_{ij}\bar{D}_{ij}, \rho_{56} = \bar{D}_{22}\beta_{ii} + \mu\bar{D}_{26}\beta_{jj}, \\
 \rho_{57} &= \bar{D}_{26}\beta_{ii} + \mu\bar{D}_{12}\beta_{jj}, \rho_{63} = -\mu\beta_{ij}\bar{D}_{ij}, \\
 \rho_{66} &= \bar{D}_{26}\beta_{ii} + \mu\bar{D}_{66}\beta_{jj}, \rho_{67} = \bar{D}_{66}\beta_{ii} + \mu\bar{D}_{16}\beta_{jj}, \\
 \rho_{71} &= -\mu\beta_{ij}\bar{D}_{ij}, \rho_{76} = \mu k\bar{A}_{44}\beta_{ij}, \\
 \rho_{77} &= \mu k\bar{A}_{45}\beta_{ij}, \rho_{78} = k(\bar{A}_{44}\beta_{ii} + \mu\bar{A}_{45}\beta_{jj}), \\
 \rho_{82} &= -\mu\beta_{ij}\bar{D}_{ij}, \rho_{86} = \mu k\bar{A}_{45}\beta_{ij}, \\
 \rho_{87} &= \mu k\bar{A}_{55}\beta_{ij}, \rho_{88} = k(\bar{A}_{45}\beta_{ii} + \mu\bar{A}_{55}\beta_{jj}), \\
 \text{other } \rho_{tp} &= 0
 \end{aligned}$$

Appendix B

$$\begin{aligned}
a_{1i0i1} &= a_{3i0i2} = a_{4i0i3} = 1, & a_{6i0i4} &= a_{7i0i5} = a_{8i0i6} = 1, & \bar{q}_{pij} &= \sum_{t=1}^8 \left\{ \sum_{k=0}^i \beta_{ik} A_{pt} [\bar{q}_{tk0} - \bar{q}_{tkj}(1 - \delta_{ki})] \right. \\
b_{20jj1} &= b_{30jj2} = b_{50jj3} = 1, & b_{60jj4} &= b_{70jj5} = b_{80jj6} = 1, & & + \sum_{l=0}^j \beta_{jl} B_{pt} [\bar{q}_{t0l} - \bar{q}_{til}(1 - \delta_{lj})] \\
b_{30002} &= 0, & & & & + \sum_{k=0}^i \sum_{l=0}^j \beta_{ik} \beta_{jl} C_{ptkl} \bar{q}_{tkl} (1 - \delta_{ki} \delta_{lj}) \Big\} \\
a_{pijfd} &= \sum_{t=1}^8 \left\{ \sum_{k=0}^i \beta_{ik} A_{pt} [a_{tk0fd} - a_{tkjfd}(1 - \delta_{ki})] \right. \\
&\quad + \sum_{l=0}^j \beta_{jl} B_{pt} [a_{t0lfd} - a_{tilfd}(1 - \delta_{lj})] \\
&\quad + \sum_{k=0}^i \sum_{l=0}^j \beta_{ik} \beta_{jl} C_{ptkl} a_{tklfd} (1 - \delta_{ki} \delta_{lj}) \Big\} \\
b_{pijfd} &= \sum_{t=1}^8 \left\{ \sum_{k=0}^i \beta_{ik} A_{pt} [b_{tk0gd} - b_{tkjgd}(1 - \delta_{ki})] \right. \\
&\quad + \sum_{l=0}^j \beta_{jl} B_{pt} [b_{t0lgd} - b_{tilgd}(1 - \delta_{lj})] \\
&\quad + \sum_{k=0}^i \sum_{l=0}^j \beta_{ik} \beta_{jl} C_{ptkl} b_{tklgd} (1 - \delta_{ki} \delta_{lj}) \Big\} \\
\bar{q}_{pij} &= \sum_{t=1}^8 \left\{ \sum_{k=0}^i \beta_{ik} A_{pt} [\bar{q}_{tk0} - \bar{q}_{tkj}(1 - \delta_{ki})] \right. \\
&\quad + \sum_{l=0}^j \beta_{jl} B_{pt} [\bar{q}_{t0l} - \bar{q}_{til}(1 - \delta_{lj})] \\
&\quad + \sum_{k=0}^i \sum_{l=0}^j \beta_{ik} \beta_{jl} C_{ptkl} - A_{p1} u_{iq} u_{jr} \Big\}
\end{aligned}$$

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