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Numerical Solution of an Under-determined Problem of the Laplace Equatin

不足決定系ラプラス方程式の数値解法

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The purpose of this paper is to present an attempt at a numerical treatment of a kind of under-determined problem of the Laplace equation in two spatial dimensions. A resolution is sought for the problem in which the Dirichlet and Neumann data are arbitrarily imposed on each part of the boundary of the domain. This new problem can be regarded as a boundary inverse problem, in which the proper boundary conditions are to be identified for the rest of the boundary. The solution of this problem is not unique. The treatment is based on the direct variational method, and a functional is minimized by the method of the steepest descent. The minimization problem is recast into successive primary and dual boundary value problems of the Laplace equation. After numerical computations by using the boundary elements, it is concluded that our scheme is stable, but the numerical solutions converge to the nearest local minimum.

Key Words: *inverse analysis, Laplace equation, direct variational method, numerical optimization, boundary element method*

1 Problem of the Laplace equation

Let $\Omega \subset \boldsymbol{R}^2$ be a domain bounded by a smooth curve Γ with the rectangular coordinates $\boldsymbol{x} = (x_1, x_2)$. We consider the Laplace equation

$$\Delta u(\boldsymbol{x}) = 0 \qquad \text{in} \quad \Omega. \qquad (1)$$

Let $\boldsymbol{n}$ denote the outward unit normal to the boundary Γ, and let $q = \dfrac{\partial u}{\partial n}$ on the boundary.

Assume that the Dirichlet data $\bar{u}$ are prescribed on a non-zero measure part of the boundary Γ_u of Γ, and the Neumann data $\bar{q}$ are prescribed on other non-zero measure part of the boundary Γ_q of Γ. We notice that Γ_u and Γ_q are taken arbitrarily. Here we consider the case when data are contaminated with measurement errors. We notice also that, even if the data $\bar{u}$ and $\bar{q}$ are exact, the solution $u(\boldsymbol{x})$ of the Laplace equation is not uniquely determined provided that $\Gamma_u \cap \Gamma_q = \phi$. We consider such new problem by reflecting a real situation in practice of the measurement. We call the problem of this kind *boundary inverse problem*, because the problem falls essentially on identification of the proper boundary values.

Basic idea for resolution of the problem consists of identifying a proper boundary value $u = \omega$ for the rest of the boundary $\Gamma_u^c = \Gamma \backslash \Gamma_u$ so that the solution $u(\boldsymbol{x})$ of the Laplace equation also satisfies the boundary conditions $q = \bar{q}$ given on Γ_q. If $\Gamma_u \cup \Gamma_q = \Gamma$, then the corresponding problem is the conventional boundary value problem of the Laplace equation. When Γ_q coincides with Γ_u, the corresponding problem reduces to the conventional Cauchy problem of the Laplace equation, whose variational approach was investigated in [1]. The problem considered in this paper is a relaxation of those two problems, and we follow the approach presented in [1] in order to resolve this relaxed problem.

2 Direct variational approach

We write $u(\boldsymbol{x}) = u(\boldsymbol{x};\omega)$ to stress explicitly the dependence of the solution of the Laplace equation Eq.(1) on the boundary value ω to be identified on Γ_u^c. By the words *proper boundary values,* we mean the boundary values ω which minimize the object functional

$$\begin{aligned} J(\omega) &= \int_{\Gamma_q} |q(\boldsymbol{x};\omega) - \bar{q}(\boldsymbol{x})|^2 d\Gamma \\ &\quad + \eta \int_\Omega |\nabla u(\boldsymbol{x};\omega)|^2 d\Omega \\ &\quad \text{with} \quad u|_{\Gamma_u} = \bar{u}\,. \end{aligned} \tag{2}$$

Here we insist $J : H^{1/2}(\Gamma_u^c) \ni \omega \longmapsto \boldsymbol{R}_+ = [0,+\infty)$. The Dirichlet integral added in Eq.(2) as a regularizer with the regularization parameter $\eta > 0$ guarantees existence of the local minimum of the functional $J(\omega)$.

In fact, let the variation $\delta\omega$ of $\omega(\boldsymbol{x})$ on Γ be of the form $\omega(\boldsymbol{x}) = \omega^{(0)}(\boldsymbol{x}) + \varepsilon\omega^{(1)}(\boldsymbol{x})$ with arbitrary real number ε, where $\omega^{(0)}(\boldsymbol{x}) = \bar{u}(\boldsymbol{x})$ and $\omega^{(1)}(\boldsymbol{x}) = 0$ on Γ_u. The solution $u(\boldsymbol{x})$ of Eq.(1) must satisfy the Green's identity

$$\begin{aligned} u(\boldsymbol{\xi}) &= -\int_\Gamma u(\boldsymbol{x}) \frac{\partial G}{\partial n(\boldsymbol{x})}(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x}) \\ &\quad + \int_\Gamma \frac{\partial u}{\partial n}(\boldsymbol{x}) G(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x})\,, \qquad \boldsymbol{\xi} \in \Omega\,, \end{aligned} \tag{3}$$

and the boundary integral equation

$$\begin{aligned} \frac{1}{2}u(\boldsymbol{\xi}) &+ \int_\Gamma u(\boldsymbol{x}) \frac{\partial G}{\partial n(\boldsymbol{x})}(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x}) \\ &= \int_\Gamma \frac{\partial u}{\partial n}(\boldsymbol{x}) G(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x})\,, \qquad \boldsymbol{\xi} \in \Gamma\,, \end{aligned} \tag{4}$$

where $G(\boldsymbol{x};\boldsymbol{\xi})$ is the fundamental solution to the Laplacian Δ; $-\Delta G(\boldsymbol{x};\boldsymbol{\xi}) = \delta(\boldsymbol{x}-\boldsymbol{\xi})$ with the Dirac measure $\delta(\cdot)$ at the point $\boldsymbol{\xi}$. Due to the variation $\delta\omega = \varepsilon\omega^{(1)}$ on Γ_u^c we assume that $u(\boldsymbol{x}) = u^{(0)}(\boldsymbol{x}) + \varepsilon u^{(1)}(\boldsymbol{x})$ on $\bar{\Omega}$. Substituting this expression into Eqs.(3) and (4), and noticing the arbitrariness of ε, we obtain

$$\begin{aligned} u^{(j)}(\boldsymbol{\xi}) &= -\int_\Gamma u^{(j)}(\boldsymbol{x}) \frac{\partial G}{\partial n(\boldsymbol{x})}(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x}) \\ &\quad + \int_\Gamma \frac{\partial u^{(j)}}{\partial n}(\boldsymbol{x}) G(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x})\,, \qquad \boldsymbol{\xi} \in \Omega\,, \\ \frac{1}{2}u^{(j)}(\boldsymbol{\xi}) &+ \int_\Gamma u^{(j)}(\boldsymbol{x}) \frac{\partial G}{\partial n(\boldsymbol{x})}(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x}) \\ &= \int_\Gamma \frac{\partial u^{(j)}}{\partial n}(\boldsymbol{x}) G(\boldsymbol{x};\boldsymbol{\xi}) d\Gamma(\boldsymbol{x})\,, \qquad \boldsymbol{\xi} \in \Gamma \end{aligned}$$

for $j = 0, 1$. This implies that $q(\boldsymbol{x}) = \frac{\partial u}{\partial n}(\boldsymbol{x})$ on Γ has the form $q^{(0)}(\boldsymbol{x}) + \varepsilon q^{(1)}(\boldsymbol{x})$, and we can see that

$$\begin{aligned} & J(\omega + \delta\omega) \\ = & \int_{\Gamma_q} |q^{(0)}(\boldsymbol{x}) + \varepsilon q^{(1)}(\boldsymbol{x}) - \bar{q}(\boldsymbol{x})|^2 d\Gamma \\ & \quad + \eta \int_\Omega |\nabla \left(u^{(0)}(\boldsymbol{x}) + \varepsilon u^{(1)}(\boldsymbol{x})\right)|^2 d\Omega \\ = & \int_{\Gamma_q} |q^{(0)}(\boldsymbol{x}) - \bar{q}(\boldsymbol{x})|^2 d\Gamma + \eta \int_\Omega |\nabla u^{(0)}(\boldsymbol{x})|^2 d\Omega \\ & \quad + 2\varepsilon[\int_{\Gamma_q} \left\{q^{(0)}(\boldsymbol{x}) - \bar{q}(\boldsymbol{x})\right\} q^{(1)}(\boldsymbol{x}) d\Gamma \\ & \quad + \eta \int_\Omega \nabla u^{(0)}(\boldsymbol{x}) \cdot \nabla u^{(1)}(\boldsymbol{x}) d\Omega] \\ & \quad + \varepsilon^2[\int_{\Gamma_q} |q^{(1)}(\boldsymbol{x})|^2 d\Gamma + \eta \int_\Omega |\nabla u^{(1)}(\boldsymbol{x})|^2 d\Omega] \\ = & J(\omega) + \ < J'(\omega), \delta\omega > \\ & \quad + \varepsilon^2[\int_{\Gamma_q} |q^{(1)}(\boldsymbol{x})|^2 d\Gamma + \eta \int_\Omega |\nabla u^{(1)}(\boldsymbol{x})|^2 d\Omega]\,. \end{aligned}$$

For $\eta > 0$, the term involving ε^2 is always positive. This guarantees that $J(\omega)$ has at least one local minimum.

With a suitable choice of positive real numbers α_k for $k = 0, 1, 2, \cdots$, we consider the minimizing process;

$$\omega_{k+1}(\boldsymbol{x}) = \omega_k(\boldsymbol{x}) - \alpha_k J'(\omega_k) \tag{5}$$

with some given initial ω_0, where the gradient $J'(\omega) \in H^{-1/2}(\Gamma_u^c)$ is defined from the first variation;

$$J(\omega + \delta\omega) - J(\omega) = \ < J'(\omega), \delta\omega > + \ o(\| \delta\omega \|) \tag{6}$$

with a real-valued function $o(\| \delta\omega \|)$ of higher order than $\| \delta\omega \|$, as it tends to zero in the $L^2(\Gamma_u^c)$–norm;

$$\| \varphi \| := \left\{ \int_{\Gamma_u^c} |\varphi|^2 d\Gamma \right\}^{1/2}$$

We insist $J'(\omega) \in H^{1/2}(\Gamma_u^c)$. To seek a concrete expression of $J'(\omega)$, we notice

$$\begin{aligned} & J(\omega + \delta\omega) - J(\omega) \\ = & \int_{\Gamma_q} \left\{ |q(\boldsymbol{x};\omega + \delta\omega) - \bar{q}(\boldsymbol{x})|^2 \right. \\ & \quad \left. - |q(\boldsymbol{x};\omega) - \bar{q}(\boldsymbol{x})|^2 \right\} d\Gamma \\ & \quad + \eta \int_\Omega \left\{ |\nabla u(\boldsymbol{x};\omega + \delta\omega)|^2 - |\nabla u(\boldsymbol{x};\omega)|^2 \right\} d\Omega \\ = & \int_{\Gamma_q} \{q(\boldsymbol{x};\omega + \delta\omega) + q(\boldsymbol{x};\omega) - 2\bar{q}(\boldsymbol{x})\} \\ & \quad \{q(\boldsymbol{x};\omega + \delta\omega) - q(\boldsymbol{x};\omega)\} \, d\Gamma \end{aligned}$$

$$+\eta\int_\Omega[\left\{\frac{\partial u}{\partial x_1}(\boldsymbol{x};\omega+\delta\omega)+\frac{\partial u}{\partial x_1}(\boldsymbol{x};\omega)\right\}$$
$$\left\{\frac{\partial u}{\partial x_1}(\boldsymbol{x};\omega+\delta\omega)-\frac{\partial u}{\partial x_1}(\boldsymbol{x};\omega)\right\}$$
$$+\left\{\frac{\partial u}{\partial x_2}(\boldsymbol{x};\omega+\delta\omega)+\frac{\partial u}{\partial x_2}(\boldsymbol{x};\omega)\right\}$$
$$\left\{\frac{\partial u}{\partial x_2}(\boldsymbol{x};\omega+\delta\omega)-\frac{\partial u}{\partial x_2}(\boldsymbol{x};\omega)\right\}]d\Omega$$
$$= \int_{\Gamma_q}\{q(\boldsymbol{x};\omega+\delta\omega)-q(\boldsymbol{x};\omega)$$
$$+2[q(\boldsymbol{x};\omega)-\bar{q}(\boldsymbol{x})]\}\,\delta q(\boldsymbol{x};\omega)d\Gamma$$
$$+\eta\int_\Omega\left[\left\{\frac{\partial\delta u}{\partial x_1}(\boldsymbol{x};\omega)+2\frac{\partial u}{\partial x_1}(\boldsymbol{x};\omega)\right\}\frac{\partial\delta u}{\partial x_1}(\boldsymbol{x};\omega)\right.$$
$$\left.+\left\{\frac{\partial\delta u}{\partial x_2}(\boldsymbol{x};\omega)+2\frac{\partial u}{\partial x_2}(\boldsymbol{x};\omega)\right\}\frac{\partial\delta u}{\partial x_2}(\boldsymbol{x};\omega)\right]d\Omega$$
$$= \int_{\Gamma_q}\{\delta q(\boldsymbol{x};\omega)+2[q(\boldsymbol{x};\omega)-\bar{q}(\boldsymbol{x})]\}\,\delta q(\boldsymbol{x};\omega)d\Gamma$$
$$+\eta\left[\int_\Gamma\left\{\frac{\partial\delta u}{\partial n}(\boldsymbol{x};\omega)+2\frac{\partial u}{\partial n}(\boldsymbol{x};\omega)\right\}\delta u(\boldsymbol{x};\omega)d\Gamma\right.$$
$$\left.-\int_\Omega\{\Delta\delta u(\boldsymbol{x};\omega)+2\Delta u(\boldsymbol{x};\omega)\}\,\delta u(\boldsymbol{x};\omega)d\Omega\right]$$
$$= \int_{\Gamma_q}2[q(\boldsymbol{x};\omega)-\bar{q}(\boldsymbol{x})]\delta q(\boldsymbol{x};\omega)d\Gamma$$
$$+\int_{\Gamma_q}|\delta q(\boldsymbol{x};\omega)|^2d\Gamma$$
$$+\eta\int_{\Gamma_u^c}2\frac{\partial u}{\partial n}(\boldsymbol{x};\omega)\delta\omega(\boldsymbol{x};\omega)d\Gamma$$
$$+\eta\int_{\Gamma_u^c}\delta q(\boldsymbol{x};\omega)\delta\omega(\boldsymbol{x};\omega)d\Gamma\,. \qquad (7)$$

Here we put $\delta u(\boldsymbol{x};\omega)=u(\boldsymbol{x};\omega+\delta\omega)-u(\boldsymbol{x};\omega)$, and accordingly $\delta q(\boldsymbol{x};\omega)=q(\boldsymbol{x};\omega+\delta\omega)-q(\boldsymbol{x};\omega)$. We notice that $\Delta(\delta u)=0$ in Ω, $\delta u=0$ on Γ_u, and $\delta u=\delta\omega$ on Γ_u^c.

We now consider $v(\boldsymbol{x})\in H^2(\Omega)$ as a solution of the Laplace equation

$$\Delta v(\boldsymbol{x})=0 \quad \text{in} \quad \Omega\,, \qquad (8)$$

subject to the boundary conditions

$$v|_{\Gamma_q} = 2\{q(\boldsymbol{x};\omega)-\bar{q}(\boldsymbol{x})\}\,, \qquad (9)$$
$$v|_{\Gamma_q^c} = 0\,, \qquad (10)$$

where $\Gamma_q^c=\Gamma\backslash\Gamma_q$. From Green's integral formula;

$$\int_\Omega(\Delta v)\delta u d\Omega = \int_\Gamma\frac{\partial v}{\partial n}\delta u d\Gamma - \int_\Gamma v\frac{\partial\delta u}{\partial n}d\Gamma+\int_\Omega v\Delta(\delta u)d\Omega\,,$$

we have

$$0=\int_{\Gamma_u^c}\frac{\partial v}{\partial n}\delta\omega d\Gamma-\int_{\Gamma_q}2(q-\bar{q})\delta q d\Gamma\,. \qquad (11)$$

Consequently we know that

$$J(\omega+\delta\omega)-J(\omega)$$
$$=\int_{\Gamma_u^c}\frac{\partial v}{\partial n}\delta\omega d\Gamma+\eta\int_{\Gamma_u^c}2\frac{\partial u}{\partial n}\delta\omega d\Gamma+o(\|\delta\omega\|)$$
$$=\left(\frac{\partial v}{\partial n}+2\eta\frac{\partial u}{\partial n},\ \delta\omega\right)_{L^2(\Gamma_u^c)}+o(\|\delta\omega\|)\,. \qquad (12)$$

Therefore we obtain the explicit form

$$J'(\omega)=\frac{\partial v}{\partial n}(\boldsymbol{x};\omega)+2\eta\frac{\partial u}{\partial n}(\boldsymbol{x};\omega) \quad \text{on} \quad \Gamma_u^c\,. \qquad (13)$$

Our algorithm can be summarized as follows:

<u>*Algorithm*</u>

Given $\omega_0|_{\Gamma_u^c}$.

For $k=0,1,2,\cdots$, *until satisfied, do:*

Solve $\Delta u_k(\boldsymbol{x})=0$ *in* Ω *with* $u_k|_{\Gamma_u}=\bar{u}$, $u_k|_{\Gamma_u^c}=\omega_k$ *to find* $q_k(\boldsymbol{x};\omega_k)$ *on* Γ_q.

Solve $\Delta v_k(\boldsymbol{x})=0$ *in* Ω *with* $v_k|_{\Gamma_q}=2\{q_k(\boldsymbol{x};\omega_k)-\bar{q}(\boldsymbol{x})\}$, $v_k|_{\Gamma_q^c}=0$ *to find* $J'(\omega_k)=\frac{\partial v_k}{\partial n}+2\eta\frac{\partial u_k}{\partial n}$ *on* Γ_u^c.

Update $\omega_{k+1}=\omega_k-\alpha_kJ'(\omega_k)$ *on* Γ_u^c.

We shall discuss a suitable choice of the sequence of positive real numbers $\{\alpha_k\}$. To this end, we employ the Armijo criterion in mathematical programming, that guarantees for the sequence $\{\omega_k\}$ to satisfy

$$J(\omega_k-\alpha_kJ'(\omega_k)) \ \leq\ J(\omega_k)-\xi\alpha_k\,\|\,J'(\omega_k)\,\|^2$$

with a constant ξ $(0<\xi<1/2)$.

<u>*Controlling the step size* α_k</u>

Given parameters $0<\xi<\frac{1}{2}$, $0<\tau<1$ (*say,* $\xi=0.1$, $\tau=0.5$), *and* $\varepsilon=10^{-4}$.

If $\|\,J'(\omega_k)\,\|<\varepsilon$, *then stop.*

else $\beta_0:=1$.

For $m=0,1,2,\cdots$, *do:*

If $J(\omega_k-\beta_mJ'(\omega_k))\leq J(\omega_k)-\xi\beta_m\,\|\,J'(\omega_k)\,\|^2$, *then* $\alpha_k:=\beta_m$.

else $\beta_{m+1}:=\tau\beta_m$

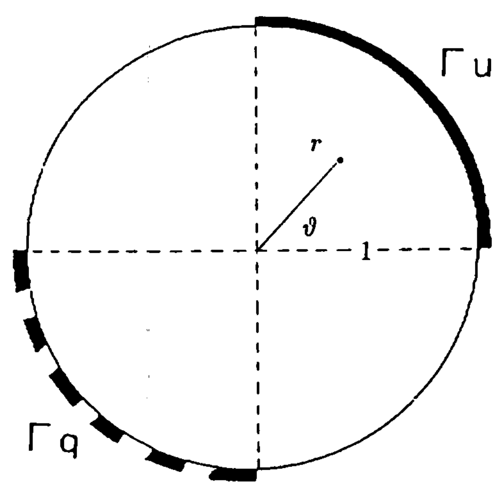
3 A numerical example

The collocation boundary element method with C^0 linear finite element is used for the numerical approximation of solutions of the first-kind boundary value problems of the Laplace equation involved in the algorithm.

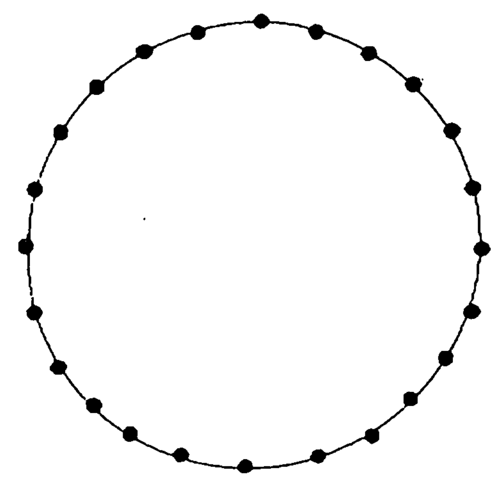
We notice that the function $u^*(x_1, x_2) = x_1^2 - x_2^2 = r^2\cos(2\vartheta)$ in terms of the polar coordinates (r, ϑ) is harmonic. Suppose that this function u^* is unknown, and consider the Laplace equation (1) in the unit circle $\Omega = \{ (r,\vartheta) \mid 0 \le r = \sqrt{x_1^2 + x_2^2} < 1,\ 0 \le \vartheta < 2\pi \}$ with the exact Dirichlet data; $\bar{u} = \cos(2\vartheta)$ on $\Gamma_u = \{ (1,\vartheta) \mid 0 \le \vartheta \le \frac{\pi}{2} \}$ and the exact Neumann data; $\bar{q} = 2\cos(2\vartheta)$ on $\Gamma_q = \{ (1,\vartheta) \mid \pi \le \vartheta \le \frac{3}{2}\pi \}$ as shown in Fig.1(a). The boundary Γ is divided uniformly into 24 boundary elements as shown in Fig.1(b). As an initial guess, we take

$$\omega_0 = -\cos\left(\frac{2}{3}(\vartheta - \frac{\pi}{2}\right)$$

on $\Gamma_u^c = \{ (1,\vartheta) \mid \frac{\pi}{2} \le \vartheta \le 2\pi \}$. We examine the results for each $\eta = 0$, 0.05, 0.1, 0.15, and 0.2. Among them we find the best fitted ω_k at $\eta = 0.15$ by inspection. Calculated ω_k together with the exact boundary values of u^* on Γ_u^c is presented in Fig.2. The numerical process was stable, and the numerical solutions rapidly converge to a local minimum solution other than u^*.



(a)



(b)

Figure 1. Boundary Γ_u and Γ_q (a), boundary elements (b).

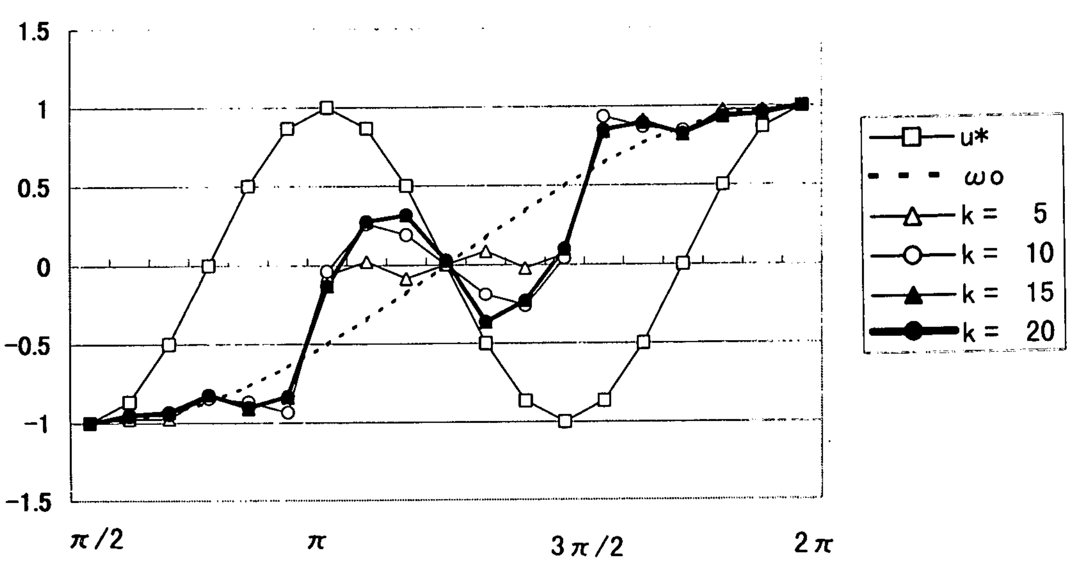


Figure 2. Calculated ω_k v.s. u^* on Γ_u^c.

We take another initial guess

$$\omega_0 = \cos 2\vartheta \left[\frac{2}{3\pi}(\vartheta - \frac{\pi}{2})\left\{\frac{2}{3\pi}(\vartheta - \frac{\pi}{2}) - 1\right\} + 1\right]$$

in order to confirm that our numerical solutions are apt to converge to the nearest local minimum. As Fig.3 shows, this initial guess is quite in the vicinity of u^* on Γ_u^c. However, we can observe that the calculated ω_k at $\eta = 0.2$ converges to a curve other than u^*.

4 Concluding remarks

We considered an under-determined problem of the Laplace equation by the boundary data, regarded as a boundary inverse problem. The problem is solvable , but the solution is not unique. Using the method of the steepest descent for a functional to be minimized, our problem is led to an iterative process consisting of the solution of primary and dual boundary value problems of the Laplace equation. The boundary element method is applied for numerical solution of the primary and dual problems. Simple numerical examples suggest that our numerical solutions rapidly converge to the nearest local minimum of the functional, but out result indicates that scrutiny of the treatment is further required in order to find the global minimum.

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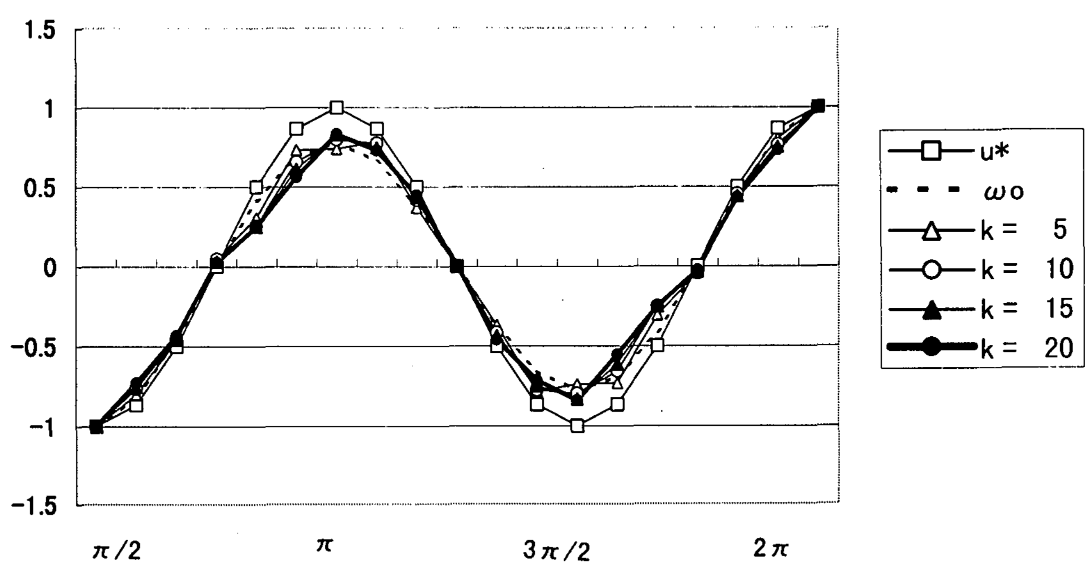

Figure 3. Calculated ω_k , starting with ω_0 in the vicinity of u^*.