# Application of Adaptive Time Step Integration Strategy in Nonlinear Structural Dynamic Analysis

## 非線形動的構造解析における適応型時間積分法の応用

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The choice of proper time step is very important in solving nonlinear structural dynamic problems with proper integration precision and reasonable computational efficiency. In this paper, a program is developed, with MATLAB, adopting an adaptive time integration strategy to analyze the dynamic responses of bridge piers. The time step in each time instant can be adjusted automatically according to the modal components of dynamic behavior of the system. The influences of some important controlling parameters on the computation accuracy are studied. It can not only assure the accuracy and keep valuable information but also improve the computational efficiency considerably.

***Key Words:** adaptive time step integration, dynamic analysis, bridge pier, MATLAB*

## 1. Introduction

Nonlinear dynamic analysis is inherently time-consuming in ordinary finite element computation system. When structure is divided into many elements, the computational efficiency becomes a major issue. Direct time step integration is one of the most commonly used methods in dynamic analysis. Usually the time step in direct integration should be small enough to obtain good accuracy in the solution, however, the time step should not be too small since this would mean that the solution is more costly than actually required. Therefore selection of an appropriate time step in direct time integration is of first importance in dynamic analysis.

Practically, in addition to considering computing time, the user may take much time to do some preliminary analysis in order to determine the range of frequency response excited by the loading. This estimation is related to the natural periods and the variation of the loading on the structure, and perhaps to a Fourier analysis of the loading to identify its significant harmonic components. Especially in the case of nonlinear dynamic problems, the choice of time step will be more complicated because the natural frequency of the structure may vary in different time transient. And what is more, conventional nonlinear dynamic analysis programs usually require the user to re-evaluate the structural stiffness[1]. These issues mentioned above not only take much time of the user, but also demand the user to have enough experience and relative knowledge to input data such as the time step in integration algorithm for good results. The adaptive time step strategy[2),3)] is very helpful for relieving the user from these tasks, and ensuring the computational accuracy and efficiency at the same time. Using this strategy, the time step can vary automatically according to the modal components of dynamic response and the nonlinear behavior of the structural system. A good accuracy of results can be obtained even a rough initial time step is chosen at first by the user.

Coded in MATLAB environment, the adaptive strategy in this research includes an automatic time step determination technique based on the "dominant frequency" concept, together with consideration of the real structure's nonlinear responses. The nonlinear behavior is also considered through the determination of time instants with stiffness re-evaluation resulting from the material deterioration.

The Newmark method and the Modified Newton-Raphson (MNR) procedures are used for the solution of nonlinear equation in order to obtain reliable dynamic responses. Special attention has been paid, in the present paper, to the convergence criteria which play an important role in determining both the validity and efficiency of a nonlinear dynamic analysis. If the convergence tolerances are ineffective, the true conditions might not be accurately represented. On the other hand, too strict convergence criteria may cause excessive computations to be performed.

Effectiveness of the adaptive method is first checked through a typical dynamic example compared with the conventional method. A bridge pier model is then analyzed for structural responses under the assumption of a bi-linear material approximation. The influences of some important controlling parameters on the computation efficiency of the adaptive method are studied based on the computation experience in this paper.

## 2. Description of the Adaptive Time Integration Strategy

The following special functions are to be included automatically in adaptive method:

a. controlling the numbers of iteration within a reasonable range in each time step;
b. re-evaluating the stiffness in each time step; and
c. calculating the adaptive time step in each time instant.

### 2.1 Estimation of time step

For nonlinear dynamic problems, the equilibrium equation of motion can be written as :

$$M\ddot{u} + C\dot{u} + S(u) = F(t) \qquad (1)$$

where: $M$ is the mass matrix, $C$ is the damping matrix, $S$ is a vector of structural reactions containing the nonlinear behavior, $\ddot{u}$, $\dot{u}$ and $u$ are acceleration, velocity and displacement vectors of the finite element assemblage, respectively, and $F(t)$ is a vector of external loading.

The increment−iteration forms of the Eq.(1) are simply expressed as:

$$M\ddot{u}^k_{t+\Delta t} + C\dot{u}^k_{t+\Delta t} + K_T\Delta u^k = F_{t+\Delta t} - S(u^{(k-1)}_{t+\Delta t})$$
$$u^k_{t+\Delta t} = u^{(k-1)}_{t+\Delta t} + \Delta u^k \qquad (2)$$

where $K_T$ is a tangent stiffness matrix, and $\Delta t$ is the time step.

In solving these equations, the Newmark integration method and MNR iterative method are employed in each incremental iteration. In this study, two types of convergence check are performed simultaneously. First, the Euclidean norm of the incremental displacements is required to be within a specified tolerance of the current displacement:

$$\frac{norm(\Delta u^k)}{norm(u^k_{t+\Delta t})} < \text{displacement tolerance value} \qquad (3a)$$

In the conventional method with a constant time step, convergence of the incremental displacements usually provide a reliable indication of whether the current displacements are sufficiently close to their equilibrium values. In our previous computations with the adaptive method, however, the check through only Eq.(3a) did not necessarily give converged solutions. Therefore the second convergence criterion is introduced in this study, requiring the Euclidean norm of the current residual force vector $\Delta r$ to be within a specified tolerance of the current external loading:

$$\frac{norm(\Delta r)}{norm(F_{t+\Delta t})} < \text{force tolerance value} \qquad (3b)$$

$$\text{then} \quad u^k_{t+\Delta t} = u^{(k-1)}_{t+\Delta t} + \Delta u^k,$$
$$u_{t+\Delta t} = u^k_{t+\Delta t}.$$

$$\ddot{u}_{t+\Delta t} = a_0(u_{t+\Delta t} - u_t) - a_2\dot{u}_t - a_3\ddot{u}_t \qquad (4)$$

$$\dot{u}_{t+\Delta t} = \dot{u}_t + a_6\ddot{u}_t + a_7\ddot{u}_{t+\Delta t}$$

where $a_0$, $a_2$, $a_3$, $a_6$ and $a_7$ are Newmark constants.

An estimation of the dominant frequency[4)] at $t+\Delta t$ is:

$$\omega^2_{t+\Delta t} = \frac{\Delta u^T K_T \Delta u}{\Delta u^T M \Delta u} \qquad (5)$$

This value reflects the modal components of the response at present time step. A corresponding characteristic period value can be calculated from:

$$T^* = \frac{2\pi}{\omega_{t+\Delta t}} \qquad (6)$$

In order to adequately integrate all available components of the response in this time instant, an estimation of time step can be obtained as a fraction $\lambda$ of the characteristic period T*:

$$\Delta t^* = \frac{T^*}{\lambda} \quad (7)$$

the lower limit value of $\lambda$ is decided by the correlation between time step estimation and error in the solution obtained by the direct time integration algorithm. The finite element idealization has to be chosen in such a way that the lowest p frequencies and mode shapes of the structure are predicted accurately, where p is determined by the distribution and frequency content of the loading. So in direct integration method, we are only interested in the first p order mode responses, and it is known that a time step $\Delta t = T_p/10$, where $T_p$ is the natural period of p-th mode response, will generally give reliable result. $T^*$ is the characteristic period which is normally greater than $T_p$, therefore if the lower limit of $\lambda$ values is equal to or greater than 20 in Eq.(7), a good accuracy could be expected for Newmark method.

The time step for each discrete time instant of the integration process has to be estimated because the modal components of dynamic response of the structure are variable in different time transient according to Eq.(7). But the equation may lead to the time steps in an unreasonable range, either to sharp alterations or to very small and unnecessary alterations. Furthermore, the choice of the time step will affect the re-evaluation and decomposition of the effective matrix in each time instant. In order to avoid such difficulties for keeping new time step estimated in an effective range, the following rules are employed[3]:

$$\xi = \frac{\Delta t^*}{\Delta t_t} \quad (8)$$

If $\xi_1 < \xi < \xi_2$, then maintain $\Delta t_{t+\Delta t} = \Delta t_t$;

else $\Delta t_{t+\Delta t} = \Delta t^*$

If $\xi < \xi_{min}$ then $\Delta t_{t+\Delta t} = \xi_{\min} \Delta t_t$

If $\xi > \xi_{max}$ then $\Delta t_{t+\Delta t} = \xi_{max} \Delta t_t$

The above rules are merely heuristic, and therefore the parameters $\xi_{min}$, $\xi_1$, $\xi_2$ and $\xi_{max}$ may be adjusted for different nonlinear problems. Typical values of these parameters have been given[3], for example, as $\xi_{min}$ =0.5, $\xi_1$=0.625, $\xi_2$=1.6 and $\xi_{max}$=1.8, respectively. Through the limited number of applications studied by the authors, those parameters were found to be rather insensitive to the analysis results. Variation of the time steps were constrained in a reasonable range although the initial value of the time step was a very rough estimation in the subsequent examples.

## 2.2 Influence of the nonlinear behavior

Consideration of the nonlinear behavior in adaptive time strategy focuses on two aspects: the first is calculation of the time step, and the second is determination of time instant with stiffness re-evaluation.

The effect of the nonlinear behavior on the calculation of time steps is reflected by varying the $\lambda$ parameter. The variation of $\lambda$ is automatically performed by monitoring the number of iterations $N_{it}$ required for iterative convergence of the MNR process at each time step. The general rules for defining the variation of $\lambda$ are as follows:
(a) At time instants when the iterative MNR process requires more iterations for convergence, the value of $\lambda$ is increased; (b) Conversely, when nonlinear effects are less severe, the value of $\lambda$ is reduced; (c) The value of $\lambda$ can not be either reduced beyond $\lambda_{\min}$ or increased beyond $\lambda_{\max}$. The effect of the $\lambda$ parameter in the adaptive time step strategy, including the choice of $\lambda_{\min}$ and $\lambda_{\max}$, are discussed in detail in the following examples.

Since the change of the dominant frequency reflects not only the variation of the modal composition of the response, but also the variation of the stiffness of the structural system, the re-evaluation of the effective stiffness will affect the estimation value of the new time step remarkably.

## 3. Examples and Discussions

## 3.1 A simple example

A simple nonlinear structural dynamic problem is chosen to check the performance of the strategy. The physical model is shown in Fig.1, which is a circular elastic bar with lumped masses. The length of the beam, *L*, is 4(m), the radius is 0.05(m) and the Young's modulus is $2\times 10^{10}$(kgf/m$^2$); the density is $7.85\times 10^3$(kg/m$^3$). F(t)=Fsin(10$\pi$t) is an external force acting on the tip of the cantilever beam, K is stiffness of the springs which are not initially connected to the beam, C is the damping, and $m_i$ ( i = 1, 2, 3, 4 ) are concentrated masses.

Assumed nonlinear relationship between force and displacement at the beam tip is shown in Fig.2, in which $k_1$ is the stiffness of the beam itself and $k_2$ indicates the combined stiffness of the beam and spring.

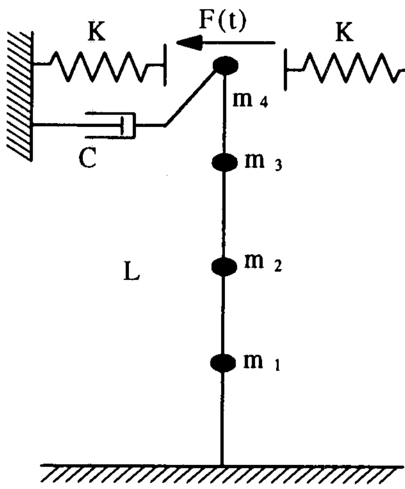



Fig.1 A physical model

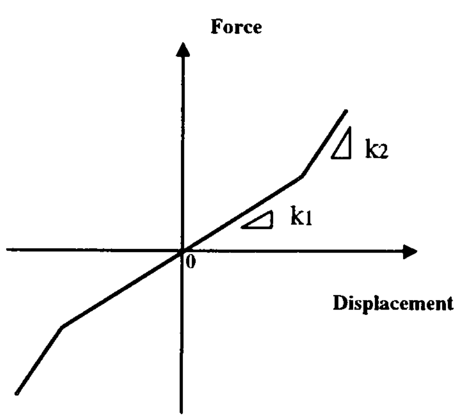


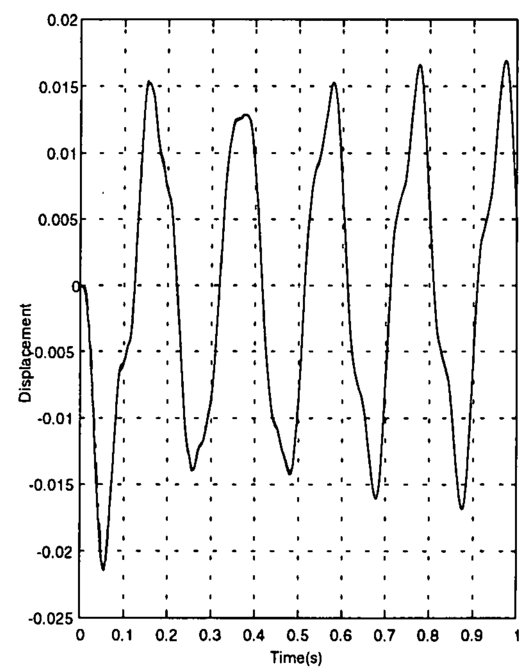

Fig.2 Conceptual relationship between force and displacement

To study the difference between the conventional method (keeping constant time step) and adaptive strategy, a reference convergent solution is taken as the comparison criterion. The reference convergent solution is obtained when the problem is solved by smaller and smaller time steps until the response converges to a fixed-time curve in conventional Newmark method.

In this work, the cantilever beam is divided into four beam elements. The mass matrix is deduced by adding concentrated masses on consistent mass matrix. Damping is added on the beam tip. For this system $h$ = 0.03 is chosen as the damping ratio.

## (1) Results comparison between conventional method and adaptive strategy

The results of the adaptive strategy and the conventional method are compared in Fig.3, where the initial value of adaptive time step is 0.01(s), and the constant time step in conventional method is 0.0001(s). Although the initial adaptive time step is rather rough, it can be seen from Fig.3 that the accuracy of results is very similar, but the adaptive strategy is more effective than conventional method for the time steps can be adjusted to the proper values automatically.




— conventional method as time step is 0.0001s
- - adaptive method as initial time step is 0.01s

Fig.3 Results comparison between adaptive method and conventional method

Fig.4 gives out the variation of adaptive time steps. It can be found that in adaptive strategy, the time steps are controlled according to the dominant frequency of the nonlinear structural dynamic response in every time instant, in such a way the variations of the time step are limited in a reasonable range. The losses of the valuable components or waste of computational time can be avoided.

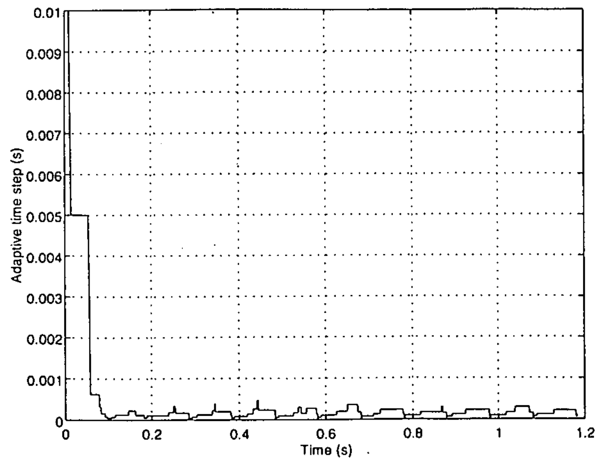



Fig.4 Variation of the adaptive time steps

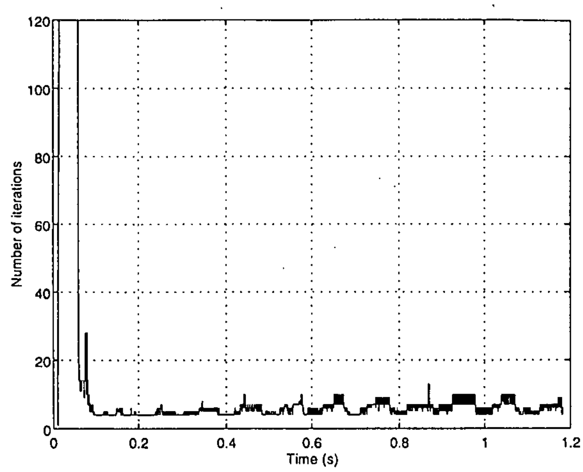



Fig.5 Variation of numbers of iterations in adaptive time step strategy

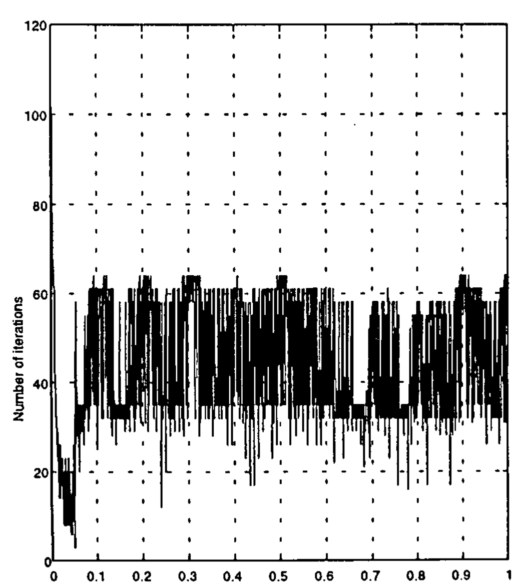
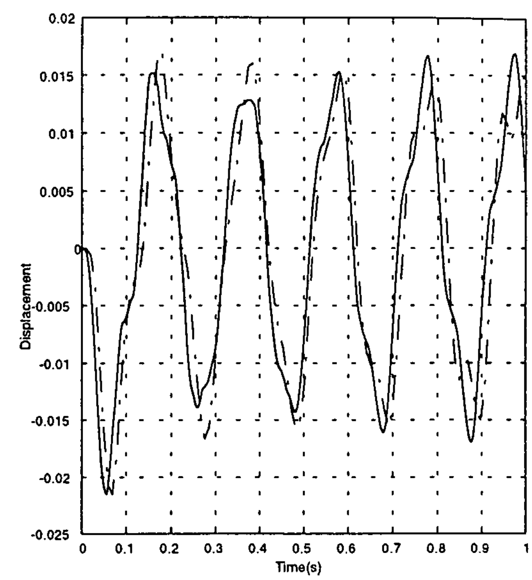



Fig.6 Variation of numbers of iterations in conventional method

Since the time steps are adjusted to the proper values, the numbers of iterations decrease sharply, therefore the computational efficiency is considerably improved as it can be seen from the comparison between the following Fig.5 and Fig.6.




— convergent solution
--- conventional method with rough time step

Fig.7 Comparison between convergent solution and that with rough time step in conventional method

Fig.5 and Fig.6 give out the variations of numbers of iterations for the convergence at each time step in adaptive strategy and conventional method during the time history, respectively. The numbers of the iterations range from 30 to 60 times in conventional method, much greater than those of less than 10 times in most cases in adaptive step strategy.

Under the same condition, the results indicate that different initial values of time step give out the very similar results in the adaptive time step strategy, and the good accuracy can be obtained although the initial value of time step may be rather rough. Namely, the user doesn't need do much work for choosing a proper time step in advance for adaptive method. However, if the time step is a roughly estimated value in conventional method, good results can not be obtained, as shown in Fig.7, where the constant time step is 0.01(s).

From this example, it is also found that a good accuracy can be obtained by adopting two convergent criteria expressed in Eqs.(3a) and (3b) simultaneously.

**(2) Effect of the $\lambda$ parameter in adaptive time step strategy**

The choice of the $\lambda_{min}$ parameter is also very important for a good accuracy in Newmark adaptive time step strategy. In this paper, the results with the different $\lambda_{min}$ values are compared, in which the $\lambda_{min}$ are 20, 30 and 50, respectively. The results with different $\lambda_{min}$ values are very similar if the value of $\lambda_{min}$ is greater than 20. A good accuracy can be expected in Newmark adaptive step strategy if the value of the $\lambda_{max}$ is limited less than 500. The choice of the $\lambda_{max}$ is dependent on the estimated response extent of the structure.

## 3.2 Application of adaptive method to a bridge pier model under earthquake load

The bridge pier model was divided into four beam elements as shown in Fig.8. The Hermit beam element is adopted to represent the beam behavior.

The length of the pier is 4(m) with element beam length 1(m). Radius of the pier is 0.4(m). A consistent matrix is used for mass matrix with an extra lumped mass (20000kg) on the beam tip to represent superstructure weight. The damping matrix is assumed proportional to mass matrix, where damping ratio $h$ is also taken as 0.03. The initial Young's modulus is $2.6\times10^9$ (kgf/m$^2$), and the yielding stress is $2.6\times10^6$ (kgf/m$^2$).

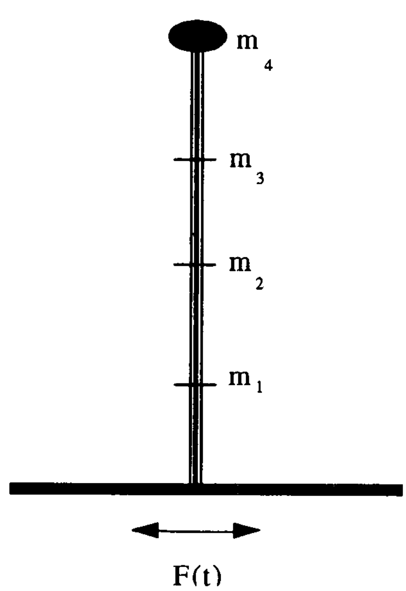



Fig.8 A concrete bridge pier model

Element stiffness matrix is[6)]

$$[k] = \frac{E(t,\varepsilon)\ I}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \qquad (10)$$

where $E(t,\varepsilon)$ is the Young's modulus, $I$ is the moment inertia of the pier and $L$ is the length of beam element.

Element's consistent mass matrix is

$$[m]=\frac{\gamma_o A\ L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \qquad (11)$$

in which $A$ is the area of beam element, and $\gamma_o$ is the density of the material ($2.4\times10^3$ kg/m$^3$).

The damping matrix is:

$$[c]=2\,h\,[\omega]\,[m_{ii}] \qquad (12)$$

with $[\omega]$ as the natural period matrix of structure.

The material property is shown in Fig.9, which is a simple bilinear approximation. The ratio of the stiffness $k_2/k_1$ is taken as 0.08. When a section of the beam is in plastic state, the whole beam element is simply assumed in plastic range, so system stiffness coefficient becomes a plastic one.

The time history of the horizontal ground motion inputted is shown in Fig.10, where the acceleration

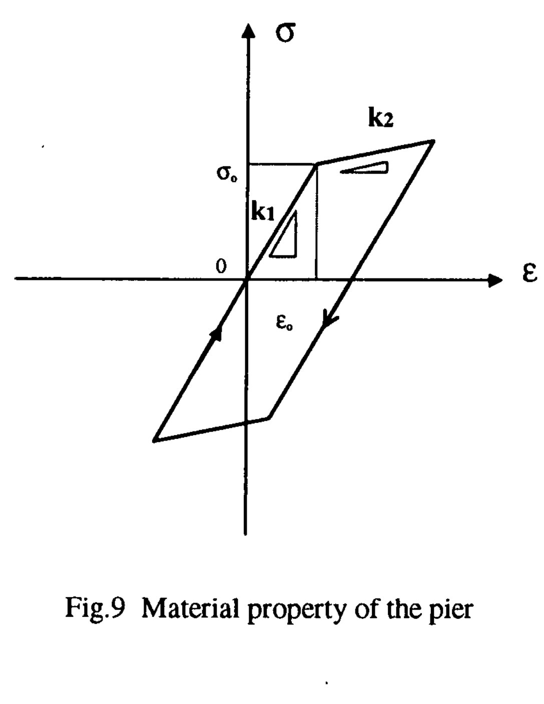



Fig.9 Material property of the pier

readings are recorded in a constant interval of 0.01s. The initial time step used is 0.03(s) in the adaptive method.

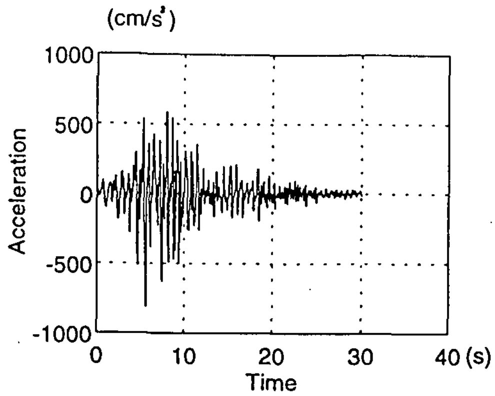



Fig.10 Acceleration history of the horizontal ground motion

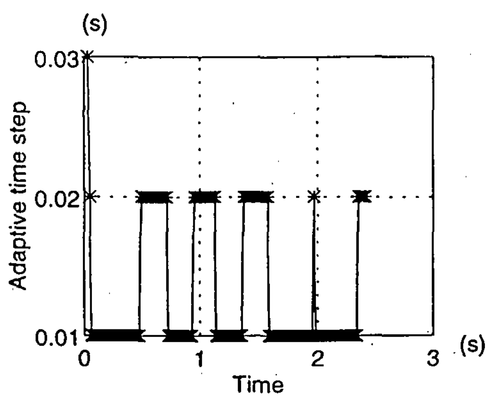



Fig. 11 Time step variation in adaptive method

The step variation is shown in Fig.11, from which it can be found that the adaptive time step can be adjusted to the proper value automatically in every time transient.

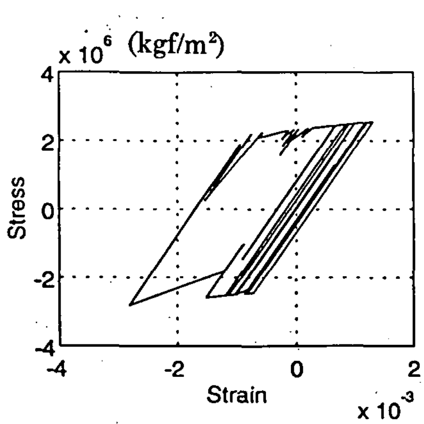



Fig. 12 Strain-stress responses of the pier

Fig.12 shows the stress-strain time history of the pier under the earthquake force. The nonlinear structural responses show clearly the variation of the elasto-plastic behavior history of the pier.

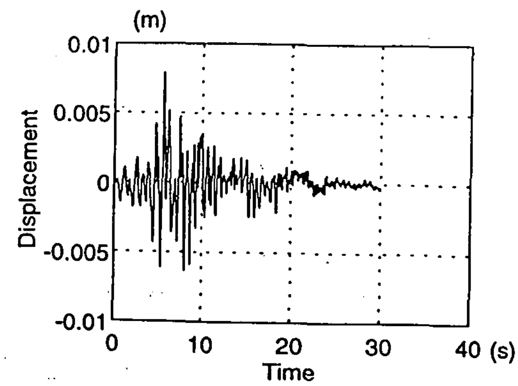



Fig.13 Response of displacements at the pier tip

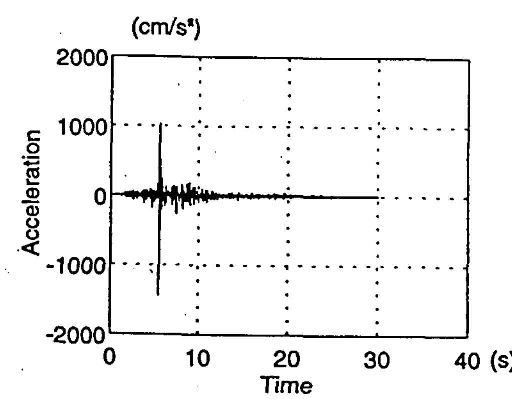



Fig.14 Response of accelerations at the pier tip

Fig.13 and Fig.14 give the responses of displacement and acceleration to time at the pier tip in adaptive method, respectively. Our research indicates that these results have the same accuracy as those obtained by conventional method with very small time step even though a rough adaptive step value is inputted at beginning.

### 3.3 Discussion on the adaptive method

A rough time step may be inputted at first in adaptive method, then it can be adjusted to a proper value automatically approaching to a satisfied result. This is the most remarkable advantage of the method.

The values of the controlling parameters adopted in this paper have been proven to be effective for getting a satisfied result.

From two examples given out above, it can be summarized that:

(1) The adaptive method is very suitable for nonlinear dynamic analysis. The natural frequency of the structure will vary in every transient when the material characteristic of the structure is in plastic state. The time step in every transient can be controlled accordingly.
(2) The method is helpful in practical engineering for sometimes engineers may not know clearly the characteristic of the structure and excitation force to choose a suitable time step, alternatively a rather rough time step is permitted.
(3) Adaptive method is available for large scale structure with many elements. The time step varies automatically to minimize the iteration numbers and the convergence time.

## 4. Concluding Remarks

Compared with the conventional method which keeps constant time step in direct time integration procedure, the adaptive strategy can give out satisfied results even though the initial value of time step inputted is a very rough estimation. This strategy can not only ensure the integration accuracy and computational efficiency, but also save the engineer's time for analyzing the data and choosing the proper time step. It can be concluded that the adaptive time step integration strategy is very effective for the larger complicated nonlinear structures.

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