Journal of Applied Mechanics Vol. 1 (August 1998) JSCE

# Boundary-Type Integral Formulation of Domain Variables for Three-Dimensional Initial Strain Problems

## 3次元初期ひずみ問題のための領域変数の境界積分定式

Hang MA* and Norio KAMIYA**
馬 杭 神谷紀生

*Nonmember, Prof. Weld. Res. Instit., Gansu Univ. Tech.(Lanzhou 730050, P.R.China)
**Nonmember, Dr.Eng. Prof. Sch. Inform. Sci., Nagoya Univ. (Nagoya 464-8601, Japan)

The boundary-type integral formulations of domain variables were presented explicitly for three-dimensional initial strain problems using boundary element method. The domain variables were first represented by complete series of polynomial expansions, then the related domain integrals were transformed into the boundary ones with the aid of the intrinsic correlation among the integral kernels as well as the high ordered fundamental solutions. The applicability of the formulation was discussed briefly.

***Key words :*** boundary element method, domain variable, fundamental solution, integral kernel, polynomial expansion, boundary-type integral formulation.

## 1. Introduction

The boundary element method has pronounced merits characterized by dividing elements only at the boundaries and with high precision of solutions, which reduces data preparation and saves computing time[1]. The internal cells in the domain, however, are usually indispensable if the problem under consideration is inhomogeneous or nonlinear. There have been many attempts and developments in order to overcome this limitations in recent years, such as the dual reciprocity method[2], the multiple reciprocity method[3] and the computing point method[4)-5)]. One of the key point in the multiple reciprocity method was the transformation of domain integrals into boundary-type via high ordered fundamental solutions[2]. In the computing point method, the inhomogeneous term in the nonlinear Poisson equation was represented by polynomial expansions and then the domain integrals were transformed into the boundary-type therefore the problem can be solved without domain discretizations[4)-5)]. All of these work show the effectiveness of the transformation in getting rid of the domain discretization.

In a general sense, the initial strain problems include the problems such as the thermo-elasticity, the residual stress problem[6], the creep problem and the elasto-plasticity in initial strain algorithm. Some of them belong to the material nonlinearity. The applicability of the procedure lies in how well the domain variables, the initial strains, can be described by polynomial approximation in terms of the space coordinates and how the transformation can be realized from domain type into boundary one.

In the previous work, the domain variables were assumed to be representative by polynomials and the transformation from domain into boundary-type integrals was realized via the intrinsic correlation among the integral kernels for two-dimensional problem[7]. The present work extended the transformation to three-dimensional case, which were realized, similarly, with the aid of both the intrinsic correlation among the integral kernels and the high ordered fundamental solutions. The formulations was given explicitly and discussed briefly.

## 2. Basic Equations

The initial strain problems can be described by two boundary integral equations, the displacement equation and the stress equation for a domain $\Omega$ with boundary $\Gamma$ as follows[1]:

$$\frac{1}{2}u_i(p) = \int_\Gamma \tau_j(q)u^*_{ij}(p,q)d\Gamma(q) - \int_\Gamma u_j(q)\tau^*_{ij}(p,q)d\Gamma(q) + \int_\Omega \varepsilon_{jk}(q)\sigma^*_{ijk}(p,q)d\Omega(q) \quad (1)$$

$$\sigma_{ij}(p) = \int_\Gamma \tau_k(q)u^*_{ijk}(p,q)d\Gamma(q) - \int_\Gamma u_k(q)\tau^*_{ijk}(p,q)d\Gamma(q) + \int_{\Omega-\Omega_\varepsilon} \varepsilon_{kl}(q)\sigma^*_{ijkl}(p,q)d\Omega(q) + \varepsilon_{kl}(p)O_{ijkl}(p) \quad (2)$$

where

$p$ — source point

$q$ — field point

$\Omega_\varepsilon$ — tiny zone centered at $p$

$u_j$ — displacement

$\tau_j$ — traction at the boundary

$\sigma_{ij}$ — stress

$\varepsilon_{ij}$ — initial strain or initial strain increment

$u^*_{ij}$ — fundamental solution of elasticity

$\tau^*_{ij}, \sigma^*_{ijk}, u^*_{ijk}, \tau^*_{ijk}, \sigma^*_{ijkl}, O_{ijkl}$ — derived fundamental solutions

In the above two equations there are domain integrals of the initial strains generated owing to material nonlinearity. If the initial strain $\varepsilon_{ij}$ is replaced by the term $\alpha\theta\delta_{ij}$ the two equations can be used for thermo-elasticity, where $\alpha$ is the thermal expansion coefficient of the material and $\theta$ is the temperature. The effects of the domain variables on the displacement and the stress equations have conventionally to be evaluated by numerical calculations over divided internal cells in the domain. Suppose the initial strain can be described by polynomial approximation in terms of the spatial coordinates as follows:

$$\varepsilon_{ij}(q) \approx \sum_{m=0,\, n=0,\, l=0}^{m+n+l=N} C_{ij}^{mnl} x_1^m(q) x_2^n(q) x_3^l(q) \quad (3)$$

where $N$ is the highest order and $C_{ij}^{mnl}$ the coefficients to be determined of the polynomials. Insert the above expression into the displacement and the stress equations, respectively, and replace the spatial coordinates by the two point variables between the source and the field points:

$$x_k = x_k(q) - x_k(p) \quad (4)$$

then any term of the polynomial expansions for the domain integrals in the displacement and the stress equations can be expressed as

$$C_{jk}^{mnl}\int_\Omega x_1^m(q)x_2^n(q)x_3^l(q)\sigma^*_{ijk}(p,q)d\Omega(q) = C_{jk}^{mnl}\sum_{s=0}^{m}\sum_{t=0}^{n}\sum_{u=0}^{l}\frac{m!n!l!}{(m-s)!s!(n-t)!t!(l-u)!u!} \times [x_1(p)]^{m-s}[x_2(p)]^{n-t}[x_3(p)]^{l-u} \times \int_\Omega x_1^s x_2^t x_3^u \sigma^*_{ijk}(p,q)d\Omega(q) \quad (5)$$

$$C_{jk}^{mnl}\left\{\int_\Omega x_1^m(q)x_2^n(q)x_3^l(q)\sigma^*_{ijkl}(p,q)d\Omega(q) + x_1^m(p)x_2^n(p)x_3^l(p)O_{ijkl}(p)\right\} = C_{jk}^{mnl}\left\{\sum_{s=0}^{m}\sum_{t=0}^{n}\sum_{u=0}^{l}\frac{m!n!l!}{(m-s)!s!(n-t)!t!(l-u)!u!}\right. \times [x_1(p)]^{m-s}[x_2(p)]^{n-t}[x_3(p)]^{l-u} \times \int_\Omega x_1^s x_2^t x_3^u \sigma^*_{ijkl}(p,q)d\Omega(q)$$

$$+ x_1^m(p)x_2^n(p)x_3^l(p)O_{ijkl}(p)\Big\} \quad (6)$$

respectively, where $m, n, l, s, t$ and $u$ are all integers. The transformation in the above two equations is linear and performed with a short subroutine. The domain integrals in the right hand side of equations (5) and (6) can then be transformed into the boundary integrals via the correlation between the kernels[7], the Cauchy's relation and the equilibrium equation shown as follows:

$$\sigma^*_{ijk}(p, q)n_k(q) = \tau^*_{ij}(p, q) \quad (7)$$

$$\sigma^*_{ijk,k}(p, q) + \delta_{ij}\delta(p, q) = 0 \quad (8)$$

where $n_k(q)$ is the outward normal. The derived fundamental solution $\sigma^*_{ijk}(p, q)$ is the stress at $q$ when there is a unit force acting at point $p$ in $i$ direction. And $\tau^*_{ij}(p, q)$ represents the corresponding traction at the same point.

### 3. Formulations for Displacement Equation

The derived fundamental solutions $\sigma^*_{ijk}(p, q)$ in the domain integral of the displacement equation is presented here explicitly in the three-dimensional form for clarity:

$$\sigma^*_{ijk}(p, q) = \frac{1}{8\pi(1-\nu)r} \times\left\{(1-2\nu)\left(\delta_{jk}r_{,i} - \delta_{ki}r_{,j} - \delta_{ij}r_{,k}\right) - 3r_{,i}r_{,j}r_{,k}\right\} \quad (9)$$

where $\nu$ is Poisson's ratio and $r$ the distance between the source and the field point

$$r = \sqrt{x_k x_k} \quad (10)$$

The boundary-type formulation of the zeroth order term of the polynomials is as follows:

$$\int_\Omega \sigma^*_{ijk}(p, q)d\Omega(q) = \int_\Gamma x_k \tau^*_{ij}(p, q)d\Gamma(q) \quad (11)$$

which has been derived under the state with a definite physical meaning, that is, the state of a traction free domain with a uniformly distributed initial strain of unity over the domain[7]. This equation can be extended to a general form with the Cauchy's relation (7) and the equilibrium equation (8) as follows:

$$\int_\Omega\left[mx_1^{m-1}x_2^n x_3^l\delta_{1k} + nx_1^m x_2^{n-1}x_3^l\delta_{2k} + lx_1^m x_2^n x_3^{l-1}\delta_{3k}\right] \times \sigma^*_{ijk}(p, q)d\Omega(q) = \int_\Gamma x_1^m x_2^n x_3^l \tau^*_{ij}(p, q)d\Gamma(q) \quad (12)$$

Then all the boundary-type formulations, except a special one, of integral kernels in boundary-type of the three-dimensional initial strain problem for the displacement equation can be obtained explicitly:

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{111}\, d\Omega(q) = \frac{1}{m+n+l+1}\int_\Gamma x_1^{m+1}x_2^n x_3^l \tau^*_{11}\, d\Gamma(q) \quad (13)$$

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{222}\, d\Omega(q) = \frac{1}{m+n+l+1}\int_\Gamma x_1^m x_2^{n+1} x_3^l \tau^*_{22}\, d\Gamma(q) \quad (14)$$

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{333}\, d\Omega(q) = \frac{1}{m+n+l+1}\int_\Gamma x_1^m x_2^n x_3^{l+1} \tau^*_{33}\, d\Gamma(q) \quad (15)$$

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{112}\, d\Omega(q) = \frac{1}{m+n+l+1}\int_\Gamma x_1^m x_2^{n+1} x_3^l \tau^*_{11}\, d\Gamma(q) \quad (16)$$

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{223}\, d\Omega(q)$$

$$= \frac{1}{m+n+l+1} \int_\Gamma x_1^m x_2^n x_3^{l+1} \tau^*_{22} \, d\Gamma(q) \tag{17}$$

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{331} \, d\Omega(q)$$

$$= \frac{1}{m+n+l+1} \int_\Gamma x_1^{m+1} x_2^n x_3^l \tau^*_{33} \, d\Gamma(q) \tag{18}$$

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{113} \, d\Omega(q)$$

$$= \frac{1}{m+n+l+1} \int_\Gamma x_1^m x_2^n x_3^{l+1} \tau^*_{11} \, d\Gamma(q) \tag{19}$$

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{221} \, d\Omega(q)$$

$$= \frac{1}{m+n+l+1} \int_\Gamma x_1^{m+1} x_2^n x_3^l \tau^*_{22} \, d\Gamma(q) \tag{20}$$

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{332} \, d\Omega(q)$$

$$= \frac{1}{m+n+l+1} \int_\Gamma x_1^m x_2^{n+1} x_3^l \tau^*_{33} \, d\Gamma(q) \tag{21}$$

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{122} \, d\Omega(q) = \frac{1}{n+1} \int_\Gamma x_1^m x_2^{n+1} x_3^l \tau^*_{12} \, d\Gamma(q)$$
$$- \frac{m}{(n+1)(m+n+l+1)} \int_\Gamma x_1^{m-1} x_2^{n+2} x_3^l \tau^*_{11} \, d\Gamma(q)$$
$$- \frac{l}{n+1} \int_\Omega x_1^m x_2^{n+1} x_3^{l-1} \sigma^*_{123} \, d\Omega(q) \tag{22}$$

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{233} \, d\Omega(q) = \frac{1}{l+1} \int_\Gamma x_1^m x_2^n x_3^{l+1} \tau^*_{23} \, d\Gamma(q)$$
$$- \frac{n}{(l+1)(m+n+l+1)} \int_\Gamma x_1^m x_2^{n-1} x_3^{l+2} \tau^*_{22} \, d\Gamma(q)$$
$$- \frac{m}{l+1} \int_\Omega x_1^{m-1} x_2^n x_3^{l+1} \sigma^*_{123} \, d\Omega(q) \tag{23}$$

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{311} \, d\Omega(q) = \frac{1}{m+1} \int_\Gamma x_1^{m+1} x_2^n x_3^l \tau^*_{31} \, d\Gamma(q)$$
$$- \frac{l}{(m+1)(m+n+l+1)} \int_\Gamma x_1^{m+2} x_2^n x_3^{l-1} \tau^*_{33} \, d\Gamma(q)$$
$$- \frac{n}{m+1} \int_\Omega x_1^{m+1} x_2^{n-1} x_3^l \sigma^*_{123} \, d\Omega(q) \tag{24}$$

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{133} \, d\Omega(q) = \frac{1}{l+1} \int_\Gamma x_1^m x_2^n x_3^{l+1} \tau^*_{13} \, d\Gamma(q)$$
$$- \frac{m}{(l+1)(m+n+l+1)} \int_\Gamma x_1^{m-1} x_2^n x_3^{l+2} \tau^*_{11} \, d\Gamma(q)$$
$$- \frac{n}{l+1} \int_\Omega x_1^m x_2^{n-1} x_3^{l+1} \sigma^*_{123} \, d\Omega(q) \tag{25}$$

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{211} \, d\Omega(q) = \frac{1}{m+1} \int_\Gamma x_1^{m+1} x_2^n x_3^l \tau^*_{21} \, d\Gamma(q)$$
$$- \frac{n}{(m+1)(m+n+l+1)} \int_\Gamma x_1^{m+2} x_2^{n-1} x_3^l \tau^*_{22} \, d\Gamma(q)$$
$$- \frac{l}{m+1} \int_\Omega x_1^{m+1} x_2^n x_3^{l-1} \sigma^*_{123} \, d\Omega(q) \tag{26}$$

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{322} \, d\Omega(q) = \frac{1}{n+1} \int_\Gamma x_1^m x_2^{n+1} x_3^l \tau^*_{32} \, d\Gamma(q)$$
$$- \frac{l}{(n+1)(m+n+l+1)} \int_\Gamma x_1^m x_2^{n+2} x_3^{l-1} \tau^*_{33} \, d\Gamma(q)$$
$$- \frac{m}{n+1} \int_\Omega x_1^{m-1} x_2^{n+1} x_3^l \sigma^*_{123} \, d\Omega(q) \tag{27}$$

All of the above formulations have the similar form with that in the two-dimensional case[7] so that they can be reduced to two-dimensional ones, but a special term $\sigma^*_{123}$ of the domain integral appears at the right hand side in some of these formulations (Equations (22)-(27)), which can be transformed via the high ordered fundamental solutions of elasticity as shown below.

## 4. Formulation of the Special Term

Let us start with the Galerkin tenser of the zeroth order for the three-dimensional elasticity[1]:

$$G^*_{ij} = G^{*(0)}_{ij} = \frac{r}{8\pi\mu} \delta_{ij} \tag{28}$$

where $\mu$ is the shear modulus. It is not difficult to write out the $N$-th ordered Galerkin tenser for elasticity:

$$G^{*(N)}_{ij} = \frac{r^{2N+1}}{4\pi\mu(2N+2)!}\delta_{ij}$$

$$(N = 0, 1, 2, \cdots) \qquad (29)$$

which has the recurrence relation of

$$G^{*(N)}_{ij,kk} = G^{*(N-1)}_{ij} \qquad (30)$$

Then the corresponding high ordered fundamental solution or Kelvin's solution can be deduced:

$$u^{*(N)}_{ij} = G^{*(N)}_{ij,kk} - \frac{1}{2(1-\nu)}G^{*(N)}_{ik,kj}$$

$$= \frac{r^{2N-1}}{16\pi\mu(1-\nu)(N+1)(2N)!}$$

$$\times \left\{\left[4(1-\nu)(N+1)-1\right]\delta_{ij} - (2N-1)r_{,i}r_{,j}\right\}$$

$$(N = 0, 1, 2, \cdots) \qquad (31)$$

With the constitutive relation of elasticity, the corresponding high ordered derived fundamental solution in the displacement equation can therefore be obtained:

$$\sigma^{*(N)}_{ijk} = \mu\left[\beta\delta_{jk}u^{*(N)}_{il,l} + (u^{*(N)}_{ij,k} + u^{*(N)}_{ik,j})\right]$$

$$= \frac{r^{2N-2}}{16\pi(1-\nu)N(N+1)(2N-2)!}$$

$$\times \left\{\left[2(1-\nu)(N+1)-1\right](\delta_{ij}r_{,k} + \delta_{ki}r_{,j})\right.$$

$$\left. + \left[2\nu(N+1)-1\right]\delta_{jk}r_{,i} - (2N-3)r_{,i}r_{,j}r_{,k}\right\}$$

$$(N = 1, 2, \cdots) \qquad (32)$$

where $\beta$ is a material constant defined by

$$\beta = \frac{2\nu}{1-2\nu} \qquad (33)$$

It is noticed that the equilibrium equation is no longer hold for the derived fundamental solution in their high ordered counterparts, but it has the following simple form:

$$\sigma^{*(N)}_{ijk,k} = \frac{r^{2N-3}}{4\pi(2N-2)!}\delta_{ij}$$

$$(N = 1, 2, \cdots) \qquad (34)$$

And the outward normal directional derivative of the derived fundamental solution is

$$\frac{\partial}{\partial n}\sigma^{*(N)}_{ijk} = \frac{r^{2N-3}}{16\pi(1-\nu)N(N+1)(2N-2)!}$$

$$\times \left\{(2N-3)\frac{\partial r}{\partial n}\left\langle\left[2(1-\nu)(N+1)-1\right](\delta_{ij}r_{,k} + \delta_{ki}r_{,j})\right.\right.$$

$$\left. + \left[2\nu(N+1)-1\right]\delta_{jk}r_{,i} - (2N-5)r_{,i}r_{,j}r_{,k}\right\rangle$$

$$- (2N-3)(r_{,j}r_{,k}n_i + r_{,k}r_{,i}n_j + r_{,i}r_{,j}n_k)$$

$$+ (\delta_{ij}n_k + \delta_{ki}n_j)\left[2(1-\nu)(N+1)-1\right]$$

$$\left. + \left[2\nu(N+1)\delta_{jk}n_i\right]\right\}$$

$$(N = 1, 2, \cdots) \qquad (35)$$

The transformation of the special term from the domain type into the boundary one can then be realized by the recurrence formula using Gaussian divergence theorem:

$$\int_\Omega x_1^m x_2^n x_3^l \sigma^{*(N)}_{ijk}\, d\Omega(q) = \int_\Omega x_1^m x_2^n x_3^l \sigma^{*(N+1)}_{ijk,ss}\, d\Omega(q)$$

$$= \int_\Gamma x_1^m x_2^n x_3^l \frac{\partial r}{\partial n}\sigma^{*(N+1)}_{ijk}\, d\Gamma(q)$$

$$- \int_\Gamma \frac{\partial r}{\partial n}(x_1^m x_2^n x_3^l)\sigma^{*(N+1)}_{ijk}\, d\Gamma(q)$$

$$+ \int_\Omega (x_1^m x_2^n x_3^l)_{,ss}\sigma^{*(N+1)}_{ijk}\, d\Omega(q)$$

$$(N = 1, 2, \cdots) \qquad (36)$$

The explicit forms of the two high ordered kernels of the special term are:

$$\sigma^{*(N)}_{123} = \frac{-(2N-3)r^{2N-2}}{16\pi(1-\nu)N(N+1)(2N-2)!}r_{,1}r_{,2}r_{,3}$$

$$(37)$$

$$\frac{\partial}{\partial n}\sigma^{*(N)}_{123} = \frac{-(2N-3)r^{2N-3}}{16\pi(1-\nu)N(N+1)(2N-2)!}$$

$$\times\left[(2N-5)\frac{\partial r}{\partial n}r_{,1}r_{,2}r_{,3}+r_{,1}r_{,2}n_3+r_{,2}r_{,3}n_1+r_{,3}r_{,1}n_2\right]$$

$$(N=1,\ 2,\ \cdots) \qquad (38)$$

With these formulations, all of the domain integrals have been represented by the boundary-type for the displacement equation.

## 5. Formulations for Stress Equation

The formulation of the zeroth order term or the constant term of the polynomials for the stress equation is deduced also from the traction free state with a uniformly distributed unit initial strain over the domain[7] as follows:

$$\int_\Omega \sigma^*_{ijkl}(p,\ q)d\Omega(q)+C(p)O_{ijkl}(p)$$
$$=\int_\Gamma x_l\tau^*_{ijk}(p,\ q)d\Gamma(q) \qquad (39)$$

The term $O_{ijkl}(p)$ owing to the strong singularity of the kernel appears only in the constant term and can be contained automatically by the boundary integral without special consideration in the programming. $C(p)$ is a coefficient depending on where the source point is located.

The transformation for the non-zeroth order term of polynomials is realized by making use of the derived constitutive relation of elasticity:

$$\sigma^*_{ijkl}(p,\ q)=-\mu\left\{\beta\delta_{ij}\sigma^*_{skl,s}+\sigma^*_{ikl,j}+\sigma^*_{jkl,i}\right\} \qquad (40)$$

Therefore, the domain integrals of the kernels relevant to the stress equation can be represented by the boundary-type as follows:

$$\frac{1}{\mu}\int_\Omega x_1^m x_2^n x_3^l\sigma^*_{ijkl}\,d\Omega(q)$$
$$=\beta\delta_{ij}\int_\Omega\Big[mx_1^{m-1}x_2^n x_3^l\sigma^*_{1kl}+nx_1^m x_2^{n-1}x_3^l\sigma^*_{2kl}$$
$$+lx_1^m x_2^n x_3^{l-1}\sigma^*_{3kl}\Big]d\Omega(q)$$
$$+\int_\Omega\Big[(mx_1^{m-1}x_2^n x_3^l\delta_{1j}+nx_1^m x_2^{n-1}x_3^l\delta_{2j}+$$
$$+lx_1^m x_2^n x_3^{l-1}\delta_{3j})\sigma^*_{ikl}+(mx_1^{m-1}x_2^n x_3^l\delta_{1i}+$$
$$+nx_1^m x_2^{n-1}x_3^l\delta_{2i}+lx_1^m x_2^n x_3^{l-1}\delta_{3i})\sigma^*_{jkl}\Big]d\Omega(q)$$
$$-\int_\Gamma x_1^m x_2^n x_3^l\Big[\beta\delta_{ij}\sigma^*_{skl}\,n_s+\sigma^*_{ikl}\,n_j+\sigma^*_{jkl}\,n_i\Big]\,d\Gamma(q)$$
$$\left(m\ge 0,\ n\ge 0,\ l\ge 0,\ m+n+l>0\right) \qquad (41)$$

The domain integrals at the right hand side of the above equation can then be replaced by the known formulations for the displacement equation and there are eighteen explicit formulations in boundary-type for the kernels in the stress equation in three-dimensional initial strain problems as follows:

$$\frac{1}{\mu}\int_\Omega x_1^m x_2^n x_3^l\sigma^*_{1111}\,d\Omega(q)=-\int_\Gamma x_1^m x_2^n x_3^l$$
$$\times\Big[(\beta+2)\sigma^*_{111}\,n_1+\beta\sigma^*_{211}\,n_2+\beta\sigma^*_{311}\,n_3\Big]d\Gamma(q)$$
$$+\frac{(\beta+2)m}{m+n+l}\int_\Gamma x_1^m x_2^n x_3^l\tau^*_{11}\,d\Gamma(q)$$
$$-\frac{\beta n(n-1)}{(m+1)(m+n+l)}\int_\Gamma x_1^{m+2}x_2^{n-2}x_3^l\tau^*_{22}\,d\Gamma(q)$$
$$-\frac{\beta l(l-1)}{(m+1)(m+n+l)}\int_\Gamma x_1^{m+2}x_2^n x_3^{l-2}\tau^*_{33}\,d\Gamma(q)$$
$$+\frac{\beta n}{m+1}\int_\Gamma x_1^{m+1}x_2^{n-1}x_3^l\tau^*_{21}\,d\Gamma(q)$$
$$+\frac{\beta l}{m+1}\int_\Gamma x_1^{m+1}x_2^n x_3^{l-1}\tau^*_{31}\,d\Gamma(q)$$
$$-\frac{2\beta nl}{m+1}\int_\Omega x_1^{m+1}x_2^{n-1}x_3^{l-1}\sigma^*_{123}\,d\Omega(q) \qquad (42)$$

$$\frac{1}{\mu}\int_\Omega x_1^m x_2^n x_3^l\sigma^*_{2222}\,d\Omega(q)=-\int_\Gamma x_1^m x_2^n x_3^l$$
$$\times\Big[(\beta+2)\sigma^*_{222}\,n_2+\beta\sigma^*_{322}\,n_3+\beta\sigma^*_{122}\,n_1\Big]d\Gamma(q)$$
$$+\frac{(\beta+2)n}{m+n+l}\int_\Gamma x_1^m x_2^n x_3^l\tau^*_{22}\,d\Gamma(q)$$
$$-\frac{\beta l(l-1)}{(n+1)(m+n+l)}\int_\Gamma x_1^m x_2^{n+2}x_3^{l-2}\tau^*_{33}\,d\Gamma(q)$$
$$-\frac{\beta m(m-1)}{(n+1)(m+n+l)}\int_\Gamma x_1^{m-2}x_2^{n+2}x_3^l\tau^*_{11}\,d\Gamma(q)$$

$$+\frac{\beta l}{n+1}\int_\Gamma x_1^m x_2^{n+1} x_3^{l-1}\tau^*_{32}\,d\Gamma(q)$$

$$+\frac{\beta m}{n+1}\int_\Gamma x_1^{m-1} x_2^{n+1} x_3^{l}\tau^*_{12}\,d\Gamma(q)$$

$$-\frac{2\beta lm}{n+1}\int_\Omega x_1^{m-1} x_2^{n+1} x_3^{l-1}\sigma^*_{123}\,d\Omega(q) \tag{43}$$

$$\frac{1}{\mu}\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{3333}\,d\Omega(q) = -\int_\Gamma x_1^m x_2^n x_3^l$$

$$\times\left[(\beta+2)\sigma^*_{333}\,n_3 + \beta\sigma^*_{133}\,n_1 + \beta\sigma^*_{233}\,n_2\right]d\Gamma(q)$$

$$+\frac{(\beta+2)\,l}{m+n+l}\int_\Gamma x_1^m x_2^n x_3^l\tau^*_{33}\,d\Gamma(q)$$

$$-\frac{\beta m(m-1)}{(l+1)(m+n+l)}\int_\Gamma x_1^{m-2} x_2^n x_3^{l+2}\tau^*_{11}\,d\Gamma(q)$$

$$-\frac{\beta n(n-1)}{(l+1)(m+n+l)}\int_\Gamma x_1^m x_2^{n-2} x_3^{l+2}\tau^*_{22}\,d\Gamma(q)$$

$$+\frac{\beta m}{l+1}\int_\Gamma x_1^{m-1} x_2^n x_3^{l+1}\tau^*_{13}\,d\Gamma(q)$$

$$+\frac{\beta n}{l+1}\int_\Gamma x_1^m x_2^{n-1} x_3^{l+1}\tau^*_{23}\,d\Gamma(q)$$

$$-\frac{2\beta mn}{l+1}\int_\Omega x_1^{m-1} x_2^{n-1} x_3^{l+1}\sigma^*_{123}\,d\Omega(q) \tag{44}$$

$$\frac{1}{\mu}\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{1122}\,d\Omega(q)$$

$$=\frac{1}{m+n+l}\int_\Gamma x_1^m x_2^n x_3^l\left(n\tau^*_{11} + m\tau^*_{22}\right)d\Gamma(q)$$

$$-\int_\Gamma x_1^m x_2^n x_3^l\left(\sigma^*_{221}\,n_1 + \sigma^*_{112}\,n_2\right)d\Gamma(q) \tag{45}$$

$$\frac{1}{\mu}\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{2233}\,d\Omega(q)$$

$$=\frac{1}{m+n+l}\int_\Gamma x_1^m x_2^n x_3^l\left(l\tau^*_{22} + n\tau^*_{33}\right)d\Gamma(q)$$

$$-\int_\Gamma x_1^m x_2^n x_3^l\left(\sigma^*_{332}\,n_2 + \sigma^*_{223}\,n_3\right)d\Gamma(q) \tag{46}$$

$$\frac{1}{\mu}\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{3311}\,d\Omega(q)$$

$$=\frac{1}{m+n+l}\int_\Gamma x_1^m x_2^n x_3^l\left(m\tau^*_{33} + l\tau^*_{11}\right)d\Gamma(q)$$

$$-\int_\Gamma x_1^m x_2^n x_3^l\left(\sigma^*_{113}\,n_3 + \sigma^*_{331}\,n_1\right)d\Gamma(q) \tag{47}$$

$$\frac{1}{\mu}\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{1112}\,d\Omega(q) = \int_\Gamma x_1^m x_2^n x_3^l\tau^*_{21}\,d\Gamma(q)$$

$$+\frac{n}{m+n+l}\int_\Gamma x_1^{m+1} x_2^{n-1} x_3^l(\tau^*_{11} - \tau^*_{22})\,d\Gamma(q)$$

$$-\int_\Gamma x_1^m x_2^n x_3^l\left(\sigma^*_{111}\,n_2 + \sigma^*_{211}\,n_1\right)d\Gamma(q)$$

$$-l\int_\Omega x_1^m x_2^n x_3^{l-1}\sigma^*_{123}\,d\Omega(q) \tag{48}$$

$$\frac{1}{\mu}\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{2223}\,d\Omega(q) = \int_\Gamma x_1^m x_2^n x_3^l\tau^*_{32}\,d\Gamma(q)$$

$$+\frac{l}{m+n+l}\int_\Gamma x_1^m x_2^{n+1} x_3^{l-1}(\tau^*_{22} - \tau^*_{33})\,d\Gamma(q)$$

$$-\int_\Gamma x_1^m x_2^n x_3^l\left(\sigma^*_{222}\,n_3 + \sigma^*_{322}\,n_2\right)d\Gamma(q)$$

$$-m\int_\Omega x_1^{m-1} x_2^n x_3^l\sigma^*_{123}\,d\Omega(q) \tag{49}$$

$$\frac{1}{\mu}\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{3331}\,d\Omega(q) = \int_\Gamma x_1^m x_2^n x_3^l\tau^*_{13}\,d\Gamma(q)$$

$$+\frac{m}{m+n+l}\int_\Gamma x_1^{m-1} x_2^n x_3^{l+1}(\tau^*_{33} - \tau^*_{11})\,d\Gamma(q)$$

$$-\int_\Gamma x_1^m x_2^n x_3^l\left(\sigma^*_{333}\,n_1 + \sigma^*_{133}\,n_3\right)d\Gamma(q)$$

$$-n\int_\Omega x_1^m x_2^{n-1} x_3^l\sigma^*_{123}\,d\Omega(q) \tag{50}$$

$$\frac{1}{\mu}\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{1113}\,d\Omega(q) = \int_\Gamma x_1^m x_2^n x_3^l\tau^*_{31}\,d\Gamma(q)$$

$$+\frac{l}{m+n+l}\int_\Gamma x_1^{m+1} x_2^n x_3^{l-1}(\tau^*_{11} - \tau^*_{33})\,d\Gamma(q)$$

$$-\int_\Gamma x_1^m x_2^n x_3^l\left(\sigma^*_{111}\,n_3 + \sigma^*_{311}\,n_1\right)d\Gamma(q)$$

$$-n\int_\Omega x_1^m x_2^{n-1} x_3^l\sigma^*_{123}\,d\Omega(q) \tag{51}$$

$$\frac{1}{\mu}\int_\Omega x_1^m x_2^n x_3^l \sigma^*_{2221}\,d\Omega(q) = \int_\Gamma x_1^m x_2^n x_3^l\tau^*_{12}\,d\Gamma(q)$$

$$+\frac{m}{m+n+l}\int_\Gamma x_1^{m-1} x_2^{n+1} x_3^l(\tau^*_{22} - \tau^*_{11})\,d\Gamma(q)$$

$$-\int_\Gamma x_1^m x_2^n x_3^l\left(\sigma^*_{222}\,n_1 + \sigma^*_{122}\,n_2\right)d\Gamma(q)$$

$$-l\int_{\Omega} x_1^m x_2^n x_3^{l-1}\sigma^*_{123}\,d\Omega(q) \tag{52}$$

$$\begin{aligned}\frac{1}{\mu}\int_{\Omega} x_1^m x_2^n x_3^l\sigma^*_{3332}\,d\Omega(q) &= \int_{\Gamma} x_1^m x_2^n x_3^l\tau^*_{23}\,d\Gamma(q)\\
&+\frac{n}{m+n+l}\int_{\Gamma} x_1^m x_2^{n-1} x_3^{l+1}(\tau^*_{33}-\tau^*_{22})d\Gamma(q)\\
&-\int_{\Gamma} x_1^m x_2^n x_3^l\left(\sigma^*_{333}n_2+\sigma^*_{233}n_3\right)d\Gamma(q)\\
&-m\int_{\Omega} x_1^{m-1} x_2^n x_3^l\sigma^*_{123}\,d\Omega(q)\end{aligned} \tag{53}$$

$$\begin{aligned}\frac{1}{\mu}\int_{\Omega} x_1^m x_2^n x_3^l\sigma^*_{1223}\,d\Omega(q) &= n\int_{\Omega} x_1^m x_2^{n-1} x_3^l\sigma^*_{123}\,d\Omega(q)\\
&+\frac{m}{m+n+l}\int_{\Gamma} x_1^{m-1} x_2^n x_3^{l+1}\tau^*_{22}\,d\Gamma(q)\\
&-\int_{\Gamma} x_1^m x_2^n x_3^l\left(\sigma^*_{123}n_2+\sigma^*_{223}n_1\right)d\Gamma(q)\end{aligned} \tag{54}$$

$$\begin{aligned}\frac{1}{\mu}\int_{\Omega} x_1^m x_2^n x_3^l\sigma^*_{2331}\,d\Omega(q) &= l\int_{\Omega} x_1^m x_2^n x_3^{l-1}\sigma^*_{123}\,d\Omega(q)\\
&+\frac{n}{m+n+l}\int_{\Gamma} x_1^{m+1} x_2^{n-1} x_3^{l+1}\tau^*_{33}\,d\Gamma(q)\\
&-\int_{\Gamma} x_1^m x_2^n x_3^l\left(\sigma^*_{123}n_3+\sigma^*_{331}n_2\right)d\Gamma(q)\end{aligned} \tag{55}$$

$$\begin{aligned}\frac{1}{\mu}\int_{\Omega} x_1^m x_2^n x_3^l\sigma^*_{3112}\,d\Omega(q) &= m\int_{\Omega} x_1^{m-1} x_2^n x_3^l\sigma^*_{123}\,d\Omega(q)\\
&+\frac{l}{m+n+l}\int_{\Gamma} x_1^m x_2^{n+1} x_3^{l-1}\tau^*_{11}\,d\Gamma(q)\\
&-\int_{\Gamma} x_1^m x_2^n x_3^l\left(\sigma^*_{123}n_1+\sigma^*_{112}n_3\right)d\Gamma(q)\end{aligned} \tag{56}$$

$$\begin{aligned}\frac{1}{\mu}\int_{\Omega} x_1^m x_2^n x_3^l\sigma^*_{1123}\,d\Omega(q) &= -\frac{nl}{(m+1)(m+n+l)}\\
&\times\int_{\Gamma} x_1^{m+2} x_2^{n-1} x_3^{l-1}(\tau^*_{22}+\tau^*_{33})d\Gamma(q)\\
&+\frac{l}{m+1}\int_{\Gamma} x_1^{m+1} x_2^n x_3^{l-1}\tau^*_{21}\,d\Gamma(q)\\
&+\frac{n}{m+1}\int_{\Gamma} x_1^{m+1} x_2^{n-1} x_3^l\tau^*_{31}\,d\Gamma(q)\\
&-\int_{\Gamma} x_1^m x_2^n x_3^l(\sigma^*_{211}n_3+\sigma^*_{311}n_2)d\Gamma(q)\\
&-\frac{l(l-1)}{m+1}\int_{\Omega} x_1^{m+1} x_2^n x_3^{l-2}\sigma^*_{123}\,d\Omega(q)\\
&-\frac{n(n-1)}{m+1}\int_{\Omega} x_1^{m+1} x_2^{n-2} x_3^l\sigma^*_{123}\,d\Omega(q)\end{aligned} \tag{57}$$

$$\begin{aligned}\frac{1}{\mu}\int_{\Omega} x_1^m x_2^n x_3^l\sigma^*_{2231}\,d\Omega(q) &= -\frac{lm}{(n+1)(m+n+l)}\\
&\times\int_{\Gamma} x_1^{m-1} x_2^{n+2} x_3^{l-1}(\tau^*_{33}+\tau^*_{11})d\Gamma(q)\\
&+\frac{m}{n+1}\int_{\Gamma} x_1^{m-1} x_2^{n+1} x_3^l\tau^*_{32}\,d\Gamma(q)\\
&+\frac{l}{n+1}\int_{\Gamma} x_1^m x_2^{n+1} x_3^{l-1}\tau^*_{12}\,d\Gamma(q)\\
&-\int_{\Gamma} x_1^m x_2^n x_3^l(\sigma^*_{322}n_1+\sigma^*_{122}n_3)d\Gamma(q)\\
&-\frac{m(m-1)}{n+1}\int_{\Omega} x_1^{m-2} x_2^{n+1} x_3^l\sigma^*_{123}\,d\Omega(q)\\
&-\frac{l(l-1)}{n+1}\int_{\Omega} x_1^m x_2^{n+1} x_3^{l-2}\sigma^*_{123}\,d\Omega(q)\end{aligned} \tag{58}$$

$$\begin{aligned}\frac{1}{\mu}\int_{\Omega} x_1^m x_2^n x_3^l\sigma^*_{3312}\,d\Omega(q) &= -\frac{mn}{(l+1)(m+n+l)}\\
&\times\int_{\Gamma} x_1^{m-1} x_2^{n-1} x_3^{l+2}(\tau^*_{11}+\tau^*_{22})d\Gamma(q)\\
&+\frac{n}{l+1}\int_{\Gamma} x_1^m x_2^{n-1} x_3^{l+1}\tau^*_{13}\,d\Gamma(q)\\
&+\frac{m}{l+1}\int_{\Gamma} x_1^{m-1} x_2^n x_3^{l+1}\tau^*_{23}\,d\Gamma(q)\\
&-\int_{\Gamma} x_1^m x_2^n x_3^l(\sigma^*_{133}n_2+\sigma^*_{233}n_1)d\Gamma(q)\\
&-\frac{n(n-1)}{l+1}\int_{\Omega} x_1^m x_2^{n-2} x_3^{l+1}\sigma^*_{123}\,d\Omega(q)\\
&-\frac{m(m-1)}{l+1}\int_{\Omega} x_1^{m-2} x_2^n x_3^{l+1}\sigma^*_{123}\,d\Omega(q)\end{aligned} \tag{59}$$

$$\left(m\ge 0,\ n\ge 0,\ l\ge 0,\ m+n+l>0\right)$$

The special term $\sigma^*_{123}$ of domain integrals appears also in most of the above formulations which can be evaluated by the boundary-type recurrence equation (36).

## 6. Discussions

The applicability of the procedure in getting rid of domain discretizations by polynomial expansions lies in how the transformation can be realized from domain type into boundary one and how well the domain variables, the initial strains, can be described by polynomial approximation in terms of the field coordinates. In the present work, the field variable $x_k(q)$ has been replaced by the two point variable $x_k$ in the integrals through transformations (5) and (6), therefore the intrinsic correlation between the kernels can be made use of to avoid the term by term integrals except the special term in the three-dimensional case. It may be come from the geometrical symmetries.

As stated above, the transformations (5) and (6) can be performed via a short subroutine. There are other merits, for example, the term $O_{ijkl}$ owing to the singularity appears only in the zeroth order or constant term of the polynomials and will be embraced in the boundary integral (39) without special consideration so that the programming can be simplified.

Even for the zeroth order or constant term of the polynomials, the order of singularity in the boundary-type integrals is reduced by one. And because these formulations are hold no matter where the source point locates, inside or outside the domain or on the boundary, they might be used to evaluate the domain integrals with strong singularities when domain discretization is performed.

In performing the computing point methods[4)-5)], the polynomial can be expanded to any order conveniently using these formulations to meet the precision requirements. The formulations can be reduced to deal with the problem of thermo-elasticity by replacing the initial strain $\varepsilon_{ij}$ with the term $\alpha\theta\delta_{ij}$, for example, suppose

$$\alpha\theta H^*_i(p,q) = \alpha\theta\delta_{jk}\sigma^*_{ijk}(p,q) \qquad (60)$$

$$\alpha\theta H^*_{ij}(p,q) = \alpha\theta\delta_{kl}\sigma^*_{ijkl}(p,q) \qquad (61)$$

where $H^*_i$ and $H^*_{ij}$ stand for the domain integral kernels in the displacement and the stress equations, which can be derived by the combination of $\delta_{jk}\sigma^*_{ijk}$ and $\delta_{kl}\sigma^*_{ijkl}$, respectively, for the problem of thermo-elasticity. Then the boundary-type formulations for the zeroth order term can be obtained as follows:

$$\int_\Omega H^*_i(p,q)d\Omega(q) = \int_\Gamma x_j\tau^*_{ij}(p,q)d\Gamma(q) \qquad (62)$$

$$\int_\Omega H^*_{ij}(p,q)d\Omega(q) + \delta_{kl}O_{ijkl}(p) = \int_\Gamma x_k\tau^*_{ijk}(p,q)d\Gamma(q) \qquad (63)$$

For residual stress problems[6)], the initial strains formed by material nonlinearity can be approximated by polynomials and expected to be solved without domain discretizations.

## 7. Conclusion

With the aid of the intrinsic correlation among the integral kernels as well as the high ordered fundamental solutions, the explicit form of boundary-type integral formulations was deduced for the three-dimensional initial strain problems using boundary element method in which the domain variables were represented by complete series of polynomial expansions.

## REFERENCES

1)Brebbia,C.A.,Telles,J.C.F and Wrobel,L.C.: *Boundary Element Techniques—Theory and Application in Engineering,* Springer-Verlag, Heideberg, Berlin, 1984.

2)Cheng,A.H.-D., Lafe,O. and Grilli,S.: Dual-reciprocity BEM based on global interpolation functions, *Engineering Analysis with Boundary Elements,* 1994, **13**, 303-311.

3)Nowak,A.J.: Application of multiple reciprocity BEM to nonlinear potential problems, *Engineering Analysis with Boundary Elements,* 1995, **16**, 323-332.

4)Xu,S.Q. and Kamiya,N.: A formulation and solution for boundary element analysis of inhomogeneous-nonlinear problem, *Computational Mechanics,* 1998 (in press).

5)Xu,S.Q. and Kamiya,N.: A formulation and solution for boundary element analysis of inhomogeneous-nonlinear problem, 2nd report: the case involving derivatives of unknown function, *Engineering Analysis with Boundary Elements* (submitted).

6)Ma,H. and Deng,H.L.: Nondestructive determination of welding residual stresses by boundary element method, *Advances in Engineering Software*, 1998 (in press).

7)Ma,H., Kamiya,N. and Xu,S.Q.: Complete polynomial expansion of domain variables at boundary for two-dimensional elasto-plastic problems, *Engineering Analysis with Boundary Elements*, 1998 (in press)

**(Received April 24,1998)**