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A Degenerate Finite Element Formulation for Large Displacement Analysis of Thin-Walled Beams

薄肉断面梁の有限変位解析のためのディジェネレイト有限要素定式化

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A finite element formulation for the large displacement analysis of thin-walled beams is presented. The degeneration approach is taken in this formulation, so that the nodal degrees-of-freedom in the beam element thus developed are all vector quantities. It is also noteworthy that the concept of bimoment which is significant in the classical theory of thin-walled beams needs not be referred to. The present formulation is therefore simple and straightforward. Numerical examples are solved and excellent agreement with analytical solutions is obtained, confirming the validity of the present formulation.

Key Words : *degeneration approach, finite element method, thin-walled member*

1. Introduction

Because of their structural effectiveness, thin-walled members are used extensively in steel structures. Therefore, much research has been conducted and now we may state that the theory is well established[1)-4)]. The theory is, however, rather involved and only simple thin-walled structural problems can be solved analytically. For practical problems, we must resort to a numerical method to obtain solutions. In this conjunction, the finite element method has been used exclusively.

In the classical finite element approach, the governing equations in the theory of thin-walled beams are discretized. A typical beam element thus derived has two nodes and seven degrees-of-freedom are assigned to each node[5),6)]: a derivative of rotation with respect to the longitudinal direction of a beam, three translations and three rotations. The derivative of rotation is associated with the so-called bimoment that causes the warping of a thin-walled cross-section.

There exists another class of finite element formulation, which is called the degeneration approach[7)-11)]. This approach treats structural member as a special case of continuum: the governing equations for continuum are directly discretized by the finite element method and the characteristics of a structural member are implemented only in the discretization procedure.

Since finite rotations are not vector quantities, the large displacement analysis of three-dimensional beams is not a simple task, as far as the nodal variables of a beam element include rotations. To this end, various techniques such as Euler angles have been devised[12),13)]. Because of its consistency with three-dimensional solid mechanics, the degeneration approach enables us to exclude rotations from nodal variables. In fact, for the large displacement analysis of three-dimensional solid beams, we have developed a degenerate beam element which has no rotational degrees-of-freedom at nodes[11)].

The objective of the present research is to present a simple and accurate finite element formulation for the large displacement analysis of thin-walled

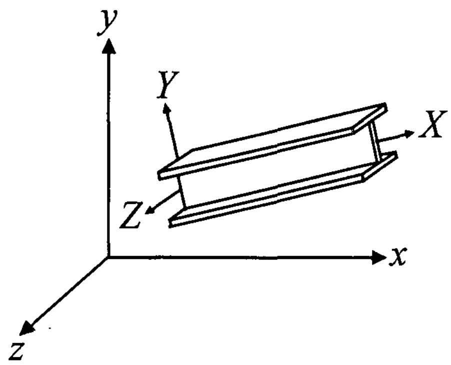


Fig. 1 Coordinate systems

beams. To this end, we extend our previous work[9),11)], in which the degeneration approach is employed and rotations are excluded from nodal variables. The proposed formulation is free from the difficulties associated with finite rotations and also does not require the concept of bimoment, the physical meaning of which is not always obvious.

2. Formulation

We utilize two sets of coordinate systems in the present formulation: spatial coordinates and material coordinates[14)]. In what follows, x, y and z denote the former while X, Y and Z the latter. The tensor notation is also employed in the present description, so that we may use x_i and X_I to represent spatial coordinates and material coordinates, respectively. Furthermore, we let the lower-case and upper-case subscripts designate the association with the spatial and material coordinate systems, respectively. We set the X (X_1)-axis in the longitudinal direction of a beam, as is shown in Fig. 1.

We resort to the total Lagrangian formulation in the present study[15)]. Therefore, the description would be in terms of the 2nd Piola-Kirchhoff stress S_{IJ} and the Green strain E_{IJ}.

2.1 Basic Assumptions

In the present study, we employ the following beam assumptions:

a) cross sections do not deform in their respective planes:
b) the cross section of each constituent plate of a thin-walled beam remains plane; and
c) only three stress components are significant.

Using the Green strain components, we can mathematically express Assumption a) as

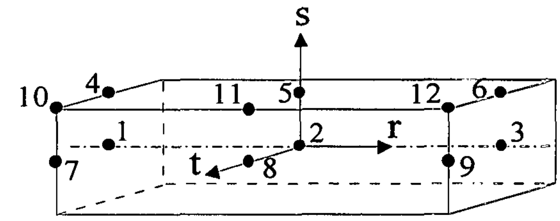
$$E_{YY} = E_{ZZ} = E_{YZ} = 0 \tag{1}$$

Assumption b) is implemented by devising a new finite element, which we shall describe in the subsequent section.

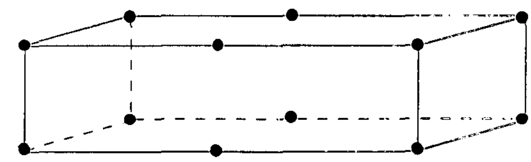
Assumption c) means

$$S_{YY} = S_{ZZ} = S_{YZ} = 0 \tag{2}$$

The remaining stress components S_{XX}, S_{XY} and S_{XZ} are related to the three nontrivial strain components E_{XX}, E_{XY} and E_{XZ}, respectively, through Young's modulus E and the shear modulus G.



(a) Beam element



(b) 12-node solid element

Fig. 2 Finite elements

2.2 Beam Element

To implement Assumption b), a beam element is developed from a 12-node three-dimensional isoparametric solid element. These two elements are illustrated in Fig. 2. The beam element consists of twelve nodes: three reference nodes on the beam axis (Nodes 1 to 3) and nine relative nodes on the beam surface (Nodes 4 to 12).

We assign three displacement components to each of the reference nodes as nodal variables while we assign to each relative node three components of the relative displacement with respect to the reference node located on the same cross-section. No rotations are involved in nodal variables. As a consequence, the present beam element is free from the complexity due to finite rotations.

We describe geometry of the beam element by a set of natural curvilinear coordinates (r,s,t). The displacement vector u_j of a point at (r,s,t) in a beam element can be expressed in terms of nodal variables as

$$u_j = \sum_{a=1}^{12} N^a(r,s,t) U_j^a \quad (3)$$

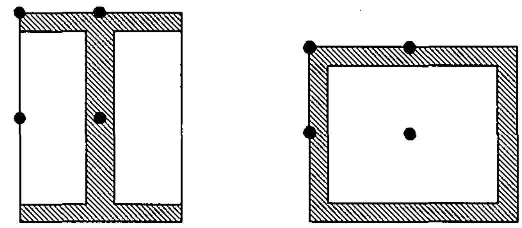
where U_j^a denotes the absolute displacement vector at Node "a" for $a = 1 \sim 3$ and the relative displacement vector at Node "a" for $a = 4 \sim 12$. The shape function N^a consistent with this definition of U_j^a is derived in the way similar to that of the degenerate shell element due to Kanok-Nukulchai et al.[7] and given by

$$N^a(r,s,t) = \begin{cases} \frac{1}{2} r^a r \left(1 + r^a r\right) & \text{for } a = 1,3 \\ 1 - r^2 & \text{for } a = 2 \\ \frac{1}{2} r^a r \left(1 + r^a r\right)\left(s^a s + t^a t\right) & \text{for } a = 4,6,7,9 \\ \left(1 - r^2\right)\left(s^a s + t^a t\right) & \text{for } a = 5,8 \\ \frac{1}{2} ts \left(1 + r^a r\right) & \text{for } a = 10,12 \\ ts\left(1 - r^2\right) & \text{for } a = 11 \end{cases} \quad (4)$$

where

$$r^a = \begin{cases} -1 & a = 1,4,7,10 \\ 0 & a = 2,5,8,11 \\ 1 & a = 3,6,9,12 \end{cases}$$
$$s^a = \begin{cases} 0 & a = 1 \sim 3, 7 \sim 9 \\ 1 & a = 4 \sim 6, 10 \sim 12 \end{cases} \quad (5)$$
$$t^a = \begin{cases} 0 & a = 1 \sim 6 \\ 1 & a = 7 \sim 12 \end{cases}$$

This element is isoparametric provided that relative position vector is input for each relative node. The relative position vector is therefore defined in the same way as the relative displacement vector and we have



(a) I-section (b) Box-section

Fig. 3 Integration zones

$$x_j = \sum_{a=1}^{12} N^a(r,s,t) X_j^a \quad (6)$$

where X_j^a with $a = 1 \sim 3$ is the position vector of a reference node while X_j^a with $a = 4 \sim 12$ is the relative position vector of a relative node with respect to the reference node located on the same cross-section.

The beam element in Fig. 2(a) is solid. To analyze a thin-walled beam with this element, the technique that was employed in the linear analysis of thin-walled beams by conventional brick elements[9] is utilized herein: for instance, in the analysis of a typical thin-walled beam with an I-section or a box-section, only the shaded zone shown in Fig. 3 is assumed to have rigidity and therefore the integration for the construction of an element stiffness matrix is carried out over the shaded zone only.

2.3 Governing Discretized Equations

For the basis of finite element formulation, we employ the principle of virtual work derived from the governing equations for three-dimensional continuum[15]. In order to impose Eq. (1), we further resort to the penalty method[16]. Hence, we start the formulation with the following equation:

$$\begin{aligned} W = & \int_{V_0} S_{XX} \delta E_{XX} dV + \int_{V_0} 2S_{XY} \delta E_{XY} dV \\ & + \int_{V_0} 2S_{XZ} \delta E_{XZ} dV - \int_{V_0} \rho_0 b_j \delta u_j dV \\ & - \int_{A_{0t}} t_{0j} \delta u_j dA + \int_{V_0} kE_{YY} \delta E_{YY} dV \\ & + \int_{V_0} kE_{ZZ} \delta E_{ZZ} dV + \int_{V_0} kE_{YZ} \delta E_{YZ} dV = 0 \end{aligned} \quad (7)$$

where ρ_0 the mass density, b_j the body force, t_{0j} the prescribed traction, δE_{IJ} the virtual strain, δu_j the virtual displacement and k the penalty number. V_0 is the body under consideration and A_{0t} the boundary surface with the prescribed traction. The subscript 0 indicates the original state.

Employing the same shape functions for δu_j as those for u_j, we arrive at

$$W = \sum_{e=1}^{n} W^e = 0 \tag{8}$$

where

$$W^e = \sum_{b=1}^{12} \delta U_j^b \left[K_j^b - R_j^b \right] \tag{9}$$

$$\begin{aligned} K_j^b = \int_{V_0^e} [& S_{XX} F_{jX} N^b_{,X} \\ & + S_{XY}(F_{jX} N^b_{,Y} + F_{jY} N^b_{,X}) \\ & + S_{XZ}(F_{jX} N^b_{,Z} + F_{jZ} N^b_{,X}) \\ & + kE_{YY} F_{jY} N^b_{,Y} + kE_{ZZ} F_{jZ} N^b_{,Z} \\ & + \frac{1}{2} kE_{YZ}(F_{jY} N^b_{,Z} + F_{jZ} N^b_{,Y})] \, dV \end{aligned} \tag{10}$$

$$R_j^b = \int_{V_0^e} \rho_0 b_j N^b dV + \int_{A_{0t}^e} t_{0j} N^b dA \tag{11}$$

where F_{jJ} is the deformation gradient, and n in Eq. (8) stands for the number of elements.

We evaluate Eqs. (10) and (11) for each element, and assemble all those individual element contributions. Since the nodal virtual displacement is arbitrary, we end up with

$$\mathbf{K} - \mathbf{R} = \mathbf{0} \tag{12}$$

where $\mathbf{K}$ and $\mathbf{R}$ are the assemblage of K_j^b and R_j^b, respectively.

Since Eq. (12) is a nonlinear algebraic system, we must have recourse to some numerical methods. In the present study, we utilize the Newton-Raphson technique, and the following linearized equation is solved repeatedly until convergence is attained:

$$\mathbf{K_T} \Delta \mathbf{U}^{(m)} = \mathbf{R} - \mathbf{K}^{(m)} \tag{13}$$

where the superscript (m) denotes the number of iterations. $\Delta \mathbf{U}^{(m)}$ is the iterative increment of the nodal displacement at the m th iteration. $\mathbf{K_T}$ is the tangent stiffness matrix, which can be evaluated by taking the derivative of $\mathbf{K}$ with respect to the nodal displacement $\mathbf{U}$. To that end, consistent linearization[7),17)] is performed and the tangent stiffness matrix is obtained for an element as

$$\begin{aligned} K_{Tji}^{ba} = \int_{V_0^e} \Big[& F_{jX} N^b_{,X} E F_{iX} N^a_{,X} \\ & + \left(F_{jX} N^b_{,Y} + F_{jY} N^b_{,X}\right) G \left(F_{iX} N^a_{,Y} + F_{jY} N^a_{,X}\right) \\ & + \left(F_{jX} N^b_{,Z} + F_{jZ} N^b_{,X}\right) G \left(F_{iX} N^a_{,Z} + F_{jZ} N^a_{,X}\right) \\ & + F_{jY} N^b_{,Y} k F_{iY} N^a_{,Y} + F_{jZ} N^b_{,Z} k F_{iZ} N^a_{,Z} \\ & + \left(F_{jY} N^b_{,Z} + F_{jZ} N^b_{,Y}\right) \frac{k}{4} \left(F_{iY} N^a_{,Z} + F_{jZ} N^a_{,Y}\right) \Big] dV \\ + \delta_{ij} \int_{V_0^e} \Big[& S_{XX} N^b_{,X} N^a_{,X} + S_{XY}\left(N^b_{,X} N^a_{,Y} + N^b_{,Y} N^a_{,X}\right) \\ & + S_{XZ}\left(N^b_{,X} N^a_{,Z} + N^b_{,Z} N^a_{,X}\right) \\ & + kE_{YY} N^b_{,Y} N^a_{,Y} + kE_{ZZ} N^b_{,Z} N^a_{,Z} \\ & + \frac{k}{2} E_{YZ}\left(N^b_{,Y} N^a_{,Z} + N^b_{,Z} N^a_{,Y}\right) \Big] dV \end{aligned} \tag{14}$$

3. Numerical Examples

Numerical examples are solved to test the validity of the proposed formulation. In all the analyses, five beam elements are employed for the discretization and the calculations are performed on Sun SPARCstation 2 using double precision.

3.1 Cantilever Beam under End Torque

We first conduct the linear analysis of a cantilever beam subjected to a concentrated torque T at its free end (Fig. 4). The length of the beam and the material properties are assumed as: $L = 150$, $E = 2.0 \times 10^6$ and $G = 1.0 \times 10^6$.

Two different cross-sections are considered: an I-section and a box-section. Their dimensions are described in Fig. 5. The magnitude of the applied torque is 0.04 for the I-beam and 400 for the box-beam.

As a numerical result, the variation of warping

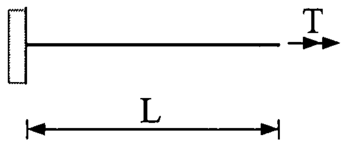
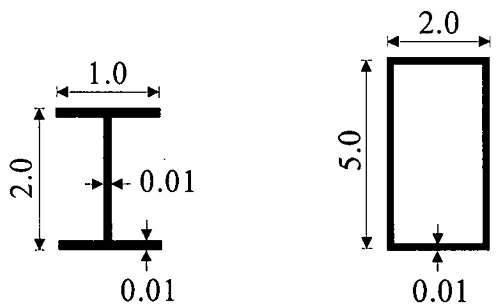


Fig. 4 Cantilever beam under end torque



(a) I-section (b) Box-section

Fig. 5 Cross sections (end-torque problem)

displacement is presented together with the analytical solution in Fig. 6. Very good agreement is observed all along the beam length.

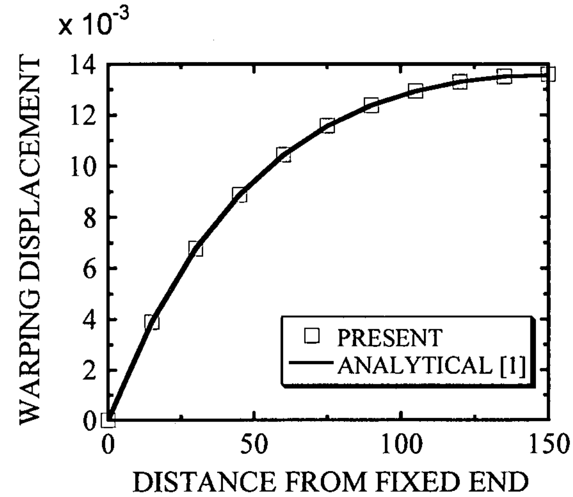
3.2 Cantilever Beam under End Load

The large displacement analysis of a cantilever beam (Fig. 7) is performed. The length of the beam and the material properties are: $L = 10$, $E = 2.0 \times 10^6$ and $G = 1.0 \times 10^6$. This is a well-known benchmark problem for two-dimensional large displacement analysis, and the analytical solution by means of elliptic integrals is available[18].

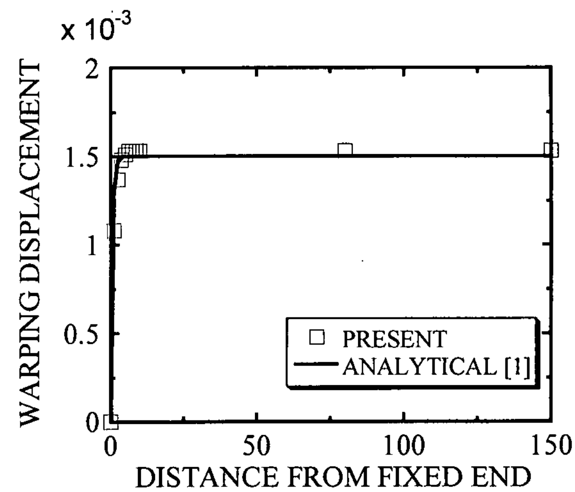
Two cross-sections given in Fig. 8 are assumed in this problem. The numerical results for the two cross-sections, which are depicted in Fig. 9 as the load P-vertical displacement v relationship, are very similar to each other. The analytical solution is also plotted in this figure. As can be seen there, the present numerical results are in excellent agreement with the analytical solution all the way to the end of the computation: the maximum error involved in the present results is found merely 0.37%.

3.3 Lateral Buckling of Simply Supported I-Beam

The simply supported I-beam illustrated in Fig. 10 is considered. The conditions are: $L = 130$; $E = 9.3 \times 10^6$; and $G = 3.72 \times 10^6$. In addition to the vertical load P, a small transverse load of



(a) I-section



(b) Box-section

Fig. 6 Variation of warping displacement

$P/2000$ is applied to initiate the out-of-plane displacement.

While the transverse load is always applied at the centroid of the cross section, three different loading points are employed for the vertical load P: P is applied at the centroid of the cross section in Beam C; at the midpoint of the top flange in Beam T; and at the midpoint of the bottom flange in Beam B. The critical loads for the lateral buckling of these beams have been obtained analytically[19].

Due to the symmetry of the problem, only a half of the beam is analyzed. The load P-transverse displacement w relationship at the loading point is given in Fig. 11, in which the critical loads are indicated by dotted lines and P_{cr} is the critical load

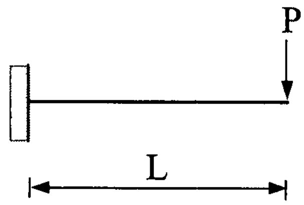
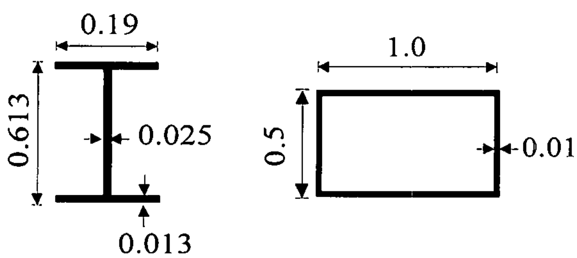


Fig. 7 Cantilever beam under end load



(a) I-section (b) Box-section

Fig. 8 Cross sections (end-load problem)

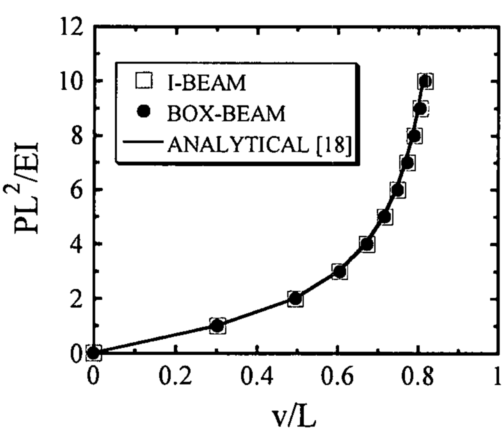
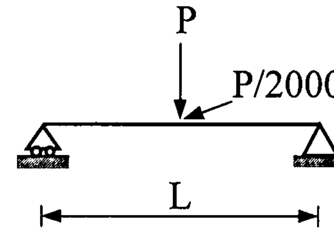
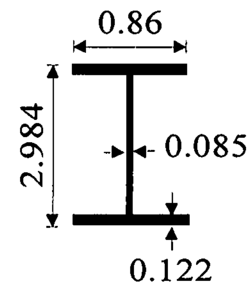


Fig. 9 Load-displacement relationship (end-load problem)



(a) Side view



(b) Cross section

Fig. 10 Simply supported I-beam

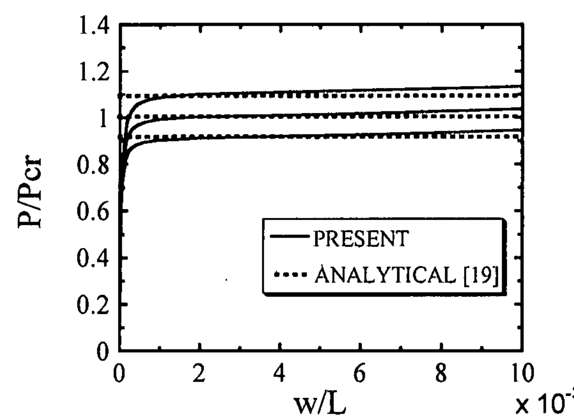


Fig. 11 Load-displacement relationship (lateral-buckling problem)

of Beam C. In each beam, a sharp increase of the displacement is clearly observed, as the critical load is approached.

4. Concluding Remarks

A finite element formulation for the large displacement analysis of thin-walled beams is presented. Employing the degeneration approach, the rotational degrees-of-freedom are excluded from nodal variables and the formulation becomes simple and straightforward. Moreover, the concept of bimoment, which is important in the theory of thin-walled beams but the physical meaning of which is not always obvious, needs not be used in the present formulation.

Three problems were solved. In the linear and the in-plane problems, the present results are compared with the analytical solutions and very good agreement is observed for both I- and box-beams. In the analysis of the lateral buckling of an I-beam, we could clearly observe the sharp increase of the displacement in the vicinity of the respective buckling load. The validity of the present formulation is therefore confirmed.

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