7. An Application of Parametric Curves to Highway Alignment

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Abstract: The authors examined an application of parametric curves used as spatial curves to highway alignment. Both of centrifugal acceleration and its rate of increase while driving the highway alignment designed by several kinds of parametric curve were compared. As a result of the examination, it was shown that cubic B-spline curves are applicable to the case of high- grade of alignment design and quadratic B-spline curves are applicable to the case of low-grade of alignment design.

Keywords: *highway alignment, parametric curve, B-spline curve, centrifugal acceleration, CAD*

1. INTRODUCTION

Highway a lignment i s u sually s pecified in terms of two separate two-dimensional alignments, horizontal and vertical. The Road Structure Ordinance, which provides for some of the technical aspects of highway design in Japan, describes a set of geometric design standards for each of these types of two-dimensional alignment. Meeting these design standards, however, does not guarantee t hat t he r esultant t hree-dimensional highway alignment will be an ideal one.

Problems like this will not occur if highway alignment can be defined as a three-dimensional alignment. One way to achieve this is a design method that uses an "adjustable curve."[1] In this method, an adjustable curve made of elastic material is used to create a road alignment as a 3D curve. In this method, 3D geometry can be easily visualized, and, because points of inflection of horizontal and vertical alignments coincide, it is possible to create a harmonious combination of horizontal and vertical alignments. There are also drawbacks, however. For example, converting spatial data defined by an adjustable curve into graphic or digital data is not easy, and matching to topography is difficult to achieve.

Research efforts are now underway, particularly in the field of mechanical engineering, to use parametric curves to simulate, on a computer, curves created by the adjustable curve method, and work is already in progress to implement the parametric curve method on a CAD system[2]. Parametric curves make it possible to reproduce the adjustable-curve-based highway alignment design method on a computer[3]. If this is done, however, a problem arises: whether or not highway alignments drawn with parametric curves satisfy the requirements of the vehicle and the driver.

This paper discusses applicability of parametric curves, which are often used to define free curves in 3D CAD, to highway alignment.

2. PARAMETRIC CURVES AND THEIR MATHEMATICAL REPRESENTEATION

(1) Parametric representation of curves

In graphic representation on a computer, straight line segments connecting two given points are used as basic elements, and a curve is represented as a series of straight line segments. Since the coordinates of the end point of each line segment on the curve needs to be found, parametric curves are used as a means of doing this.

A set of coordinates on a parametric curve is represented by one parameter. A given set of coordinates on a quadratic curve can be expressed, using a parameter t, as follows:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad \text{(Eq. 1)}$$

Various methods for numerical representation of free curves have been realized on CAD systems. In this study, some of the commonly used curves, namely, cubic spline curves, Bézier curves, and quadratic and cubic B-spline curves, are considered. These parametric curves are generated based on control points defined in space (Figure 1).

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(2) Mathematical representation of parametric curves

a) Cubic spline curve

It is generally thought that, if an adjustable curve is assumed to be an elastic bar, the geometry of curve segments connecting control points can be expressed by a cubic polynomial. The coordinates (x, y, z) of a 3D cubic spline curve drawn using four points, $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, $P_4(x_4, y_4, z_4)$, as boundary conditions are given by Eq. (2a)[4]:

$$\begin{cases} x = \sum_{i=1}^{4} x_i B_i(t) = x_1 B_1(t) + x_2 B_2(t) + x_3 B_3(t) + x_4 B_4(t) \\ y = \sum_{i=1}^{4} y_i B_i(t) = y_1 B_1(t) + y_2 B_2(t) + y_3 B_3(t) + y_4 B_4(t) \\ z = \sum_{i=1}^{4} z_i B_i(t) = z_1 B_1(t) + z_2 B_2(t) + z_3 B_3(t) + z_4 B_4(t) \end{cases} \quad \text{(Eq. 2a)}$$

where t ($t_1 \leq t \leq t_2$) is a parameter; t_1 and t_2 are parameter values for the start and end points; and $B_i(t)$ is a coefficient.

b) Bézier curve

A curve consisting of the position vectors $P_i(k)$ ($k = 1, 2, \ldots, n$) of control points given as vertices of a polygon that has n vertices can be expressed as an (n-1) th degree function. Let $R(t)$ be the position vector of a given point on the curve. A Bézier curve, then, can be defined by

$$R(t) = \sum_{i=0}^{n} P_i B_{n,i}(t) \qquad (0 \leqq t \leqq 1) \qquad \text{(Eq. 3a)}$$

The basis function $Bn,i(t)$, called a "Bernstein function," can be expressed by

$$B_{n,i}(t) = \frac{n!}{(n-1)! \cdot i!} \cdot t^i \cdot (1-t)^{n-i} \qquad (0 \leqq t \leqq 1) \qquad \text{(Eq. 3b)}$$

c) B-spline curves

Position vector $R(t)$ (t: parameter) on a curve defined by the position vectors Pi (i = 1, 2, ..., n+1) of n+1 control points can be expressed by

$$R(t) = \sum_{i=1}^{n+1} N_{i,k}(t) P_i \qquad \text{(4a)}$$

$$(t_{min} \leqq t \leqq t_{max} \quad 2 \leqq k \leqq n+1)$$

where $N_{i,k}(t)$ is a normalized B-spline basis function of k th order ((k-1) th degree).

The basis function $N_{i,k}$ can be expressed, using the Cox-deBoor recursion formula, as follows:

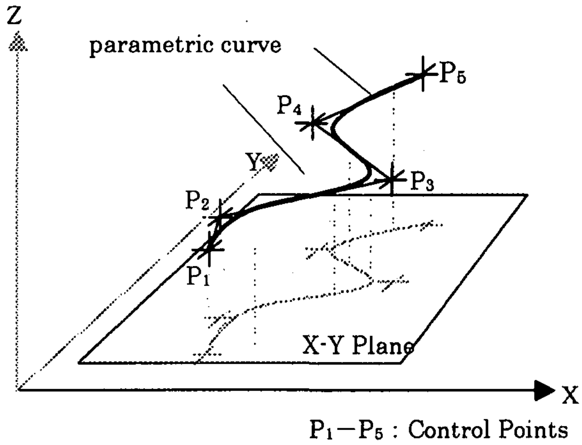

Figure 1 A Model of Parametric Curve

$$N_{i,0} = \begin{cases} 1 & x_i \leqq t < x_{i+1} \\ 0 & \text{For other values of } t \end{cases}$$

$$N_{i,k}(t) = \frac{t - x_i}{x_{i+k-1} - x_i} N_{i,k-i}(t) + \frac{x_{i+k} - t}{x_{i+k} - x_{i+1}} N_{i+1,k-1}(t)$$

(Eq. 4b)

where the values of x_i are vector elements of knot sequences that satisfy the relation $x_i \leq x_i+1$, and the number of knots to be found equals the number of control points n plus the order k.

3. EVALUATING OF APPLICABLITY TO HIGHWAY ALIGNMENT

It is generally known that drivers are sensitive to centrifugal acceleration and its rate of change, so that when centrifugal acceleration or its rate of change has exceeded a certain level, drivers begin to feel uncomfortable and initiate operations so as to increase the stability of the vehicle[5]. Since the curvature that determines centrifugal acceleration is governed by the radius of the circular curve in the horizontal alignment, in design this problem can be solved by changing its size in such a manner that similarity is maintained. Curvature varies depending on the presence or absence of transition curves and on their performance.

For transition curves, clothoids defined so that the rate of change of centrifugal acceleration is constant are used widely, and a smooth transition is achieved between a straight line and a circular curve or between circular curves. In order to apply parametric curves to highway alignment, therefore, it is necessary to evaluate centrifugal acceleration and its rate of change in the

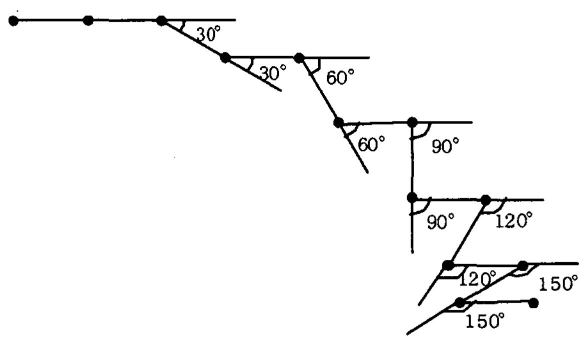

Figure 2 A model for the evaluation [Model-1]

evaluation of riding qualities.

In the riding quality evaluation, the horizontal component is also evaluated partly because motor vehicle drivers are sensitive to centrifugal acceleration, as mentioned above, and partly because it is common practice in highway alignment design. Since the x-, y-, and z-coordinates of a curve are found independently in the case of a parametric curve, the curve performance of vertical component is comparable to that of horizontal component. Curve performance, therefore, can be evaluated by evaluating that of either component. In this study, only the horizontal components of parametric curves are evaluated.

(1) Relationship between curvature of road and changes in acceleration

This section looks at the relationship between the degree of curvature of highway alignment and riding qualities that can be achieved with parametric curves.

In defining a parametric curve, passing points were defined at regular intervals; control points adjusted so that the passing points were actually passed; and highway alignment determined accordingly. Passing points were defined so that neighboring passing points were equally spaced apart (100m was adopted for convenience of numerical calculation). The passing points were located as shown in Figure 2. These settings make it possible to compare riding quality that can be achieved with different degrees of curvature in highway alignments.

The locations of control points were determined so that curve length was minimized and, therefore, control points and curve vertices coincided. For all curves except cubic spline curves, control points excluding end points were not on the curves. Control points, therefore, were

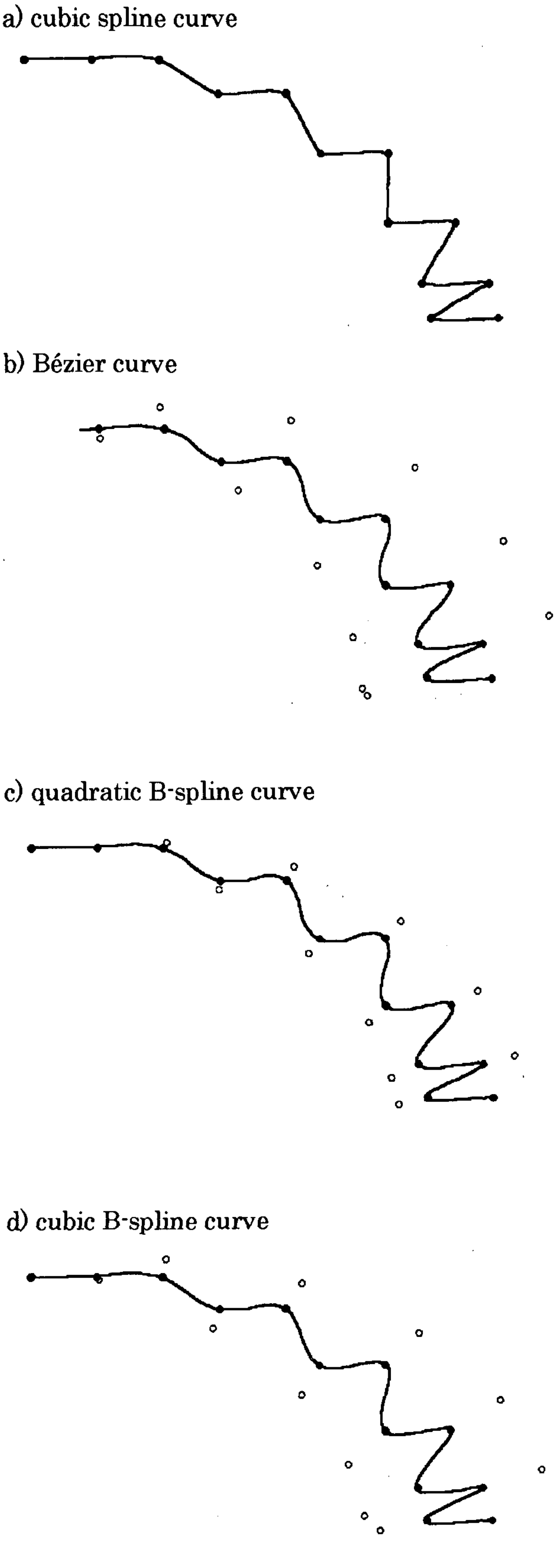

Figure 3 Parametric curves [Model-1]

● : passing point, ○:control point,

An interval of the passing points is 100m.

manipulated so that they were located on the curves. The curves thus obtained are shown in Figure 3. Centrifugal acceleration and its rate of change are evaluated for these curves.

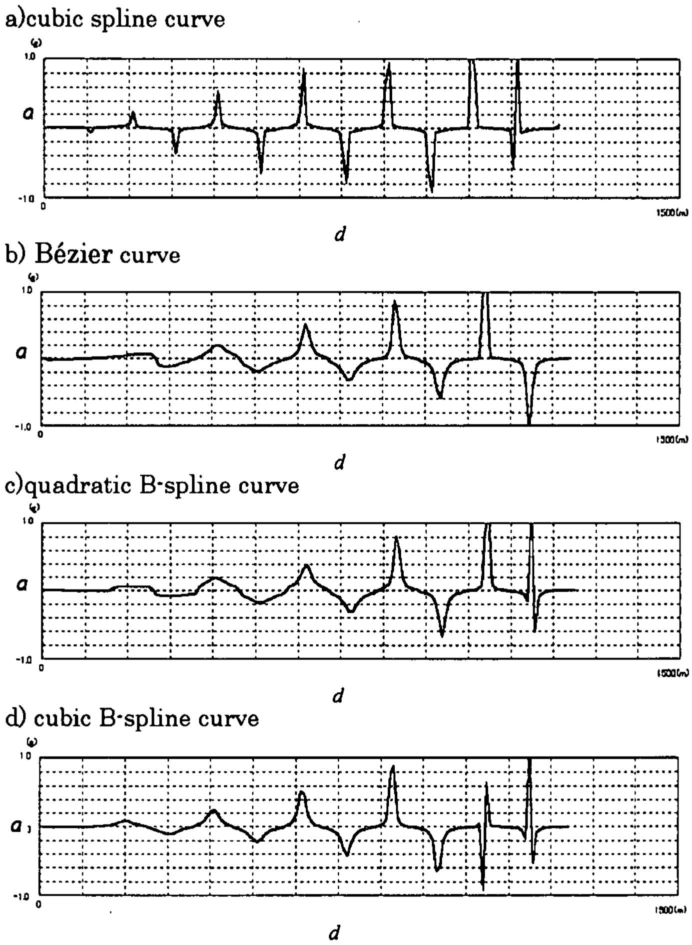

Figure 4 The change in centrifugal acceleration

Centrifugal acceleration is calculated as described below.

A vehicle rounding a curve and its driver are subject to centrifugal acceleration due to centrifugal forces. The magnitude of centrifugal acceleration can be calculated from the radius of the circular curve and the travel speed. Therefore, centrifugal force a (m/s^2) can be expressed as

$$a = \frac{v^2}{r} \quad \text{(Eq. 5)}$$

where v is the travel speed (m/s) and r is the radius of the circular curve.

Three-dimensional alignment is composed of a series of points drawn as a parametric curve and is not defined in terms of the radius of curvature. When the radius of curvature is expressed in terms of angular velocity of circular motion, Eq. (5) can be rewritten as follows:

$$a = v\,\omega \quad \text{(Eq. 6)}$$

where w is the angular velocity (rad/s).

Let P_{n-2}, P_{n-1}, and P_n be a series of three points on a parametric curve, and let us calculate the angle between two straight line segments P_{n-2} - P_{n-1} and P_{n-1} - P_n. Since the spacing between rows of coordinate points of a parametric curve is not constant, if the turning angle when the vehicle travels along Pn-1-Pn is represented by q (rad), and the distance traveled is represented by d (m), then the angular velocity w at point P_n can be calculated by the equation

$$\omega = 2\,v\,sin(\,\theta\,/\,2\,)\,/\,d \quad \text{(Eq. 7)}$$

Thus, centrifugal acceleration an (m/s^2) at point P_n can be calculated by the equation

$$a_n = 2\,v^2\,sin(\,\theta\,/\,2\,)\,/\,d \quad \text{(Eq. 8)}$$

In the above calculation, the design speed of 30 km/h is assumed, and centrifugal acceleration corresponding to the distance traveled in one second is calculated.

Figure 4 shows changes in acceleration for different parametric curves. In the case of the cubic spline curve, acceleration changes abruptly at curve connections, indicating that a cubic spline curve is not suitable for application to highway alignment. In the sections corresponding to the turning angle of 150°, all curves cause high acceleration. In the cases of the spline and B-spline curves, positive-negative oscillation occurs. This is thought to be due to local torsion of the curves. It can be said, therefore, that control point settings involving a turning angle exceeding 120° should be avoided.

In the sections corresponding to turning angles of 0 to 120°, the quadratic B-spline curve shows the minimum value of acceleration. The curve that shows smooth acceleration changes and symmetry of acceleration changes in the vicinity of points of inflection is the cubic B-spline curve.

(2) Effect of the ratio between spacings of passing points on changes in acceleration

The evaluation in Section (1) assumed constancy of distance between passing points. In real-world 3D design, however, the distance between passing points is not necessarily constant. In this section, characteristics of parametric curves are evaluated in cases where the curvature of road alignment and the ratio between distances between passing points are varied, that is, in cases where nonsymmetrical passing points are given.

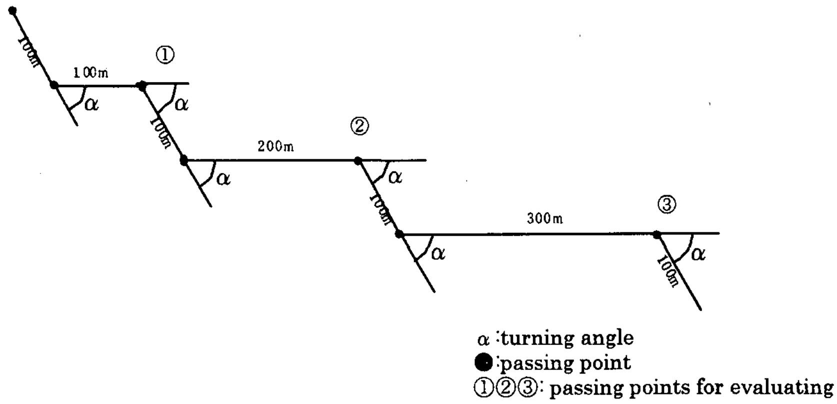

Figure 5 The model for evaluating [Model-2]

For the purposes of evaluation in this study, spacings between passing points are assumed to be 100m:100m, 100m:200m, and 100m:300m, and assumptions are made, as shown in Figure 5. The upper limit of turning angle of 120° is imposed, and evaluation is made at every 30°. Figure 6 shows examples of curves associated with the passing points assumed. The turning angle in the examples shown is 60°.

The maximum values (maxima of absolute values) of centrifugal acceleration and the rate of change of acceleration at the travel speed of 30 km/h in curved sections, including the reverse curve sections at and around passing points (1), (2) and (3) shown in Figure 5, are tabulated in Tables 1 and 2. These values indicate the following:

1) In the case of the cubic spline curve, both centrifugal acceleration and the rate of change of acceleration take maximum values in almost all sections.
2) At the turning angle of 30°, the quadratic B-spline curve and the Bézier curve show the minimum of the maximum centrifugal acceleration in the 100m:100m and 100m:200m cases; the cubic B-spline curve shows the minimum of the maximum centrifugal acceleration in the 100m:300m case.
3) At the turning angles of 60, 90 and 120°, the quadratic B-spline curve shows the minimum of the maximum centrifugal acceleration in the 100m:100m and 100m:200m cases, followed by the Bézier curve and the cubic B-spline curve. In the 100m:300m case, the cubic B-spline curve shows the minimum value, followed by the quadratic B-spline and the Bézier curve.
4) The minimum of the maximum rate of change of centrifugal acceleration at the turning angle of 60° is shown by the Bézier curve in the 100m:100m case and by the cubic B-spline curve in the 100m:200m case. In all other cases, the quadratic B-spline curve shows the minimum value, followed by the Bézier curve and the cubic B-spline curve.

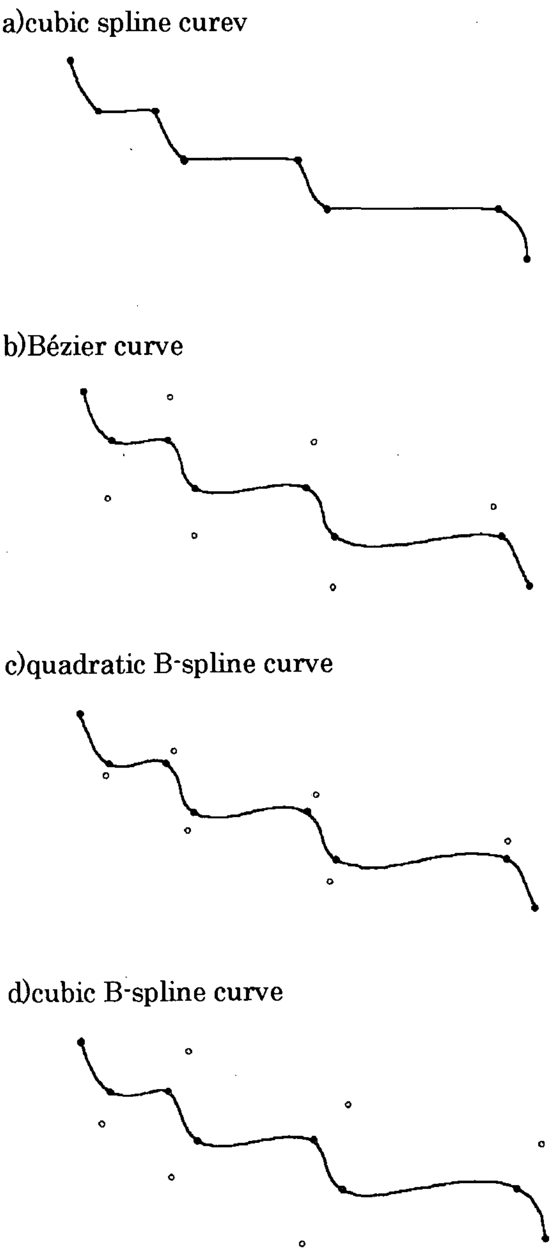

Figure 6 Paramtric curves [Model-2]

5) At the 100m:300m ratio, the cubic B-spline curve shows the minimum value in all cases, followed by the quadratic B-spline curve and the Bézier curve.

Tables 1 and 2 can be divided into two areas: (1) the area corresponding to the turning angle of 30° and the 100m:300m case and (2) the area corresponding to the turning angles of 60, 90 and 120° and the 100m:100m and 100m:200m cases. For the purposes of this study, the former is referred to as Area A (areas in Table 1 and Table 2 encircled by a thick line; high-standard area), and the latter as Area B (areas in Table 1 and Table 2 encircled by a broken line; low-standard area).

In Area A, smaller values of centrifugal acceleration are shown by the quadratic B-spline curve and the Bézier curve at the turning angle of 30° in the 100m:100m and 100m:200m cases. In the other cases, the cubic B-spline curve shows the minimum value. As for the rate of change of centrifugal acceleration, the cubic B-spline curve shows the minimum value.

High-standard highway alignment requires smoother transition and lower rates of change of centrifugal acceleration. In this sense, the cubic B-spline curve can be considered the best alignment in Area A.

Area B is characterized by large turning angles and low control point spacing ratios. In Area B, in almost all cases the quadratic B-spline curve shows the minimum value for both centrifugal acceleration and its rate of change, followed by the Bézier curve and the cubic B-spline. It can be said, therefore, that the quadratic B-spline curve is most suitable for Area B applications.

From the above observations, the following conclusion can be drawn:

The cubic B-spline curve makes for the smoothest transition for roads with a turning angle smaller than 60°, while the quadratic B-spline curve enables the drawing of road alignments with low acceleration and small rates of its change at larger turning angles. The Bézier curve is somewhere between the cubic and quadratic B-spline curves. The cubic spline curve is considered to be not suitable for application to road alignment.

Application of parametric curves to 3D highway alignment requires the evaluation of their goodness of fit for use as vertical alignment curves. Since the x-, y- and z-coordinates on a parametric curve can be found independently, vertical alignments are comparable in curve performance to horizontal alignments. Judging from the rates of change of centrifugal acceleration at the low turning angle (30°) case shown in Table 2, therefore, either the quadratic or the cubic B-spline curve is considered suitable for application. Although vertical alignment requires consideration of restrictions imposed by sight distance, this problem can be addressed by evaluating it on the design system as a problem associated with the locations of control points for parametric curves[6)].

Table 1 The maximum value of the centrifugal accelerations

degree	kind of curve	maximum value of the centrifugal acceleralation		
		100m:100m	100m:200m	100m:300m
30°	cubic spline	0.32	0.23	0.15
	Bézier	0.10	0.08	0.11
	quadratic B-spline	0.07	0.08	0.10
	cubic B-spline	0.12	0.09	0.03
60°	cubic spline	0.64	0.65	0.24
	Bézier	0.23	0.16	0.24
	quadratic B-spline	0.19	0.14	0.15
	cubic B-spline	0.33	0.21	0.09
90°	cubic spline	0.65	1.01	0.42
	Bézier	0.51	0.30	0.39
	quadratic B-spline	0.37	0.28	0.24
	cubic B-spline	0.56	0.44	0.17
120°	cubic spline	1.16	1.08	0.71
	Bézier	0.99	0.68	0.74
	quadratic B-spline	0.79	0.62	0.51
	cubic B-spline	1.02	0.87	0.38

bold-line: Area A dashed-line: Area B

Table 2 The maximum value of rates of change rate of centrifugal acceleration

degree	kind of curve	maximu value of the rate of change of centrifugal accelaration		
		100m:100m	100m:200m	100m:300m
30°	cubic spline	1.98	1.64	1.21
	Bézier	1.22	1.04	1.04
	quadratic B-spline	1.07	0.99	0.67
	cubic B-spline	1.22	0.23	0.04
60°	cubic spline	5.78	6.18	1.72
	Bézier	0.60	1.20	1.90
	quadratic B-spline	0.77	1.24	0.85
	cubic B-spline	1.27	0.58	0.09
90°	cubic spline	5.94	8.94	2.88
	Bézier	2.05	0.84	2.22
	quadratic B-spline	1.23	0.67	0.95
	cubic B-spline	3.10	1.59	0.31
120°	cubic spline	15.27	13.12	6.82
	Bézier	6.18	4.16	3.92
	quadratic B-spline	3.66	2.95	1.94
	cubic B-spline	7.72	5.29	1.35

bold-line: Area A dashed-line: Area B

From these results, it can be concluded that the quadratic and cubic B-spline curves are suitable for application to highway alignment.

4. APPLICABILITY OF PARAMETRIC CURVES

The current Japanese design standards concerning curved highway sections are based on centrifugal acceleration and its rate of change exerted upon the driver, and vertical alignment is specified in terms of highway gradient, sight distance and other factors. On the design system, therefore, the designer proceeds with design while making goodness-of-fit evaluation for given parametric curves by calculating centrifugal acceleration and its rate of change. If evaluation results are questionable, the designer can move control points so that the required degree of fit can be achieved. Since three-dimensional highway alignments do not depend on

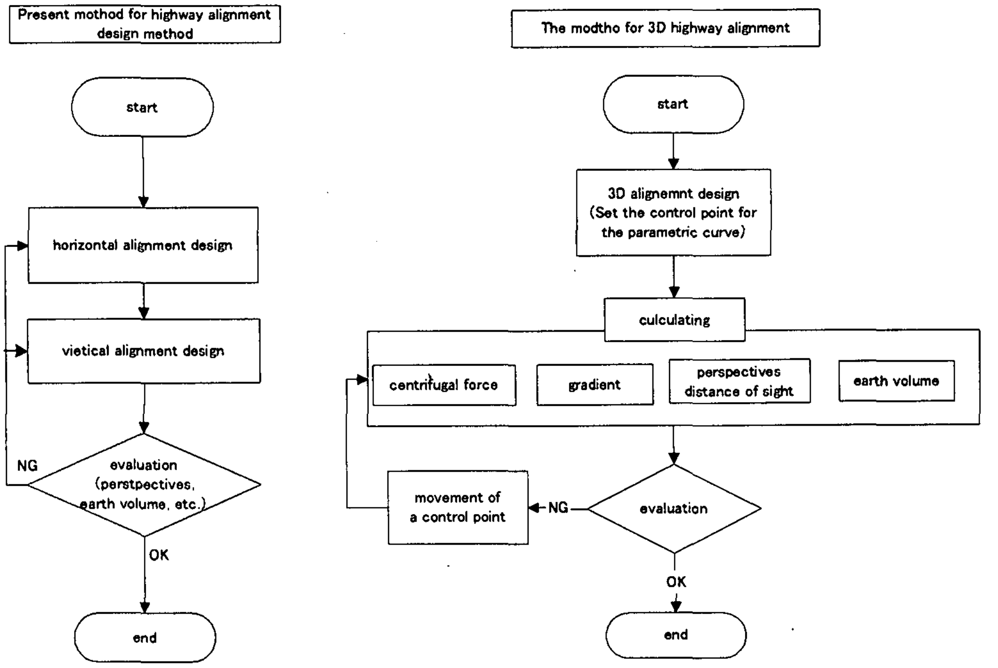

Figure.7 Comparing the highway alignment design methods

drawings, the angle of view and the method of projection can be changed freely. Linking 3D alignment data to topographic information enables simultaneous earthwork computation. Thus, application of parametric curves makes it possible to establish an evaluation-to-design feedback system in which the designer can proceed with design while making computer-assisted evaluation, as necessary, instead of combining linear elements conforming t o de sign standards as in the conventional alignment design process.

Figure 7 compares the conventional alignment design process and an alignment design process using 3D highway alignments.

Figure 8 shows an example of evaluation of centrifugal acceleration and the rate of change of centrifugal acceleration for a 3D highway alignment (using a quadratic B-spline curve) on a 3D route planning system3) developed by the author. The design speed for the highway under consideration here is 40 km/h; the allowable m aximum c entrifugal a cceleration, 0.15*g*; and the allowable maximum rate of change of centrifugal acceleration, 0.75 m/s^2 [5]. For the alignment assumed, the route planning system is capable of displaying centrifugal acceleration, highway gradient and resultant acceleration corresponding to the design speed in real time. The original highway alignment (Figure 8(a)) includes two sections in which the upper limits mentioned above are exceeded, and two sections that show values close to those limits. Control points in the problem sections, therefore, were fine-tuned, and the highway alignment that falls within the allowable limits for the design speed, as shown in Figure 8(b), was obtained.

5. CONCLUDING REMARKS

In this study, applicability of parametric curves to highway alignment has been investigated by evaluating riding qualities achievable by use of cubic spline curves, Bézier curves and B-spline curves. Through the evaluation of riding qualities from the viewpoint of centrifugal acceleration and its rate of change, the study showed that a cubic B-spline curve is suitable for application to highway alignments with a relatively large radius of curvature, and that a quadratic B-spline curve is suitable f or application to alignments with a relatively small radius of curvature.

Parametric curves make it possible to draw 3D free curves easily on a computer. Application of pa rametric curves to highway a lignment, i f m ade po ssible, c an b e expected to help establish a more sophisticated computer-assisted design method. Many studies are now being devoted to parametric curves.

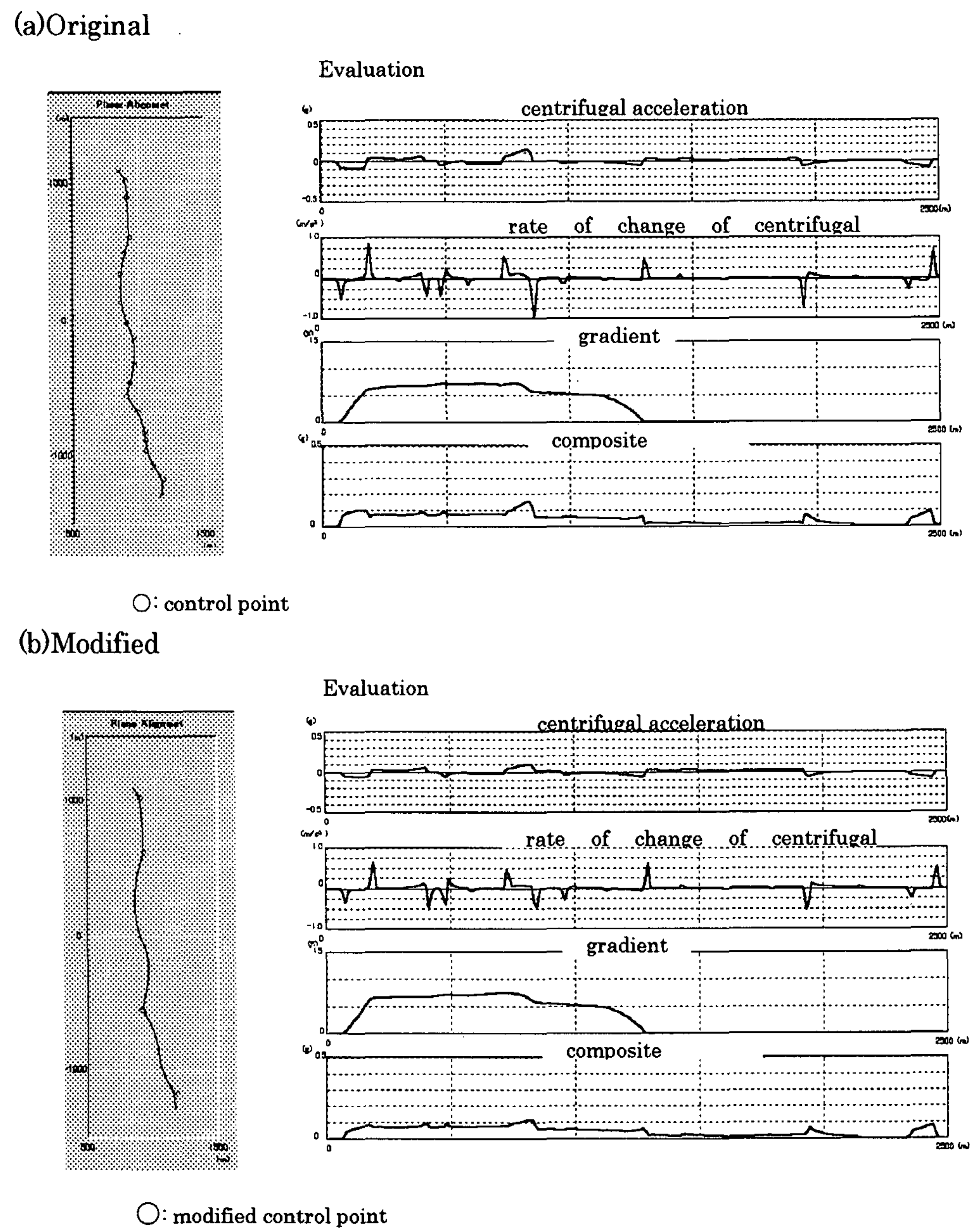

Figure 8 An application of the parametric curve to highway alignment design

The next step to be taken will be to derive a method for defining three-dimensional alignments more suitable for application to highway alignment including B-spline curves, which have been found to be useful through the present study, and to evaluate the practicality of the method thus developed.

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