Weak Imposition of Slip Boundary Condition in the Stabilized Finite Element Formulation of Shallow Water Equations

National Institute of Technology, Hachinohe College Student member ○ Joseph Galbreath
National Institute of Technology, Hachinohe College Member Akira Maruoka

1. Introduction

When simulating shore lines and rivers with a relatively coarse mesh, since the boundary is curved, the slip boundary condition is often used. The slip boundary condition is relatively easy to use in an explicit scheme, but in an implicit scheme, such as the stabilized finite element formulation, it is more difficult. The objective of this study was to apply a weak imposition of the slip boundary condition on curved boundaries for shallow water flow utilizing Nitsche's method, which was proposed by José *et al* (2014) for Stokes flow[1]. We researched an SUPG finite element formulation of the shallow water equations. Specifically, we looked into the weak imposition of the slip boundary condition in a curved boundary and the effects on it when the penalty parameter for stabilization was changed. For the simulations in this paper we used a theoretical river modeled after a sine curve and with similar slope and Manning's roughness coefficient as those of a real river.

2. Methods

In the shallow water equations we simplify the Navier-Stokes Equations to use depth averaged velocity. The general form of the shallow water equations can be expressed as:

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}_i \frac{\partial \mathbf{U}}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\mathbf{K}_{ij}\frac{\partial \mathbf{U}}{\partial x_j}\right) - \mathbf{R} = \mathbf{0}, \tag{1}$$

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ gH - u_1^2 & 2u_1 & 0 \\ -u_1u_2 & u_2 & u_1 \end{bmatrix},\ \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 1 \\ -u_1u_2 & u_2 & u_1 \\ gH - u_2^2 & 0 & 2u_2 \end{bmatrix}, \tag{2}$$

$$\mathbf{K}_{11} = \begin{bmatrix} 0 & 0 & 0 \\ -2\nu u_1 & 2\nu & 0 \\ -\nu u_2 & 0 & \nu \end{bmatrix},\quad \mathbf{K}_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\nu u_1 & \nu & 0 \end{bmatrix},$$
$$\mathbf{K}_{21} = \begin{bmatrix} 0 & 0 & 0 \\ -\nu u_2 & 0 & \nu \\ 0 & 0 & 0 \end{bmatrix},\quad \mathbf{K}_{22} = \begin{bmatrix} 0 & 0 & 0 \\ -\nu u_1 & \nu & 0 \\ -2\nu u_2 & 0 & 2\nu \end{bmatrix}, \tag{3}$$

$$\mathbf{R} = \begin{bmatrix} 0 \\ -gH\frac{\partial z}{\partial x_1} - \frac{gn^2\sqrt{u_1^2+u_2^2}}{H^{\frac{1}{3}}}u_1 \\ -gH\frac{\partial z}{\partial x_2} - \frac{gn^2\sqrt{u_1^2+u_2^2}}{H^{\frac{1}{3}}}u_2 \end{bmatrix}, \tag{4}$$

where $\mathbf{U} = (H,\ u_1H,\ u_2H)^T = (U_1,\ U_2,\ U_3)^T$ is the vector of the conservation variables, n is Manning's roughness coefficient, g is gravitational acceleration, ν is horizontal kinematic viscosity and z is the height of the bottom surface from the mean water level. H is the total water depth, and $\mathbf{u} = (u_1, u_2)^T$ is the depth averaged velocity. $\mathbf{A}_i$ and $\mathbf{K}_{ij}$ are derived from the Euler and viscous flux vectors and $\mathbf{R}$ represents all other components that might enter the equations, including external forces.

The boundary conditions used in our simulations can be defined as:

$$\mathbf{U} = \mathbf{G} \qquad \text{on } \Gamma_G, \tag{5}$$

$$\mathbf{K}_{ij}\frac{\partial \mathbf{U}}{\partial x_j}n_i = \mathbf{H} \qquad \text{on } \Gamma_H, \tag{6}$$

$$\mathbf{n}_+ \cdot \mathbf{U} = G_S \ \text{ and } \ \mathbf{t}_+ \cdot \mathbf{K}_{ij}\frac{\partial \mathbf{U}}{\partial x_j}n_i = H_S \qquad \text{on } \Gamma_S. \tag{7}$$

Equation (5) and equation (6) are the Dirichlet and Neumann boundary conditions respectively. Equation (7) is the slip boundary condition. Here $\mathbf{n} = (n_1, n_2)^T$ is a normal vector to the boundary and $\mathbf{t} = (t_1, t_2)^T = (-n_2, n_1)^T$ is a tangent vector. In equation (7) we define $\mathbf{n}_+$ and $\mathbf{t}_+$ as such, $\mathbf{n}_+ = (0, n_1, n_2)^T$, $\mathbf{t}_+ = (0, t_1, t_2)^T$. Specifically, we use the free slip boundary condition where $G_S = 0$ and $H_S = 0$.

The semi-discrete SUPG formulation of equation (1) with the Nitsche's type weak imposition of the slip boundary condition can be written as:

$$\begin{aligned}
&\int_\Omega \mathbf{W}^h \cdot \left(\frac{\partial \mathbf{U}^h}{\partial t} + \mathbf{A}_i^h \frac{\partial \mathbf{U}^h}{\partial x_i} - \mathbf{R}^h\right) d\Omega \\
&+ \int_\Omega \frac{\partial \mathbf{W}^h}{\partial x_i} \cdot \mathbf{K}_{ij}^h \frac{\partial \mathbf{U}^h}{\partial x_j} d\Omega - \int_{\Gamma_H} \mathbf{W}^h \cdot \mathbf{H}^h d\Gamma \\
&+ \sum_{e=1}^{n_{\text{el}}} \int_{\Omega^e} \tau_{\text{SUPG}} \mathbf{A}_k^h \frac{\partial \mathbf{W}^h}{\partial x_k} \cdot \mathbf{r}\left(\mathbf{U}^h\right) d\Omega \\
&+ \sum_{e=1}^{n_{\text{el}}} \int_{\Omega^e} \nu_{\text{SHOC}} \frac{\partial \mathbf{W}^h}{\partial x_i} \cdot \frac{\partial \mathbf{U}^h}{\partial x_i} d\Omega \\
&- \int_{\Gamma_S} \left(\mathbf{n}_+ \cdot \mathbf{W}^h\right)\left(\mathbf{n}_+ \cdot \mathbf{K}_{ij}^h \frac{\partial \mathbf{U}^h}{\partial x_j} n_i\right) d\Gamma - \int_{\Gamma_S} \left(\mathbf{t}_+ \cdot \mathbf{W}^h\right) H_S \, d\Gamma \\
&- \int_{\Gamma_S} \left(\mathbf{n}_+ \cdot \mathbf{K}_{ij}^h \frac{\partial \mathbf{W}^h}{\partial x_j} n_i\right)\left(\mathbf{n}_+ \cdot \mathbf{U}^h - G_S\right) d\Gamma \\
&+ \sum_{b=1}^{n_{\text{sb}}} \frac{C_{\text{pen}}}{h_S^b} \int_{\Gamma_S^b} \nu \left(\mathbf{n}_+ \cdot \mathbf{W}^h\right)\left(\mathbf{n}_+ \cdot \mathbf{U}^h - G_S\right) d\Gamma = 0,
\end{aligned} \tag{8}$$

where

$$\mathbf{r}(\mathbf{U}^h) = \frac{\partial \mathbf{U}^h}{\partial t} + \mathbf{A}_i^h \frac{\partial \mathbf{U}^h}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\mathbf{K}_{ij}^h \frac{\partial \mathbf{U}^h}{\partial x_j}\right) - \mathbf{R}^h. \tag{9}$$

In the above equation we used the stabilization parameter τ_{SUPG} and shock-capturing parameter ν_{SHOC} proposed by Takase *et al* (2010)[2] where n_{sb} is the number of slip boundary segments and h_S^b is the length of slip boundary segments.

Key Words: *weak imposed boundary condition, Nitsche's method, shallow water equations, stabilized finite element formulation*
Contact Information: 16–1 Uwanotai, Tamonoki, Hachinohe city, Aomori, 039–1192. TEL:0178–27–7304, FAX:0178–27–7316.

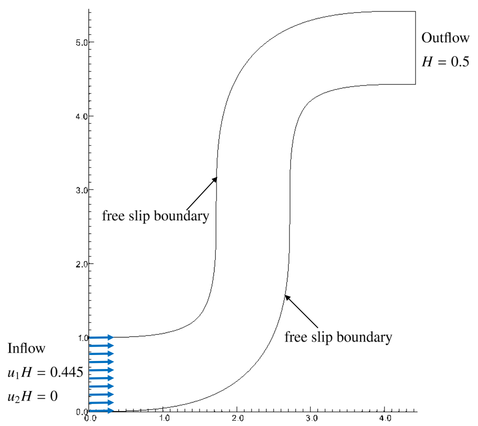


Fig. 1 Domain and boundary conditions

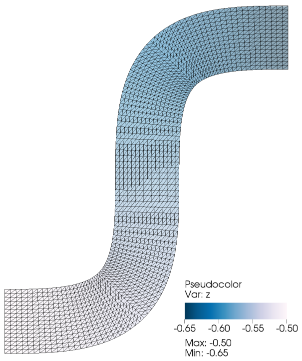


Fig. 2 Simulation target mesh

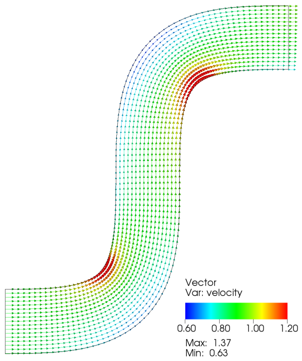


Fig. 3 Depth averaged velocity

C_{pen} is a constant penalty parameter. The last four terms of the equation are derived by a weak imposition of Nitsche's method.

Equation (8) is discretized through integrating by time with the generalized-α method, which was first proposed in Jansen *et al* (2000) and developed for the Navier-Stokes equations of incompressible flows[3].

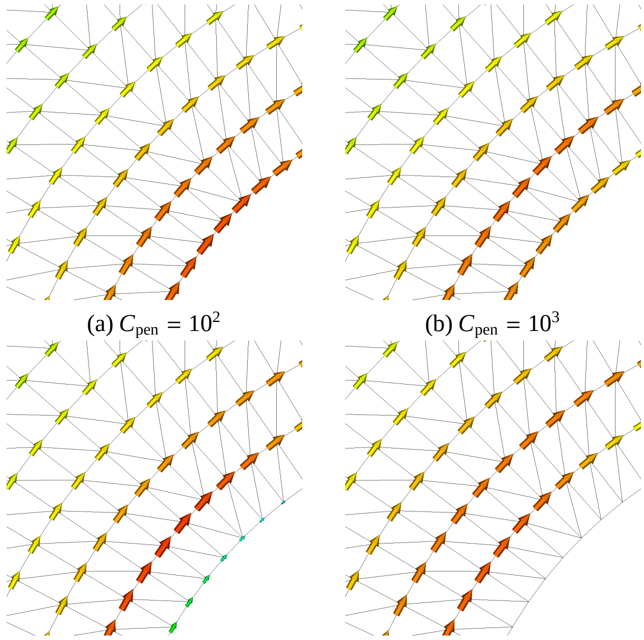
We use Gaussian quadrature to calculate integrals in our program, which is the most common numerical integration method in finite elements[4]. We use 3 quadrature points for elements and 2 quadrature points for boundary segments.

3. Result

In this paper we simulated the flow of a theoretical river and observed how the flow changed when the penalty parameter C_{pen} was changed, as well as demonstrated the effectiveness of the slip boundary condition for a rough mesh in our results.

In the simulations, we used a slope of 2%, $n = 0.1$ and $\nu = 0.001$. The domain and boundary conditions are shown in figure 1. The combination of inflow and outflow conditions corresponds to the uniform flow of a straight river. The simulation target mesh is shown in figure 2. Figure 3 shows the depth averaged velocity vectors of the entire domain when $C_{pen} = 10^2$. The results for each value of C_{pen} and the corresponding results near the boundary are shown in figure 4.

In the subsequent figure you see that the depth averaged velocity vector is tangent to the edge of the river. This shows that the slip boundary conditions are effective in the simulation. As the penalty parameter gets smaller, the effect of stabilization on the equation also gets smaller giving more accurate results. We were able to reduce the penalty parameter to 10^2 before the resulting calculations diverged.



(a) $C_{pen} = 10^2$ (b) $C_{pen} = 10^3$ (c) $C_{pen} = 10^4$ (d) $C_{pen} = 10^5$

Fig. 4 Depth averaged velocity vectors

4. Conclusion

The data presented in this study significantly improves our understanding of slip boundary conditions and how they relate to the SUPG formulation of the shallow water equations. We demonstrated that the slip boundary conditions were effective in the results of our simulations, and that we were able to obtain results that converged even when using a very small stabilizing penalty parameter. This made it possible to achieve more accurate calculations, better representing theoretical values.

As we gather a mathematical model, we can move forward with more studies to broaden our current simulation models. Further research should include comparing our numerical simulation results with that of naturally occurring rivers and improving our mathematical model to better represent naturally occurring phenomena.

References

1) José M. Urquiza, André Garon and Marie-Isabelle Farinas, Weak imposition of the slip boundary condition on curved boundaries for Stokes flow, Journal of Computational Physics, Vol. 256, pp. 748-767, 2014.
2) Shinsuke Takase, Kazuo Kashiyama, Seizo Tanaka and Tayfun E. Tezduyar, Space-time SUPG formulation of the shallow-water equations, International Journal for Numerical Methods in Fluids, Vol. 64, pp. 1379-1394, 2010.
3) Jansen K.E., Whiting C.H. and Hulbert G.M., A generalized-α method for integrating the filtered Navier-Stokes equations with a stabilized finite element method, Computer Methods in Applied Mechanics and Engineering, pp. 305-319, 2000.
4) Yuri Bazilevs, Kenji Takizawa and Tayfun E. Tezduyar, Computational fluid-structure interaction: methods and applications, John Wiley & Sons Ltd, pp. 55-58, 95-98, 2013.