Journal of Structural Engineering Vol.53A (March 2007) JSCE

Computation of SIFs for branched crack problems by scaled boundary finite element method

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In this paper, extensions of application of newly developed scaled boundary finite element method to branched crack problems are presented. In this method, stress singularity at a crack tip can be derived analytically from its stress solution. Using SBFEM's semi-analytical properties authors' have presented a direct and simple SBFEM formulation to compute fracture parameters. Stress intensity factors of branched crack are computed extending the SBFEM formulation proposed by authors. Numerical examples for a range of crack sizes are analysed to examine the effectiveness of the proposed method. In addition, the application of SBFEM is extended to crack propagation simulation in 2D linear elastic problem under mixed mode condition. Since only the domain boundary is required to discretize like in boundary element method and also it has its own unique property that the side-face boundaries and near crack tip are not necessary to discretize, the burdensome remeshing required in FEM and BEM is minimized. A mixed mode problem for crack propagation analysis is simulated. The computed SIFs and crack propagation trajectories are in remarkable agreement with available values in the literatures.

Key Words: SIFs, branched crack problem, scaled boundary finite element method, crack propagation simulation

1. Introduction

Branched cracks are very common forms of cracks in engineering structures. These cracks can adversely affect the structural integrity of components and shorten their service life. It is important therefore for engineers to be able to determine the stress distribution in the region of cracks, as failure to make proper predictions regarding the consequences of a crack could lead to catastrophic failures. The stress distribution in the vicinity of a crack depends on the value of the stress intensity factor at the crack-tip. Therefore, fast and accurate calculations of stress intensity factors (SIFs) are needed for the simulation of crack evolution and simulation based life-cycle design of engineering structures.

Over the last few decades, many researchers [1-13] have studied the problem of computing stress field around the branched cracks by analytical and various numerical techniques. Among the numerical techniques, finite element method (FEM), boundary element method (BEM), dual boundary finite method (DFEM), displacement discontinuity BEM, extended finite element method (XFEM), meshless method, dislocation method and body force method (BFM) are popular to compute SIFs of branched cracks. Even though much achievement has been made in crack modeling techniques, both simple and very accurate crack modeling techniques still need to be developed, particularly for branched crack and crack propagation problems [13].

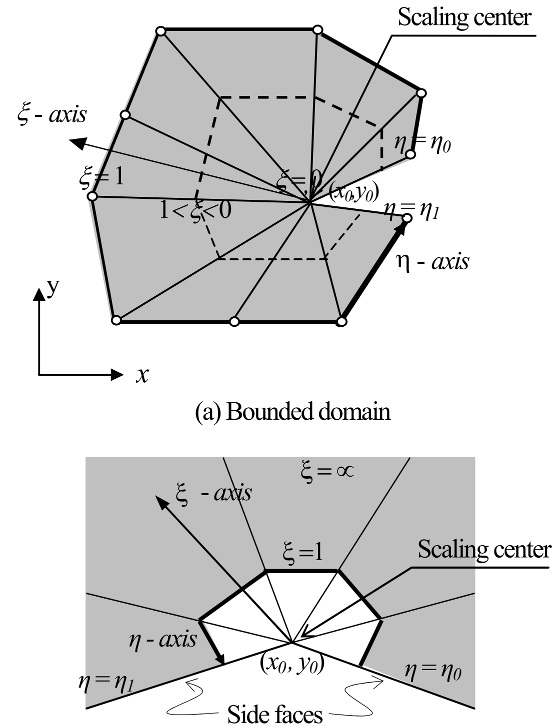
A recently developed scaled boundary finite element method (SBFEM) [14], which attempts to combine the advantages of FEM and BEM, i.e., it discretises boundaries as BEM but it does not require fundamental solutions as in the FEM, is emerging as an alternative numerical method for crack analysis. This method has ability to semi-analytically compute stress and displacement fields of singularities region at the 'scaling center' of the bounded domain when this center lies on the point of interested [14]**.** By using the semi-analytical properties, authors [15 – 17] and other researchers [15-21] have employed the SBFEM to evaluate the fracture parameters: SIFs, the *T*-stress and higher order terms of the crack -tip stress field and demonstrated the effectiveness of the SBFEM in straight crack problems. However, as per authors' knowledge, none of the previous studies have addressed the branched crack problems and the crack propagation analysis using this method.

The main purpose of this paper is to extend application of the SBFEM for computing the SIFs at the tips of branched cracks problems and crack propagation simulation. In this paper, an efficient SBFEM formulation proposed by authors[15] is extended to evaluate the SIFs of branched crack problems. In addition, a quasi-automatic procedure is presented for crack propagation simulation. Since only the domain boundary is required to discretize like in BEM and also it has a unique property that certain fixed and free (side faces) boundaries are not necessary to discretize[22], the burdensome remeshing required by FEM and BEM is minimized. By semi-analytical computation of stress singularity with high accuracy and simple remeshing requirement for crack propagation, it can be confirmed that the proposed numerical method can be applied to fracture analysis more easily with relatively coarse and simple model than other computational methods. Two branched crack problems - bounded and unbounded domain are simulated to compute SIFs, and a benchmark problem of mixed mode crack problems is considered to simulate crack propagation. The computed SIFs and crack propagation trajectory are in excellent agreement with reference results.

2. Scaled boundary finite element method

The scaled boundary finite element method is a new semi-analytical fundamental solution-less BEM based on FEM[14]. In this method, the partial differential equation of a variety of linear problems is transformed into ordinary differential equations. Then, these ordinary differential equations are solved analytically in radial direction and the coefficients of these equations are determined by the finite element approximation in the circumferential directions. The virtual work derivations of the stress and displacement fields in the method are presented in detail in Ref. [23], and authors are summarized the derivation in Ref. [17] for bounded domain. In this paper, the summary from Ref.[17] is reproduced here with some modifications for unbounded domain for convenience as follows.

In this method, a coordinate system consists of a radial direction (ξ) and a local circumferential direction (η) is used to scale the domain relative to a scaling center. The radial coordinate has a value of zero at the scaling center and a value of one on the domain boundary, which is discretized in a finite element manner. The circumferential coordinate measures the distance anticlockwise around the boundary. A bounded domain and unbounded domain with side faces are as shown in Fig 1(a) and (b) respectively. The bounded domain is described by $\eta_0 \leq \eta \leq \eta_1$ and $0 \leq \xi \leq 1$, while unbounded domain with infinite extent in the ξ direction is described by $\eta_0 \leq \eta \leq \eta_1$ and $1 \leq \xi \leq \infty$. The scaled boundary coordinates are related to Cartesian coordinates by



(a) Bounded domain

(b) Unbounded domain

Fig. 1 Scaled boundary finite element co-ordination system in (a) bounded domain and (b) unbounded domain with side faces.

$$x = x_0 + \xi x(\eta) \tag{1a}$$

$$y = y_0 + \xi y(\eta) \tag{1b}$$

where $x(\eta)$ and $y(\eta)$ are the functions describing the variation of the boundary in x and y directions as functions of η.

The basic assumption of the SBFEM is that the displacements at any point in the domain defined by scaled boundary coordinates (ξ, η) can be expressed in the form

$$\{u(\xi,\eta)\} = \sum_{i=1}^{n} N_i(\eta)\{u_i(\xi)\} = [N(\eta)]\{u(\xi)\} \tag{2}$$

where $N(\eta)$ is the shape function in the circumferential direction, which are constructed as in FEM. $u(\xi)$ defines the displacements along the radial lines. The key relations defining in SBFEM for plane problems are as follows. (See Ref. [14] for details of SBFEM).

The governing differential equation (of the static equilibrium) in the absence of body load is

$$[L]^T\{\sigma(x,y)\} = 0 \tag{3}$$

where $[L]$ is the standard linear operator in terms of Cartesian coordinate and $\{\sigma(x,y)\}=[\sigma_x, \sigma_y, \tau_{xy}]^T$ is a vector of the stresses.

The linear operator to scaled boundary coordinate system using special techniques (See for details in Ref[23]) is as follows.

$$[L] = [b^1(\eta)]\frac{\partial}{\partial \xi} + \frac{1}{\xi}[b^2(\eta)]\frac{\partial}{\partial \eta} \tag{4}$$

and the approximate stresses in the coordinate ξ, η from Eqs. (2)

to (4) leads to

$$\{\sigma(\xi,\eta)\} = [D][B^1(\eta)]\{u(\xi)\}_{,\xi} + \frac{1}{\xi}[D][B^2(\eta)]\{u(\xi)\} \quad (5)$$

where

$$[B^1(\eta)] = [b^1(\eta)][N(\eta)]$$
$$[B^2(\eta)] = [b^2(\eta)][N(\eta)]_{,\eta} \quad (6)$$

The virtual work statement is applied to introduce the equilibrium. Performing integrals over the domain using Green's Theorem and then a series of mathematical manipulations, the virtual work statement is satisfied for all virtual displacements $\{\delta u(\xi)\}$ when the following conditions are simultaneously satisfied [23).

$$[E^0]\xi^2\{u(\xi)\}_{,\xi\xi} + [[E^0]+[E^1]^T - [E^1]]\xi\{u(\xi)\}_{,\xi} - [E^2]\{u(\xi)\} = 0 \quad (7)$$

where the coefficient matrices

$$[E^0] = \int_{-1}^{1}[B^1]^T[D][B^1]|J|\,d\eta \quad (8a)$$

$$[E^1] = \int_{-1}^{1}[B^2]^T[D][B^1]|J|\,d\eta \quad (8b)$$

$$[E^2] = \int_{-1}^{1}[B^2]^T[D][B^2]|J|\,d\eta \quad (8c)$$

To perform the integration for unbounded or semi-infinite domain cases, the boundary is traversed in the opposite direction than for bounded domain case. The scaled boundary finite element equation in displacement, i.e. Eq. (7), is same as for both cases.

By inspection, the solution to the set of Euler-Cauchy differential equation represented by Eq. (7) must be of the form

$$\{u(\xi)\} = c_1\xi^{-\lambda_1}\phi_1 + c_2\xi^{-\lambda_2}\phi_2 + c_3\xi^{-\lambda_3}\phi_3 + \;...... \quad (9)$$

where the exponents λ_i and vectors $\{\phi_i\}$ are interpreted as a radial scaling factor and a displacement modes shapes. The integration constants c_i represent the contribution of each mode to the solution, and are dependent on the boundary conditions.

The displacements for each mode from Eq. (9) can be written as

$$\{u(\xi,\eta)\} = \xi^{-\lambda}\{\phi\} \quad (10)$$

Now substituting Eq. (10) and its derivations into Eq. (7) and then simplifying yields the quadratic eigenproblem.

$$[\lambda^2[E^0] - \lambda[[E^1]^T - [E^1]] - [E^2]]\{\phi\} = \{0\} \quad (11)$$

The solution of the eigenproblem is seen to yield a set of modes that span the solution spaces of both the bounded and unbounded domain simultaneously.

For bounded domain problems, only n modes with negative real component of λ lead to finite displacements at scaling center and for unbounded domain problems those mode with non-negative real component of λ are chosen to enforce finite displacements at infinity. This subset of n nodes is denoted by $[\Phi_1]$ and $[\Phi_2]$ for bounded and unbounded problems respectively.

For any set of boundary node displacements, u, the integration constants are

$$\{c\} = [\Phi_i]^{-1}\{u\} \quad (12)$$

The displacement fields and the stress field can be obtained using

$$\{u(\xi,\eta)\} = [N(\eta)]\sum_{i=1}^{n} c_i\xi^{-\lambda_i}\{\phi_i\} \quad (13)$$

$$\{\sigma(\xi,\eta)\} = [D]\sum_{i=1}^{n}[c_i\xi^{-\lambda_i-1}[-\lambda_i[B^1(\eta)] + [B^2(\eta)]]\{\phi_i\}] \quad (14)$$

Eqs. (13) and (14) are, respectively, the semi-analytical solutions for displacement and stress fields inside the domain.

3. SBFEM for fracture analysis

Computation of two fracture parameters – SIF and T-stress

To compute the fracture parameters i.e. SIFs and T-stress and higher order terms of the crack-tip stress fields, authors have presented two different formulations by comparing the classical linear elastic field solution (Williams' eigenfunction series) in the vicinity of a crack-tip with the scaled boundary finite element stress and displacement field solution at any point ahead of crack-tip in Refs.[15) and 16)] respectively. In these formulations, the so-called 'scaling center' of SBFEM is considered at the crack-tip, as shown in Fig. 2 and the stress/displacement field along the radial direction emanating from the crack-tip where the stress singularity occurs are analytically calculated to approximate the crack-tip along the line of propagation of the crack. According to Ref. [16)], the mixed mode SIFs and T-stress of the stress fields are computed by the following relations.

Stress intensity factor for mode I

$$K_I = c(\hat{\sigma}_{y'y'})\sqrt{2\pi\hat{r}} \quad (15)$$

Stress intensity factor for mode II

$$K_{II} = c(\hat{\sigma}_{x'y'})\sqrt{2\pi\hat{r}} \quad (16)$$

and T-stress

$$T = c(\hat{\sigma}_{x'x'}) \quad (17)$$

where $\hat{\sigma}_{xx}, \hat{\sigma}_{yy}$, and $\hat{\sigma}_{xy}$ are the stress components along the

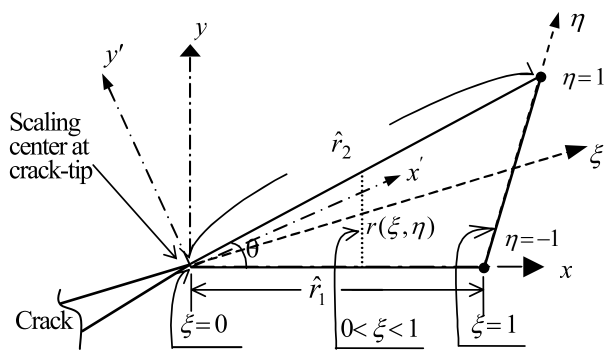


Fig. 2. SBFEM element with different coordinates

axis as shown in Fig. 2, $\hat{r}$ are the radial distances of the boundary nodes from scaling center, and c is the integration constant. In this paper only the basic equations of the proposed formulation are presented. For a more detailed description we refer to Ref[15].

Crack propagation simulation procedure

Several criteria have been proposed to predict local direction of crack propagation. Among them, one of the most commonly used is based on maximum hoop or principal stress at the crack tip [24]. In this study, the maximum principal stress criterion, which predicts the direction of crack growth from the stress state prior to the crack extension, is considered for crack propagation simulation. In this criterion, it is considered that the crack will propagate from its tip in the direction along which the maximum hoop stress $\sigma_{\theta\theta}$ occurs. The hoop (circumferential) stress in the direction of crack propagation is a principal stress. Therefore, the critical angle, θ_0, defining the radial direction of propagation can be determined by setting the shear stress $\sigma_{r\theta}$ to zero.

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta_0} = 0 \tag{18}$$

According to Ref.[25] considering both the singular (SIFs) and constant (T-stress) terms of the stress field near crack-tip on maximum circumferential stress criteria, the direction of crack propagation, θ_0, is computed by solving the following equation.

$$K_I \sin\theta_0 + K_{II}(3\cos\theta_0 - 1) - \frac{16}{3} T \sqrt{2\pi r_c} \sin\frac{\theta_0}{2} \cos\theta_0 = 0 \tag{19}$$

where K_I and K_{II} are mixed mode SIFs, and T is T-stress for any instance during the crack-growth. r_c is an additional length scale representing the fracture process zone size. When the values of K_I , K_{II} and T are known, θ_0 can be easily solved by means of Eqs. (21). Since the fracture process zone size, r_c , is generally assumed to be very small relative to the crack size and specimen dimensions, only SIFs effects are considered to compute the propagation angle θ_0 in this paper.

Considering only the singularity terms (SIFs) the Eq. (19) becomes

$$K_I \sin\theta_0 + K_{II}(3\cos\theta_0 - 1) = 0 \tag{20}$$

The simulation of crack propagation involves a number of successive analyses. Each analysis consists of the following steps.

i. A SBFEM analysis of a crack structure is performed placing a scaling center at crack-tip as shown in Fig. 3 (a). The stress and displacement fields ahead of the crack-tip are computed.
ii. A mixed mode SIFs and T-stress are computed using Eqs. (15) to (17) respectively.
iii. The direction of crack propagation is calculated from Eq. (20).
iv. A virtual increment of crack length (Δa) is defined according to the user's specifications and the location of new crack-tip is determined from the defined incremental length and computed crack propagation direction.
v. By adding two nodes locating on the opposite sides of the crack in the old crack tip, the sub-domain that includes crack-tip is further sub-divided into three sub-domains as shown in Fig. 3 (b), and then discretization of the boundaries and interfaces are updated according to the requirement for accurate computation of stress singularity. Since SBFEM has a unique property that certain fixed and free boundaries passing through scaling centers need not be discretized, the scaling centers are placed in such as way that discretization should be minimized.
vi. Step (i) to step (iv) are repeated to locate new crack-tip. Then the interfaces of sub-domains near the crack are shifted to previous crack-tip, as shown in Fig.3(c), and upgrade the discretization for analysis.
vii. Then step (i) to step (vi) are repeated for further simulation

In this procedure, the simulation of crack propagation is carried out incrementally. In every increment, users should define the incremental crack length to locate the crack-tip. Therefore, the proposed procedure is quasi-automatic.

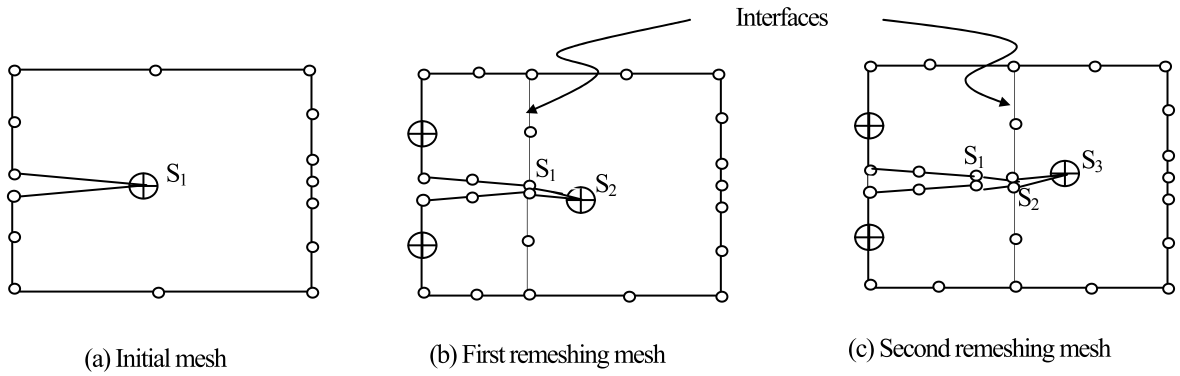


Fig. 3. The proposed remeshing procedure for crack propagation simulation
(S_i & S_{i+1} are the old and new crack-tip)

4. Numerical Examples

In this section, the proposed SBFEM was applied to evaluate SIFs of branched crack and to simulate crack propagation analysis. The following three crack problems were simulated.

i) Symmetrical branched crack in finite plate,

ii) Double symmetrical branched crack in infinite plate,

iii) A mixed mode crack problem

The first and second problems are considered as a bounded and unbounded branched crack problems to compute the SIFs at the crack-tips, and the third problem is considered to examine the effectiveness of the proposed crack propagation procedures.

4.1 Symmetrical branched crack problem in finite plate

A symmetric branched crack in a finite plate subjected to uniaxial tension perpendicular to the main crack was considered first. The schematic diagram of the problem is presented in Fig. 4 (*a*), where *H* and *W* are plate dimensions, *a* and *b* are the crack length of main crack and branched crack respectively and θ is an angle of branched crack with main crack. The problem was analysed assuming $W = 20$, $H = 16$, $\theta = 45^0$ and $b/a = 1$. The applied load was $\sigma_0 = 1$ with its units consistent with that of *E*.

The normalized SIFs at the main crack tip A and at the branched crack tip B are computed by the following equations.

$$F_I^A = K_{IA}/\sigma_0\sqrt{\pi c}$$
$$F_I^B = K_{IB}/\sigma_0\sqrt{\pi c},\ \ and\ \ F_{II}^B = K_{IIB}/\sigma_0\sqrt{\pi c} \qquad (21)$$

Regarding the discretization model, only the half portion was modeled due to symmetriy along horizontal axis by placing two scaling centers at crack tip A and B as shown in Fig 5 (a). The computed normalized SIFs at crack-tip A and B using above mentioned relations (Eq. 21) are compared with the results of the extended finite element method (X-FEM) [11] in Fig. 5 (b). The comparison shows that the SBFEM results are in good agreement with the X-FEM results.

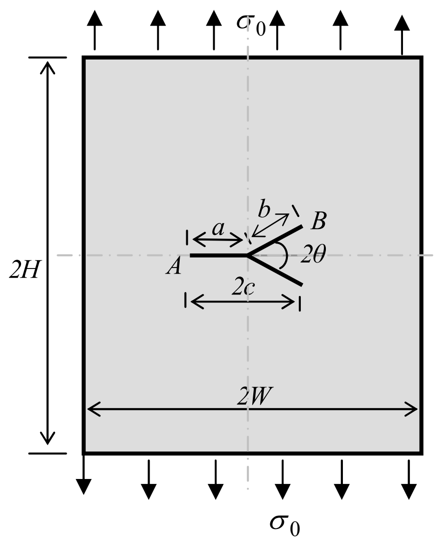


Fig. 4. Schematic diagrams of symmetric branched crack problem

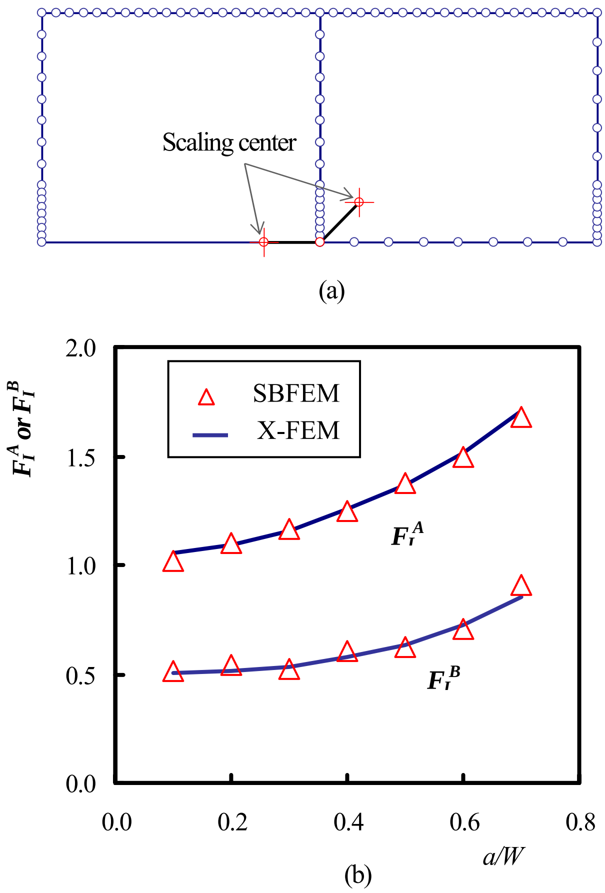


Fig 5. a) An analytical model of symmetric branched crack problem b) Comparison of computed normalized SIFs at crack tips *A* and *B*.

4.2 A double symmetrical branched crack in infinite plate

The second example is considered to be a double symmetrical branched crack (DSBC) problem in an infinite plate with uniform radial tension as shown in Fig 6 (a) where all the symbolic representations are same as in the first example.

An advantage is taken of the biaxial symmetry of the problem, and one quarter of the problem is modeled. The quarter model is divided into three scaled boundary sub-domains - an unbounded sub-domain with scaling center at the origin, o, and two bounded domains with scaling center at the crack-tip and origin. The sub–domains are connected along the discretized boundary ABC as shown in Fig. 6 (b). The SIFs at the tips of branched crack is computed from the bounded domain as in the first problem .

The computed results of SIFs normalized by applied stress, $\sigma_0(\pi c)^{0.5}$ for $\theta = 45^0$ are presented in Fig. 7. These SBFEM results are compared with those of DDBEM presented in Ref. [13].

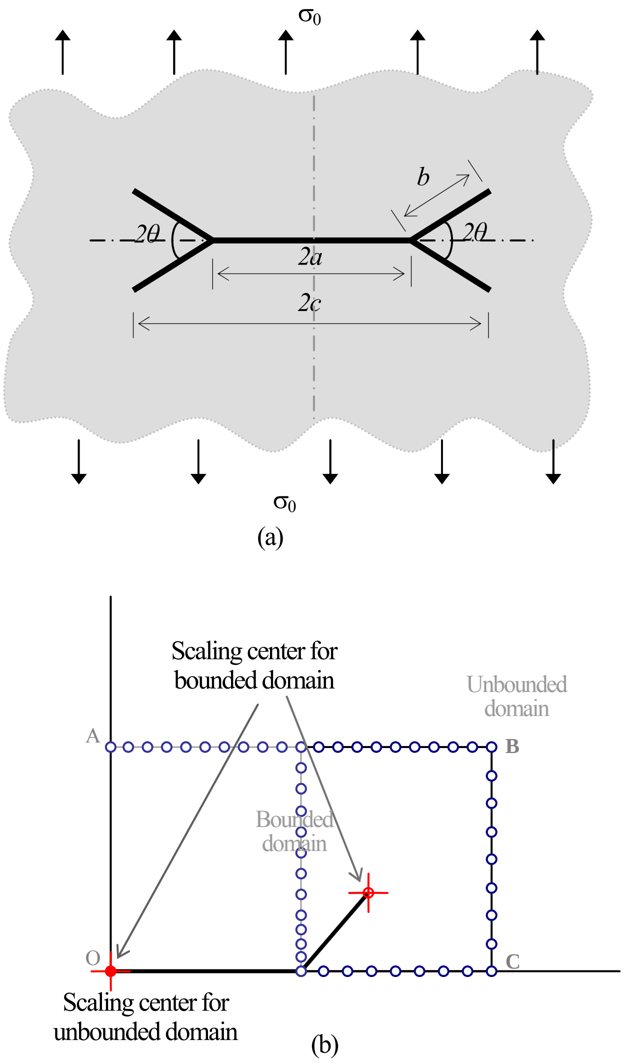


Fig. 6. a) Schematic diagrams and b) analytical model of double symmetric branched crack problem

The comparison clearly shows that the SBFEM results are in good agreement with the reference values with only 3 % deviations.

4.3 A single edged cracked plate

A single edged cracked plate with fixed at the bottom and subjected shear stress of $\tau = 1$ unit on the top was simulated for crack propagation. It is a widely used benchmark example for mixed mode crack problems. The schematic diagram is shown in Fig. 8 (a), and the parameters are: $a = 3.5$, $W = 7$ and $L = 16$. Young's modulus $E = 10^5$ and Poisson's ratio $\nu = 0.25$. All units are consistent with that of E.

For the analysis, whole structure was modeled with five sub-structures as mentioned in section 3. The SBFEM analytical model for the second remeshing (step) is as shown in Fig. 8 (b) where "+" signs are so-called the scaling centers. A scaling center was placed at crack-tip to compute the SIFs at the crack-tip as a necessary condition of SBFEM formulation presented by authors and other four scaling centers were placed at free boundaries as shown in Fig. 8 (b) so as to minimize the discretization boundaries. The computed mixed mode SIFs and T-stress values obtained from Eqs. (15) to (17) for the original case were compared with those from Refs. [27) and 26)] respectively in Table 1.

Table 1. Comparison of mixed mode SIFs and T-stress

SIFs	SBFEM	Reference	Error %
Mode I , K_I	33.95	34.0 [27)]	0.147
Mode II, K_{II}	4.542	4.55 [27]	0.176
T-stress	2.6596	2.6864 [26)]	1.00

Table 1 shows that the computed SIFs are in good agreement with the literature results with less than 1% deviation.

After computing SIFs, crack propagation direction was estimated using Eq. (22) at each increment. The initial propagation angle was -14.75 degrees. To compare the computed crack path results, the incremental crack length for each step of the crack propagation was considered as 4% of the initial crack length as in Ref.[27)]. At each step, the scaling center at the crack-tip and the interfaces of the sub-domains near the crack-tip were shifted to the new location of the crack-tip and the previous crack-tip locations respectively as mentioned in section 3. In the old crack tip, two nodes locating on the opposite sides of the crack were added. The computed crack path is compared with those of meshless method from Ref [27)]. The comparison of the computed results and references results is presented in Fig. 9. It can be seen from Fig. 9 that the computed results are in remarkable agreement with the reference results. Fig. 8 (b) clearly shows that the left, top and bottom boundaries and portion ahead of crack-tip of the problem are not necessary to discretize which leads to minimize the remeshing burdensome of FEM and BEM. Fig. 10 shows the deflected shape of the problem at 18 iterations

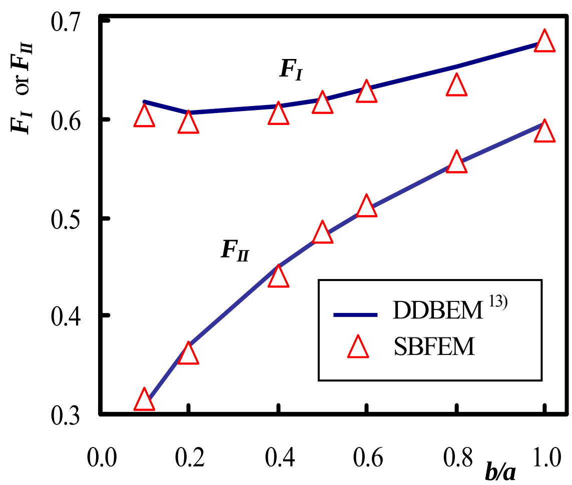


Fig. 7 Comparison of normalized SIFs at tips of branched crack for DSBC

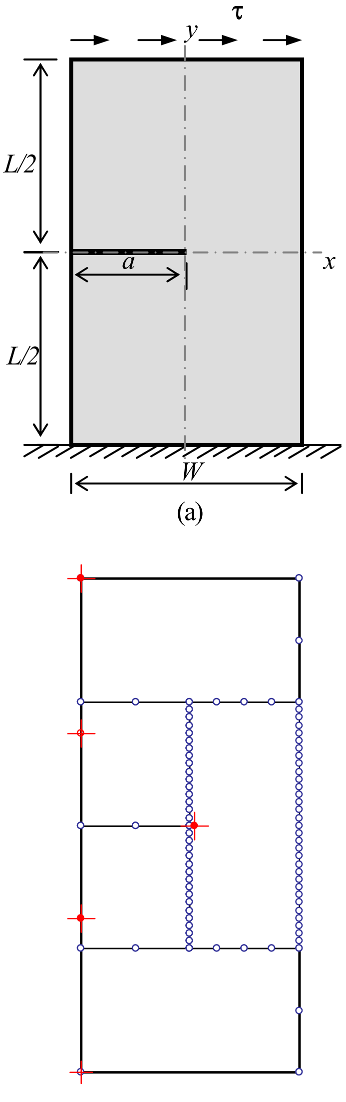


Fig.8. a) Schematic diagrams of SECPS and b) Analytical model for 1st step of remeshing.

5. Conclusion

In this paper, the applications of scaled boundary finite element method have been extended to branched crack problems and crack propagation simulation. The stress singularity i.e. SIFs of the tips of branched cracks has been computed using an SBFEM formulation proposed by authors. The accuracy of these formulations in branched crack is examined with two different examples for a range of crack sizes. It can be seen that the SBFEM computation of SIFs of the tips of the branched crack is simple and very accurate. In addition, using advantages of SBFEM over traditional FEM and BEM such as discretization of only the domain boundaries and interfaces of sub-domains, no need to discretize certain fixed and free boundaries, and freedom in sub-structuring and locations of scaling centers, a quasi-automatic procedure of the proposed method has been presented for crack propagation simulation based on linear elastic fracture mechanics. A mixed mode crack problem has been

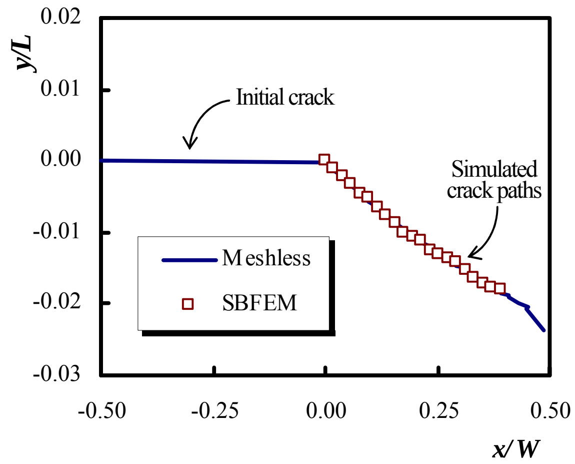


Fig. 9. Comparison of crack propagation trajectory.

examined to demonstrate the effectiveness of the proposed procedures. It can be seen that the numerical results obtained by the SBFEM formulations are in remarkable agreement with the corresponding ones in the literature. Based on the results of the study it can be confirmed that the proposed numerical method can be applied to crack problems more easily with relatively simple remeshing model than other computational methods for crack propagation analysis.

This paper has dealt with a single problem with simple geometry for crack propagation simulation considering only the SIFs effects. It can be applied to simulate problems with more complex geometry and loading cases considering both SIFs and *T*-stress effect. On the other hand, even though SBFEM has many advantages for fracture analysis over the traditional FEM and BEM, it has certain limitations such as requirement the scaling of material variations with relative to the scaling center, difficult to deal patch load within the domain, and considering linear elastic material behaviors for elasto-statics problems. Therefore, coupling of SBFEM with FEM will be more effective for fracture analysis, which is expected to appear in the authors' forthcoming publication

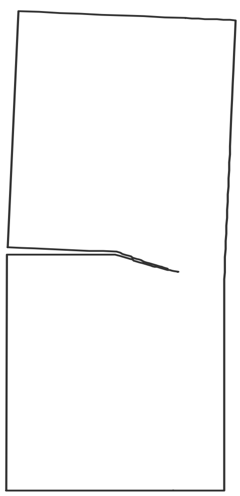


Fig. 10. Deflection shape at 18th step of remeshing

Acknowledgements

The authors would like to express their cordial thanks to Prof. Andrew J. Deeks of University of Western Australia, Australia, for providing beneficial guidelines for basic understanding the scaled boundary finite element method and to Assistant Prof. Dr. B. N. Rao of India Institute of Technology (IIT), Madras, India for proving the data of meshless method.

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(Received September 10, 2006)