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A Study on Reducing the Measured Modes in Modal Analysis Inspection for Damage Assessment of a Structure

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This paper presents a modal analysis method to assess damage for damped structures by using very few lower measured vibration modes. Different from our previous studies, all the zero-terms in global stiffness matrix are hereby excluded from the least-square estimation(LSE) procedure which is used to detect the location of damage. Therefore, the required number of vibration modes for this detection will considerably be reduced.To verify such an improvement, a ten-story shear type building with an artificially assumed damage has been investigated and the damage of the structure is consequently evaluated by using only the first(fundamental) analyzed measured mode.

Key Words : Damage assessment, modal analysis, reducing measured modes.

1. INTRODUCTION

For structures, analyzing an earthquake record or measuring modal parameters can lead to a detection for degradation of stiffness or change of damping by using various system identification techniques. Many time and frequency domains identification methods have been developed to identify structural parameters (or to detect damage) and have been widely utilized in civil engineering field. However, those methods meet some difficulties, e.g., great effect of noise on identification accuracy[1]; uniqueness problem of identification result due to location of sensors[2], as well as extremely small time intervals (time history method) required for structures with high frequencies[3], etc. To avoid those difficulties, the methods of various modal analysis techniques based directly on measured modal parameters have been considerably developed to identify structural parameters in recent years, with some improvements of vibrational testing technique[4]

There are many studies[5),6] to directly identify the stiffness matrix and to consequently evaluate the damage(degradation of stiffness) of a structure by using modal analysis methods. Among those methods, it should particularly be pointed out that Salane and Baldwin[6] successfully identified all the non-zero terms in stiffness matrix before and after damage for full-scale highway bridge structures by using measured vibration modes. In their's method, it needs a lot of measured modes if there are many those non-zero terms. It is known, however, that sometimes only the first and the second vibrational modes can satisfactorily be obtained for practical structures. Based on this reason, several kinds of modal analysis methods[7),8),9] which can assess or estimate the damage of structures by using lower measured modes have been paid attention by engineers recently. On the other hand, an exact identification of a global stiffness matrix(a symmetry full matrix) needs[10] N(N+1)/2 independent measurements, in which N represents system's degrees of freedom(DOFs). Also for LM measured vibration modes, there are LM・N known values of measurement. Therefore the required LM for the identification is equal to N(N+1)/2/N, i.e. (N+1)/2. For that reason, those kinds of modal analysis methods in which the lower measured modes are used are approximation or estimation methods of structural damage detection. In order to improve the estimation accuracy, two basic procedures for damage detection were introduced in our previous studies[8),9),11]. The first procedure is to detect the locations of damage(degradation of stiffness) by using a least-square estimation(LSE) procedure and the second procedure is to solve a certain unknown scalars(severity of damage) for those detected locations in stiffness matrix through a free-vibration equation of motion. Moreover, for general structures, it is recognized that there are many zero terms in stiffness matrix. Also these zero terms may be considered to be consistent[6] before and after damage. In order to reduce the unknown quantities in the aforementioned LSE procedure, those zero terms should be excluded from the LSE procedure. Therefore, the required modes of measurement to detect the locations of damage will further be reduced. For this consideration, based on our previous studies[8),9], this paper presents a modal analysis method in which all the zero terms are excluded from the LSE procedure for damage assessment of damped structures. To verify such an improvement of this method, a ten-story shear type building with a damage (degradation in stiffness) in some stories, is investigated by using only the first(fundamental) mode. Consequently the damage of the structure is evaluated.

In order to easily understand the real significant for degradation of member(element) stiffness in structures, the words"location and severity of damage" is used instead of the words" location and severity of stiffness degradation" in this paper.

2. MODAL ANALYSIS INSPECTION

A free-vibration equation of motion for damped structures, is described as

$$[M][\Phi][\Omega^2]+[C][\Phi][\Omega]+[K][\Phi]=[0] \tag{1}$$

where [M]=mass matrix; [C]=damping matrix; [K]=global stiffness matrix. Also, the natural frequency matrix [Ω] and the mode shape matrix [Φ] are as follows

$$\begin{aligned}&[\Omega]_{LM\times LM}=Diag(\mu_1,\mu_2,\cdots,\mu_{LM})=[\Omega_a]+i[\Omega_b]\\&[\Omega^2]_{LM\times LM}=[\Omega_a^2]-[\Omega_b^2]+2i[\Omega_a][\Omega_b]\\&[\Phi]_{LM\times N}=[\{X\}_1,\{X\}_2,\cdots,\{X\}_{LM}]\end{aligned} \tag{2}$$

in which LM=the number of measured vibration modes; N=the system's number of degrees of freedom; $i^2=-1$; $\{X\}_p$(p=1,2, · · · ,LM)=the p-th mode shapes; also

$$\mu_p=-\xi_p\omega_p+i\omega_p \tag{3}$$

$$\omega_p=\omega_p\sqrt{1-\xi_p^2} \tag{4}$$

where ω_p, ω_p and ξ_p= the system's p-th undamped natural frequency, p-th damped natural frequency and p-th damping ratio respectively. The well-known representation of the Rayleigh damping is written as follows

$$[C]=\beta_1[M]+\beta_2[K] \tag{5}$$

in which β_1, β_2=scalars regarding to the Rayleigh damping.

In this study, the mass matrix is considered as a constant matrix regardless of damage. Also initial(before damage) global stiffness matrix $[K_0]$ is considered to be a known matrix[12], e.g., structures remains in elastic stage before damage or stiffness is determined by referring the design materials. Moreover, if there are measured data from vibrational test, the matrix $[K_0]$ could be corrected or improved by using those data[13]. Denote $[\triangle K]=[K_0]-[K]$, in which [K] is degraded(after damage) global stiffness matrix and [△K] is unknown change of stiffness matrix due to damage. Substituting the [Ω], [Φ], [M], [C](see Eq.5) and $[K](=[K_0]-[\triangle K])$ into Eq.1, and arranging the obtained equations, the separated real-part of the equations can be obtained. Te equations for the real-part is

$$[\Delta K][\Gamma]=[\Xi] \tag{6}$$

in which $[\triangle K]_{N\times N}$=the unknown change of stiffness matrix due to damage;

$$\begin{aligned}&[\Gamma]_{N\times LM}=(\beta_2[\Phi][\Omega_a]+[\Phi]) \quad \text{and}\\&[\Xi]_{N\times LM}=[M]([\Phi][\Omega_a^2]+\beta_1[\Phi][\Omega_a])+[K_0][\Gamma]\end{aligned} \tag{7}$$

where N=the number of system's DOFs; LM=the number of measured modes. It should be pointed out that the change of stiffness due to damage is considered to occur only in non-zero terms[6] of global stiffness matrix [K] in this study. Therefore, there are a certain zero-terms in this unknown matrix [△K]. Those zero terms should be excluded in order to reduce the unknown terms in [△K]. Thus, the left non-zero terms in [△K] have to be written as a form of vector, i.e, Eq.6 becomes

$$[B][\Delta\vec{K}]=[\vec{Z}] \tag{8}$$

in which

$$[B]=\begin{bmatrix}B_1\\B_2\\ \cdot\\ \cdot\\ \cdot\\B_{LM}\end{bmatrix};\quad [B_t]_{N\times NF}=\begin{bmatrix}D_{1,t}&\cdots&0\\&D_{2,t}&\\&&\ddots&\\0&\cdots&D_{N,t}\end{bmatrix}; \tag{9}$$

$$D_{p,t}=\{\Gamma_{1,t}\ \Gamma_{2,t},\cdots,\Gamma_{N,t}\}\quad p=1,2,\cdots,N\quad t=1,2,\cdots,LM$$

$$\{\Delta\vec{K}\}_{NF\times 1}=\{\Delta k_{1,1},\Delta k_{1,2},\cdots,\Delta k_{1,N1},\Delta k_{2,1},\Delta k_{2,2},\cdots,\Delta k_{2,N2},\cdots,\Delta k_{N,1},\Delta k_{N,2},\cdots,\Delta k_{N,N}\}^T$$

where $N_1, N_2, \cdots, N_n$=the numbers of non-zero terms at the 1st, 2nd, · · · ,Nth rows in stiffness matrix [K] respectively; the total number of those non-zero terms $NF=N_1+N_2+\cdots+N_n$; and

$$\begin{aligned}\{\vec{Z}\}_{(LM\times N)\times 1}&=\{Z_1^{(1)},Z_2^{(1)},\cdots,Z_N^{(1)},\ Z_1^{(2)},Z_2^{(2)},\cdots,Z_N^{(2)}\\&\cdots\ Z_1^{(LM)},Z_2^{(LM)},\cdots,Z_N^{(LM)}\}^T\\&=\{\Xi_{1,1},\Xi_{2,1},\cdots,\Xi_{N,1},\ \Xi_{1,2},\Xi_{2,2},\cdots,\Xi_{N,2}\\&\cdots\ \Xi_{1,LM},\Xi_{2,LM},\cdots,\Xi_{N,LM}\}^T\end{aligned} \tag{11}$$

in which the superscripts (1), (2), · · · , (LM) in Eq.11 represent the orders of measured modes. In order to easily understand the representation of matrix [B] in Eq.9, a stiffness matrix [K] and a matrix [B] of three DOFs system are given as follows: N=3; measured modes LM=2; the number of non-zero terms(in matrix [K]) NF=7;

$$[K]=\begin{bmatrix}k_{11}&0&k_{13}\\0&k_{22}&k_{23}\\k_{31}&k_{32}&k_{33}\end{bmatrix};\quad [\Gamma]=\begin{bmatrix}\Gamma_{1,1}&\Gamma_{1,2}\\\Gamma_{2,1}&\Gamma_{2,2}\\\Gamma_{3,1}&\Gamma_{3,2}\end{bmatrix};$$

$$
[B]_{(2\times3)\times7}=\begin{bmatrix}B_1\\ B_2\end{bmatrix}=\begin{bmatrix}D_{1,1} & & 0\\ & D_{2,1} & \\ 0 & & D_{3,1}\\ D_{1,2} & & 0\\ & D_{2,2} & \\ 0 & & D_{3,2}\end{bmatrix}
$$

$$
=\begin{bmatrix}\Gamma_{1,1} & \Gamma_{3,1} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \Gamma_{2,1} & \Gamma_{3,1} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \Gamma_{1,1} & \Gamma_{2,1} & \Gamma_{3,1}\\ \Gamma_{1,2} & \Gamma_{3,2} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \Gamma_{2,2} & \Gamma_{3,2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \Gamma_{1,2} & \Gamma_{2,2} & \Gamma_{3,2}\end{bmatrix};\ [\Delta\vec{K}]=\begin{Bmatrix}\Delta k_{1,1}\\ \Delta k_{1,3}\\ \Delta k_{2,2}\\ \Delta k_{2,3}\\ \Delta k_{3,1}\\ \Delta k_{3,2}\\ \Delta k_{3,3}\end{Bmatrix}
$$

It is easy to verify that the $[B]\{\Delta\vec{K}\}$ is totally same as the $[\Delta K][\Gamma]$ in Eq.6 from observing the expressions of [B] and $\{\Delta\vec{K}\}$. Besides, it should be pointed that if $D_{p,\ i}(p=1, 2,\cdots,N)=\{0\}$ in Eq.9, the $D_{p,\ i}$ should be excluded from matrix $[B_i]$. Note, the Δk_{ij} and Δk_{ji} are hereby considered to be two independent unknown stiffness coefficients in Eq.10. According to the reason shown in Appendix Ⅰ, Eq.10 has a good estimated result for location detection of damage. On the contrary, that the Δk_{ij} and Δk_{ji} are considered to be a same coefficient(symmetry of stiffness is used) in Eq.10 will often lead to some difficulties for damage location detection(the details is given in numerical example). In order to easily understand the reason of $\Delta k_{i,j}$ and $\Delta k_{j,i}$ being considered as an independent parameter in the LSE(least-square estimate) procedure in this study, Let's focus to a node(in FEM model with N degrees of freedom), e.g., node 1 which corresponds to $k_{1,1}$ or $\Delta k_{1,1}$ in structural global stiffness matrix. It is well known that the $k_{1,1}$ is assembled by each stiffness of member corresponding to node 1. Therefore, if there is a practical damage in a member which is related to node 1, then there must be a stiffness degradation in node 1, i.e., $\Delta k_{1,1}\neq0$. Conversely, if there is $\Delta k_{1,1}=0$, then there is no stiffness degradation in node 1 surely, i.e. all the $\Delta k_{1,j}(j=1,2,...N)$ in row 1 must be equal to zero, i.e., the $\Xi_{1,1}, \Xi_{1,2},...\Xi_{1,LM}$ in row 1 must be equal to zero(LM is the number of measured modes) from observing Fig.6. Furthermore from observing Eqs.8 and 11, the $Z_1^{(1)}$, $Z_1^{(2)}$, $...Z_1^{(LM)}$(equal to $\Xi_{1,1}, \Xi_{1,2},...\Xi_{1,LM}$ respectively) must be equal to zero. Finally from observing Eq.24, the estimated $\Delta k_{1,1}$ by using the LSE procedure must be equal to zero. Therefore the conclusion can be drawn that if there is no stiffness degradation in node 1, then estimated $\Delta k_{1,1}$ must be equal to zero. On the other hand, if $\Delta k_{i,j}$ and $\Delta k_{j,i}$ are considered as a same parameter in the LSE procedure, the estimated $\Delta k_{1,1}$ may be related to other Z valsue except for those $Z_1^{(1)}, Z_1^{(2)},...Z_1^{(LM)}$(equal to $\Xi_{1,1}, \Xi_{1,2},...\Xi_{1,LM}$ respectively), i.e., $\Delta k_{1,1}$ may not be equal to zero(see Fig.5). Since only a few(lower) measured modes are used in this study, generally the $\{\Delta\vec{K}\}$ in Eq.8 can not be solved exactly, only for its approximate solution. A least-square estimation(LSE) of vector $\{\Delta\vec{K}\}$ is therefore introduced as follows

$$
\{\Delta\vec{K}\}\fallingdotseq[B]^{+}\{\vec{Z}\} \tag{12}
$$

where $[B]^{+}$=the pseudoinverse matrix[14] of [B], also the matrix $[B]^{+}$ is unique[14]. When the number of independent measurements are greater than or equal to the total number of non-zero terms NF, Eq.12 becomes an equality, i.e., the estimated $\{\Delta\vec{K}\}$ is an exact solution.

3. SOLUTION OF UNKNOWN SCALARS FOR MEMBER STIFFNESS

The location of damage could be detected(indicated) by investigating the estimated $\{\Delta\vec{K}\}$(see Eq.12), particularly the estimated change ratio $\Delta k_{ij}/k_{0ij}(k_{0ij}\neq0; i,j=1, 2,\cdots,N)$ for stiffness coefficients. It is well known that a diagonal stiffness coefficient for a node in stiffness matrix is assembled by each stiffness of member(element) concerning with this node in FEM(Finite element method) model of structures. Also the off-diagonal stiffness coefficients are made up by corresponding member(element) stiffness. The corresponding relation between a member and an off-diagonal coefficient is unique. Therefore, the damaged members(locations of damage) can be detected by investigating the change of off-diagonal coefficients in stiffness matrix. Concretely, when a member with a remarkable change ratio of stiffness is detected, the member stiffness k_E is multiplied by an unknown scalar $\alpha_k(\alpha_k\leqq1.0$, which represents the degradation of member stiffness or severity of damage). Note, it is not suggested by this study that damage may be realistically modeled as above representation of $\alpha_k\cdot k_E$. For the case of member stiffness degradation in plane frame or beam structures, both the axial rigidity EA and flexural rigidity EI should be multiplied by two unknown scalars α_{k1} and α_{k2} respectively by observing the member stiffness matrix(the length of member is considered to be regardless of damage). Furthermore, such a α_k is contained in stiffness matrix of the structure by assembling all the member stiffnesses into the stiffness matrix. For instance, if a member(element) which is joined up with nodes i and j in shear-type building structures is damaged, the off-diagonal stiffness coefficient k_{ij} becomes $\alpha_k\cdot k_{ij}$, also the diagonal stiffness coefficients k_{ii} and k_{jj} become $k_{ii}(\alpha_k)$ and $k_{jj}(\alpha_k)$ respectively, i.e.

$$
[K(\alpha^{*})]=\begin{bmatrix}\cdot & & & & \\ & \cdot & & & \\ & & k_{ii}(\alpha_k) & \alpha_k\cdot k_{ij} & \\ & & \cdot & & \\ & & & \cdot & \\ & & \alpha_k\cdot k_{ij} & k_{jj}(\alpha_k) & \\ & & & \cdot & \\ & & & & \cdot\end{bmatrix} \tag{13}
$$

where α^{*} represents the set of α_k.

In the real-part of free-vibration equation of motion for a structure, there are N equations corresponding to each vibration mode. If all those LM measured modes are used, the LM・N equations for real-part can be provided by substituting $K(\alpha^{*})$ and $C(\alpha^{*})$(see Eq.5) into this free-vibration equation of motion. Among those LM・N equations, a certain equations in which α^{*} is contained are available for solving the α^{*}. Denote the number of those available equations for solving the

α^* to be NE, and arrange those NE equations, the selected new equations regarding to α^* can be obtained as follows

$$[A]\{\alpha\}=\{C\} \tag{14}$$

in which $\{\alpha\}=\{\alpha_1, \alpha_2, \cdots, \alpha_{NU}\}^T$, NU is the total number of α^*; [A] and [C] are known NU×NU matrix and NU×1 vector respectively. When $|[A]| \neq 0$, the unique $\{\alpha\}$ can be obtained. If NU≦NE, i.e., the number of unknown scalars α^* are equal to or less than the number of available equations for α^*, then theoretically the identified $\{\alpha\}$ is exact(also if all damaged members are detected). If NU>NE, it needs additional measured modes. After all the unknown scalars are solved, the $[K(\alpha^*)]$ is the estimated result of [K]. Therefore, the severity of damage can be evaluated.

As the practical global stiffness matrix of structures is an unknown matrix, the reasonability of estimated stiffness matrix should be evaluated. In this study, a cost function[15] is employed to verify the estimated and which correspond to estimated stiffness matrix, etc. The cost function[16] is expressed as follows

$$J=\sum_{i=1}^{LM}\{\alpha_i(\omega_i^{(M)}-\omega_i^{(E)})^2 + (\eta_i^{(M)}-\eta_i^{(E)})^T H_i(\eta_i^{(M)}-\eta_i^{(E)})\} \tag{15}$$

in which LM=the number of measured modes; $\omega_i^{(M)}$ =the i-th measured undamped natural frequency; $\eta_i^{(E)}$ =the ith estimated mode shape; α_i =the corresponding weighted coefficient; H_i = a diagonal matrix that consists of weighted coefficients[16]. The superscript T means a transpose of matrix or vector. If $J<\delta$ in which δ is a scalar, the estimated results are satisfactory under the sense of condition $J<\delta$. If $J \geqq \delta$, it needs to investigate the locations of damage again or to add measured modes until the condition $J<\delta$ is satisfied. The value of δ may be taken as 0.005 ~0.01[16]

4. NUMERICAL EXAMPLE

In order to demonstrate an improvement for damage location detection in this study, a ten-story shear-type building which has been used in our previous study[9] is selected to be hereby a numerical example. The FEM model of this building with eleven nodes, ten elements and ten DOFs, is shown in Fig.1. Also the node numbering agrees with the story(floor) numbering in this figure. The mass and stiffness for each story of the structure is listed in Fig.1. Moreover, damping ratios are hereby assumed as : $\xi_1 = \xi_2 = 2\%$; the calculated coefficients[17] regarding to the Rayleigh damping are: $\beta_1 = 9.96 \times 10^{-2}$ and $\beta_2 = 2.31 \times 10^{-3}$ (see Eq.5). Because of the lack of measured data, the analyzed natural frequencies, as well as mode shapes are hereby taken as the measured ones. Besides, only the first(fundamental) measured vibrational mode is used throughout this numerical example.

To verify the availability of this method, a damage (degradation in member stiffness) event is artificially assumed at some stories in this building. Concretely, a 30% degradation of member stiffness is assumed on story-3(nodes 2-3) and story-8(nodes 7-8) respectively. Therefore the degraded member stiffnesses k_3 (story-3) and k_8 (story-8) are 39.30 and 28.39 Mn/m(see Table 1), respectively. Also the stiffness coefficients corresponding to node 2 and 3, node 7 and 8 in global stiffness matrix should be degraded. The estimated change ratios $\Delta k_{ij}/k_{0ij}$ ($k_{0ij} \neq 0$, i, j=1,2, ･ ･ ･ , N) for off-diagonal and diagonal coefficients of stiffness matrix by Eq.12 are shown in Fig.2 and 3 respectively, through using the first measured vibrational mode.

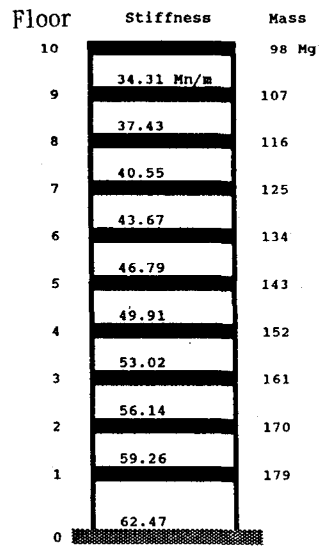

Fig.1 Ten-story shear-type building

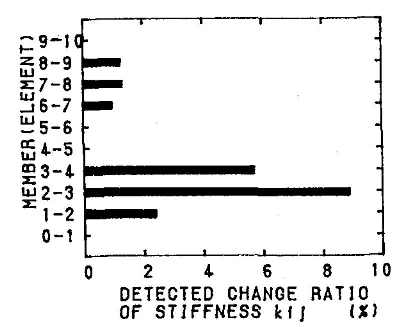

Fig.2 Damage location detection for off-diagonal coefficients in stiffness matrix.

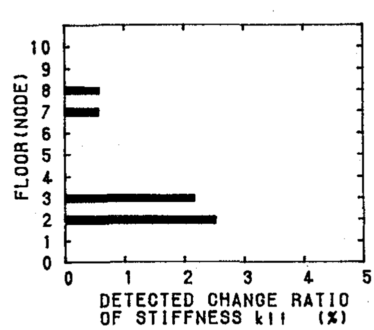

Fig.3 Damage location detection for diagonal coefficients in stiffness matrix.

Table 1 Identified member stiffness (unit: Nn/m) and error E_p

Before damage		After damage		
k_i(nodes)	Exact	Exact	Ident.	E_p
k_3(2-3)	56.14	39.30	39.30	0.0%
k_8(7-8)	40.55	28.36	28.36	0.0%

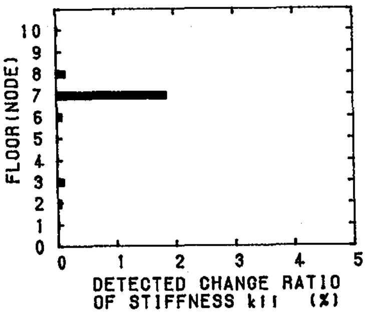

Fig.4 Damage location detection for diagonal coefficients in stiffness matrix for our previous study[9]

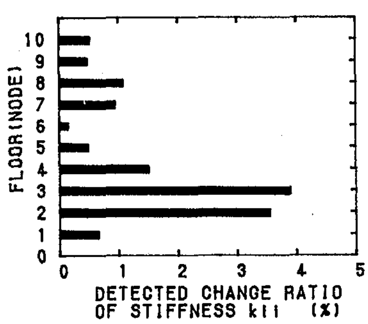

Fig.5 Damage location detection for diagonal coefficients in stiffness matrix(if symmetry of stiffness is used in Eq.10).

Note that, the estimated change ratio of off-diagonal stiffness coefficient for a member i-j is hereby equal to Max($\Delta k_{ij}/k_{0ij}$, $\Delta k_{ji}/k_{0ji}$). Furthermore, the locations of damage for members 1-2, 2-3, 3-4, 6-7, 7-8 and 8-9 are initially detected from observing Fig.2, because the change ratios of off-diagonal stiffness coefficients for those members are remarkable. From observing Fig.3, however, the change ratios of diagonal stiffness coefficients for nodes 2, 3, 7 and 8 regrading to members 2-3 and 7-8 are remarkable, the damaged members 2-3 and 7-8 are then detected. The members 1-2, 3-4, 6-7 and 8-9 are detected to be undamaged ones(see Fig.3 and Appendix Ⅰ), because the change ratios in nodes 1, 4, 6 and 9 which are assembled by members 1-2, 3-4, 6-7 and 8-9 respectively are zero. Thus, the unknown scalars α_1 and α_2 are consequently assumed on members 2-3 and 7-8 respectively. Fig.4 is the result of our previous study[9] by using the same first mode. From observing Fig.4, the damaged members 2-3 can't be detected. The main difference between the method for Fig.3 and Fig.4 is : Under the condition of same known parameters(the same measured first mode), the zero-terms(hereby being as the unknown parameters in aforementioned LSE procedure) in stiffness matrix have not been excluded in the previous study[3], On the other hand, all zero-terms have been exlluded from the LSE procedure in this study. So the number of unknown parameters in this method is much less than the one in study[9] under the condition of the same number of known parameters. Therefore, the accuracy of damage location detection in the this method is considerably improved. In this study, the percentage of an error E_p for member stiffness identification is defined as follows

$$E_p=\frac{(k_p)_{EXACT}-(k_p)_{IDENT.}}{(k_p)_{EXACT}}\times 100 \tag{16}$$

The calculated error E_p is shown in Table 1 in which the identified result is very satisfactory. Certainly the estimated results should be verified by calculating the J function. The calculated J is hereby less than 0.001, so the damaged members 2-3 and 7-8 are finally detected. Moreover, if the damaged member 2-3 has not been detected, the calculated J value is equal to 1.476 by using Eq.15($\alpha_1=0.5$, H_1 =identity matrix, refer to study[16]). Also J is equal to 0.271 in the case of member 7-8 having not been detected. Besides, Fig.5 shows a result of estimated change ratios for diagonal stiffness coefficients in case of the symmetry for stiffness matrix being considered for unknown terms in matrix [ΔK](refer to Eq.10). It is evident that a degradation seems to occur in almost all the nodes(the corresponded reason will be given in Appendix Ⅰ). So, the dealing of this symmetry of stiffness coefficients in Eq.10 is considered to be necessary.

5. CONCLUDING REMARKS

This study presents a modal analysis method which can assess both the location and severity of damage for damped structures by using very few measured modes. Different from our previous studies, all the zero-terms in global stiffness matrix are hereby excluded from the least-square estimation (LSE) procedure which is used to detect the location of damage. Therefore, the required number of vibration modes for this detection is considerably reduced. As stated in numerical example, a detection of damaged location(member of a structure) is initially decided by observing the estimated change ratio for off-diagonal stiffness coefficients $\Delta k_{ij}/k_{0ij}$ ($i\neq j$) and investigating the estimated change ratio for diagonal stiffness coefficients $\Delta k_{ii}/k_{0ii}$. Theoretically, if there is a node with non-zero $\Delta k_{ij}/k_{0ij}$ or $\Delta k_{ii}/k_{0ii}$,this node must be regarding to practical damage in this method(see Appendix Ⅰ), also this damaged location(hereby indicated by this node) can undoubtedly be decided. On the other hand, if every location(indicated by node or member) with zero values of $\Delta k_{ij}/k_{0ij}$ or $\Delta k_{ii}/k_{0ii}$ is a practical location in which stiffness has not been changed, then a conclusion can be drawn that all the practical damaged locations are detected through observing those $\triangle k_{ij}/k_{0ij}$ or $\triangle k_{ii}/k_{0ii}$. So, the estimated stiffness by using Eq.12 is unique even using lower measured modes by this method. To prove the conclusion theoretically is impossible unless the required number of independent measurements to exactly identify the global stiffness matrix are provided(see Eq.12). However, many calculated examples regarding to this study show that the conclusion are reasonable. Besides, once a practical damaged member is not detected, the estimated stiffness will evidently not be reasonable, e.g, negative stiffness or stiffness much larger than the initial values[10), 8)], etc. Also once an unknown scalar α_k is assumed on an undamaged member, the estimated α_k will be equal to 1.0 by this method under the condition without measurement error. Overall the damaged locations are finally decided after verifying aforementioned J function(see Eq.15).

Theoretically, the effects of measurement error of natural frequency, damping ratio and mode shape on identification accuracy have been investigated in our previous studies[8), 9)]. Also the application condition of this method is based on the precise measurement of vibration. However, because of the complexities of practical structures, sometimes it is difficult to obtain the precise structural vibration modes. Therefore, it is expected to verify this method for model-scale or full-scale damped structures by using practical measured vibration modes in future.

APPENDIX I.

The following is the theoretical basis (proposition) of which the $\triangle k_{i,j}$ and $\triangle k_{j,i}$ in Eq.10 should be considered as the independent parameters in LSE procedure. In the proposition, the knowledges of pseudoinverse, matrix product and matrix inverse obtaining(elementary transformation method), etc. will be introduced. Moreover, in order to describe the proposition correctively and clearly, some complicate formulae and operations had to be used.

Proposition: If there are no coefficient change at row i in practical stiffness matrix, i.e., $k_{ii}=k_{0ii}$, the estimated change of diagonal stiffness coefficient $\triangle k_{ii}$ by using aforementioned least-square estimation(LSE) procedure in Eq.9 must be equal to zero. In the other words, if one of the estimated $\triangle k_{i,1}, \triangle k_{i,2}, \cdots, \triangle k_{i,Ni} \neq 0$, at least one of the practical $\triangle k_{i,1}, \triangle k_{i,2}, \cdots, \triangle k_{i,Ni} \neq 0$, i.e. there is undoubtedly a practical change of stiffness coefficients at row i.

Proof: The pseudoinverse $[B]^+$ could be calculated by following equation[14)]

$$[B]^{+}=[B]^{T}([B][B]^{T})^{-1} \qquad (17)$$

where [B] is (LM×N)×NF matrix(see Eq.9); N is system's DOFs; NF is the total number of non-zero terms in stiffness matrix [K]; LM is the number of measured vibration modes. Thus, the square matrix $[B][B]^T$ is written as

$$[B][B]^{T}=\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{LM} \end{bmatrix} \left[B_1^T, B_2^T, \cdots, B_{LM}^T\right] = \begin{bmatrix} B_1B_1^T, B_1B_2^T, \cdots, B_1B_{LM}^T \\ B_2B_1^T, B_2B_2^T, \cdots, B_2B_{LM}^T \\ \vdots \\ B_{LM}B_1^T, B_{LM}B_2^T, \cdots, B_{LM}B_{LM}^T \end{bmatrix}_{(LM\times N)\times(LM\times N)} \qquad (18)$$

in which every diagonal submatrix $[B_iB_j^T]$(i,j=1,2,···,LM) is further obtained as follows

$$[B_iB_j^T]=\begin{bmatrix} D_{1,i} & \cdots & 0 \\ & D_{2,i} & \\ & & \ddots \\ 0 & \cdots & D_{N,i} \end{bmatrix} \begin{bmatrix} D_{1,j}^T & \cdots & 0 \\ & D_{2,j}^T & \\ & & \ddots \\ 0 & \cdots & D_{N,j}^T \end{bmatrix} = \begin{bmatrix} \sum_{p=1}^{N1}\Gamma_{p,i}\Gamma_{p,j} & & & 0 \\ & \sum_{p=1}^{N2}\Gamma_{p,i}\Gamma_{p,j} & & \\ & & \ddots & \\ 0 & & & \sum_{p=1}^{Nn}\Gamma_{p,i}\Gamma_{p,j} \end{bmatrix}_{N\times N} \qquad (19)$$

in which the $D_{1,i}, D_{2,i}, \cdots, D_{N,i}$ are listed in Eq.9. Since the subscripts i and j vary from 1 to LM, all the square submatrices $[B_iB_j^T]$(i,j=1, 2, ···, LM) are diagonal submatrices respectively. Evidently the number of those submatrices is LM×LM. Moreover, the inverse of square matrix $[B][B]^T$ can be obtained by using the well-known elementary transformation method, i.e

$$\left[\begin{array}{cccccccccccc} U_1^{(1)} & 0 & V_1^{(1)} & 0 & W_1^{(1)} & 0 & 1 & 0 & 0 & & & \\ & \ddots & & \ddots & & \ddots & & \ddots & & & & \\ 0 & U_N^{(1)} & 0 & V_N^{(1)} & 0 & W_N^{(1)} & & 1 & & & & \\ U_1^{(2)} & 0 & V_1^{(2)} & 0 & W_1^{(2)} & 0 & 0 & & 1 & & & \\ & \ddots & & \ddots & & \ddots & & & & \ddots & & \\ 0 & U_N^{(2)} & 0 & V_N^{(2)} \cdots & 0 & W_N^{(2)} & & & & 1 & & \\ & & & & \vdots & \vdots & & & & & \ddots & \\ U_1^{(LM)} & 0 & V_1^{(LM)} & 0 & W_1^{(LM)} & 0 & 0 & & & & 1 & \\ & \ddots & & \ddots & & \ddots & & & & & & \ddots \\ 0 & U_N^{(LM)} & 0 & V_N^{(LM)} & 0 & W_N^{(LM)} & & & & & & 1 \end{array}\right] \qquad (20)$$

in which the left matrix is the result of (LM×N)×(LM×N) square matrix $[B][B]^T$ and the right matrix is also a (LM×N)×(LM×N) identity matrix. Among the left matrix, the each diagonal submatrix (block) corresponds to submatries

$[B_1B_1^T], [B_1B_2^T], \cdots, [B_{LM}B_{LM}^T]$, in Eq.19 respectively. As stated before(Eq.9), the diagonal terms in the left matrix of Eq.20 are not equal to zero, i.e.,

$$U_1^{(1)}=\sum_{p=1}^{N1}\Gamma_{p,1}^2\neq 0\ ,\ V_1^{(2)}=\sum_{p=1}^{N2}\Gamma_{p,2}^2\neq 0\ ,\ W_1^{(LM)}=\sum_{p=1}^{Nn}\Gamma_{p,LM}^2\neq 0$$

where the superscripts (1),(2), $\cdots$, (LM) represent the order of measured modes. Therefore, a series of row elementary transformations for the left and right matrices can be done until the left matrix becomes identity matrix.The off-diagonal non-zero terms can be eliminated by supposing a proportion of diagonal terms corresponded with, e.g., the $U_1^{(2)}, \cdots, U_1^{(LM)}$ at row N+1, column 1, $\cdots$, row $(LM-1)\times N+1$, column 1 are eliminated by supposing $-U_1^{(1)}\cdot U_1^{(2)}/U_1^{(1)}, \cdots, -U_1^{(1)}\cdot U_1^{(LM)}/U_1^{(1)}$ respectively. Simultaneously, the zero terms at row N+1, column 1, $\cdots$, row $(LM-1)\times N+1$, column 1 in the right matrix(identity matrix) become $-U_1^{(2)}/U_1^{(1)}, \cdots, U_1^{(LM)}/U_1^{(1)}$. Repeat those suppositions until all off-diagonal non-zero terms become zero(eliminated). Furthermore the left matrix in Eq.20 becomes an identity matrix. The right matrix is then the inverse of matrix $[B][B^T]$. As those suppositions only need to be done on non-zero terms, the locations of zero terms both in the left and right matrices in Eq.20 are kept unchanged, i.e.,$([B][B^T])^{-1}$ consists of $LM\times LM$ diagonal submatrices which are similar to Eq.19. In all the following statements, a matrix [A] is similar to a matrix [B] means that the locations(indicated by row and column) of zero terms which are identically equal to zero are totally same to matrix B. The square matrix $([B][B^T])^{-1}$ is therefore written as

$$([B][B]^T)^{-1}=\begin{bmatrix} R_1^{(1)} & & 0 & S_1^{(1)} & & 0 & T_1^{(1)} & & 0 \\ & \ddots & & & \ddots & \cdots & & \ddots & \\ 0 & & R_N^{(1)} & 0 & & S_N^{(1)} & 0 & & T_N^{(1)} \\ R_1^{(2)} & & 0 & S_1^{(2)} & & 0 & T_N^{(2)} & & 0 \\ & \ddots & & & \ddots & \cdots & & \ddots & \\ 0 & & R_N^{(2)} & 0 & & S_N^{(2)} & 0 & & T_N^{(2)} \\ & \vdots & & & \vdots & & & \vdots & \\ R_1^{(LM)} & & 0 & S_1^{(LM)} & & 0 & T_1^{(LM)} & & 0 \\ & \ddots & & & \ddots & \cdots & & \ddots & \\ 0 & & R_N^{(LM)} & 0 & & S_N^{(LM)} & 0 & & T_N^{(LM)} \end{bmatrix} \quad (21)$$

$$=[G_1\ \ G_2\ \ \cdots\ \ G_{LM}]_{(LM\times N)\times(LM\times N)}$$

in which the $(LM\times N)\times N$ submatrices $[G_1], [G_2], \cdots, [G_{LM}]$ correspond to block(submatrix) $R_1^{(1)}\cdots R_N^{(LM)}$, block $S_1^{(1)}\cdots S_N^{(LM)}$, $\cdots$, block $T_1^{(1)}\cdots T_N^{(LM)}$ respectively.Since the $([B][B^T])^{-1}$ is obtained, the pseudoinverse $[B]^+$ could be further obtained, likes

$$\begin{aligned}[B]^+&=[B]^T([B][B]^T)^{-1}\\ &=[B]^T[G_1\ \ G_2\ \ \cdots\ \ G_{LM}]\\ &=[B^TG_1\ \ B^TG_2\ \ \cdots\ \ B^TG_{LM}]_{NF\times(LM\times N)}\end{aligned} \quad (22)$$

Because submatrices $[G_2], \cdots, [G_{LM}]$ are similar to $[G_1]$(see Eq.21), one of the matrix product $[B^TG_1]$ among $[B^TG_1], [B^TG_2], \cdots, [B^TG_{LM}]$, is inspected as follows

$$[B^TG_1]=\begin{bmatrix} \Gamma_{1,1} & 0 & & \Gamma_{1,LM} & 0 \\ \Gamma_{2,1} & & & \Gamma_{2,LM} & \\ \vdots & & & \vdots & \\ \Gamma_{N1,1} & & \cdots & \Gamma_{N1,LM} & \\ & \ddots & & \ddots & \\ & \Gamma_{N2,1} & & & \Gamma_{N2,LM} \\ & \Gamma_{1,1} & & & \Gamma_{1,LM} \\ & \vdots & & & \vdots \\ 0 & \Gamma_{Nn,1} & & 0 & \Gamma_{Nn,LM} \end{bmatrix}_{NF\times(LM\times N)} \begin{bmatrix} R_1^{(1)} & & 0 \\ & \ddots & \\ 0 & & R_N^{(1)} \\ & \vdots & \\ R_1^{(LM)} & & 0 \\ & \ddots & \\ 0 & & R_N^{(LM)} \end{bmatrix}_{(LM\times N)\times N}$$

$$=\begin{bmatrix} \Gamma_{1,1}R_1^{(1)}+\cdots+\Gamma_{1,LM}R_N^{(LM)} & & 0 \\ \vdots & & \\ \Gamma_{N1,1}R_1^{(1)}+\cdots+\Gamma_{N1,LM}R_N^{(LM)} & & \\ & \ddots & \\ & & \Gamma_{Nn,1}R_N^{(1)}+\cdots+\Gamma_{Nn,LM}R_N^{(LM)} \\ & & \vdots \\ 0 & & \Gamma_{Nn,1}R_N^{(1)}+\cdots+\Gamma_{Nn,LM}R_N^{(LM)} \end{bmatrix}_{NF\times N} \quad (23)$$

From observing Eq.23, it is found that the $NF\times N$ matrix $[B^TG_1]$ is just similar to $[B_1]^T$(see Eq.9). Also matrices $[B^TG_2], \cdots, [B^TG_{LM}]$ are similar to $[B_2]^T, \cdots, [B_{LM}]^T$. Therefore, the whole matrix $[B]^+$(see Eqs.22, 23) is similar to $[B]^T$, only the values of each non-zero terms are different between the matrices $[B]^+$ and $[B]^T$. Furthermore, the mark Γ in matrix $[B]^T$ can be changed into mark P in matrix $[B]^+$ in order to represent the matrix $[B]^+$. Finally, Eq.12 is expressed in detail as follows

$$\{\Delta K\}=\begin{Bmatrix}\Delta k_{1,1}\\ \Delta k_{1,2}\\ \vdots\\ \Delta k_{N,N}\end{Bmatrix}=[B]^+\{\bar{Z}\}=\begin{bmatrix} P_1^{(1)} & 0 & & P_1^{(LM)} & 0 \\ P_2^{(1)} & & & P_2^{(LM)} & \\ \vdots & & & \vdots & \\ P_{N1}^{(1)} & & \cdots & P_{N1}^{(LM)} & \\ & \ddots & & & \ddots \\ & P_1^{(1)} & & & P_1^{(LM)} \\ & P_2^{(1)} & & & P_2^{(LM)} \\ & \vdots & & & \vdots \\ 0 & P_{Nn}^{(1)} & & 0 & P_{Nn}^{(LM)} \end{bmatrix}\begin{Bmatrix}Z_1^{(1)}\\ \vdots\\ Z_N^{(1)}\\ \vdots\\ Z_1^{(LM)}\\ \vdots\\ Z_N^{(LM)}\end{Bmatrix}$$

$$=\begin{Bmatrix} P_1^{(1)}Z_1^{(1)}+\cdots+P_1^{(LM)}Z_1^{(LM)}\\ P_2^{(1)}Z_1^{(1)}+\cdots+P_2^{(LM)}Z_1^{(LM)}\\ \vdots\\ P_{Nn}^{(1)}Z_N^{(1)}+\cdots+P_{Nn}^{(LM)}Z_N^{(LM)} \end{Bmatrix} \quad (24)$$

in which the subscripts 1,2, $\cdots$,N of Z represent the row numbering of Z corresponding to matrix $[\Xi]$(see Eq.11 and Eq.6). Also the superscript (1),(2), ・・・,(LM) represent the order of measured mode. It is known from Eq.24 that the row numberings between $\Delta k_{1,1}$ and $(Z_1^{(1)}, Z_1^{(2)}, \cdots, Z_1^{(LM)})$, $\Delta k_{1,2}$ and $(Z_1^{(1)}, Z_1^{(2)}, \cdots, Z_1^{(LM)})$, ・・・,$\Delta k_{N,N}$ and $(Z_N^{(1)}, Z_N^{(2)}, \cdots, Z_N^{(LM)})$ are totally same. Therefore, if there is no change of practical stiffness coefficients at a certain row, e.g. in row 1, i.e., practical $\Delta k_{1,1}, \Delta k_{1,2}, \cdots, \Delta k_{1,N1}$ $=0$. Then $\Xi_{1,1}, \Xi_{1,2}, \cdots, \Xi_{1,LM}$ must be equal to zero(see Eqs.6 and 11, $[\Xi]$ is the result of $[\Delta K][\Gamma]$), i.e., $Z^{1\,(1)}, Z_1^{(2)}, \cdots, Z_1^{(LM)}=0$. Furthermore, the estimated $\Delta k_{1,1}, \Delta k_{1,2}, \cdots, \Delta k_{1,Ni}=0$(see Eq.24). In the other words, if one of the estimated $\Delta k_{1,1}, \Delta k_{1,2}, \cdots, \Delta k_{1,N1}$ is not equal to zero, it means that at least one of the $Z_1^{(1)}, Z_1^{(2)}, \cdots Z_1^{(LM)}$ is not equal to zero(see Eq.24). According to Eq.6, at least one of the practical $\Delta k_{1,1}, \Delta k_{1,2}, \cdots, \Delta k_{1,N1}$ is not equal to zero, i.e., undoubtedly there is a change of stiffness coefficient at row 1 in stiffness matrix. Certainly, the practical $\Delta k_{1,1}$ (at row 1) is also not equal to zero(diagonal stiffness coefficient is assembled by corresponded member stiffnesses). The proposition is therefore true.

That the $[B]^+$ is similar to $[B]^T$ is easily to be verified by the following supposed two DOFs system : N=2; measured mode LM=1; NF=4(non-zero terms in stiffness matrix [K], see Eq.10), i.e.

$$\{\Delta\vec{K}\}=\{\Delta k_{1,1}, \Delta k_{1,2}, \Delta k_{2,1}, \Delta k_{2,2}\}^T\ ;\ [\Gamma]=\{1,2\}^T.\ \text{Thus}$$

$$[B]=\begin{bmatrix}1&2&0&0\\0&0&1&2\end{bmatrix};\ [B]^T=\begin{bmatrix}1&0\\2&0\\0&1\\0&2\end{bmatrix};$$

$$[B]^+=\begin{bmatrix}\frac{1}{5}&0\\\frac{2}{5}&0\\0&\frac{1}{5}\\0&\frac{2}{5}\end{bmatrix};\ [B][B]^+=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

However, if the symmetry of stiffness coefficient is considered in Eq.10, i.e., NF=3, then

$$[B]=\begin{bmatrix}1&2&0\\0&1&2\end{bmatrix};\ [B]^T=\begin{bmatrix}1&0\\2&1\\0&2\end{bmatrix};$$

$$[B]^+=\begin{bmatrix}\frac{5}{21}&-\frac{2}{21}\\\frac{8}{21}&\frac{1}{21}\\-\frac{4}{21}&\frac{10}{21}\end{bmatrix};\ [B][B]^+=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

It is evident that the matrix $[B]^+$ is dissimilar to the matrix $[B]^T$ in the case of NF=3.

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