Unified Plastic Limit Solution to Circular Plate Under Portion Uniform Load

Ma Guo-wei*, Yu Mao-hong**,
Yutaka Miyamoto***, Shoji Iwasaki****, Hideaki Deto*****

*M. of Eng., Dept. of Civil Eng., Xian Jiaotong University, Xian,710049,China
**Professor, Dept. of Civil Eng., Xian Jiaotong University, Xian,710049,China
***Professor, Dept. of Civil and Environmental Eng., Iwate University, Ueda , Morioka, Japan
****Dr. of Eng., Dept. of Civil and Environmental Eng., Iwate University, Ueda , Morioka, Japan
*****Dept. of Civil and Environmental Eng., Iwate University, Ueda , Morioka, Japan

In this paper, we adopt Twin Shear Stress Unified Yield Criterion to seek the plastic limit solution to simply supported circular plate under portion uniform load. The solutions in view of Maximum principal stress criterion, Tresca criterion and Mises criterion are all its special cases or in its linear proximity. This solution can be applied to many engineering materials. We can acquire some new benificial knowledge from the analytical solutions and the diagrams in this paper.

Key words: *Portion Uniform Load, Simply Supported Circular Plate, Plastic Limit analysis, Unified Yield Criterion*

1. Introduction

The procedure of elasto-plastic analysis of structures contains elastic stage, elastic limit stage and elasto-plastic stage. The stresses reach Plastic limit state in the end when loads increase further. Analytic solutions of the whole elasto-plastic stage can only be obtained for few simple applications. For complex structures, it is difficult to get the analytic complete solutions because of the complication of mathematical formula. Sometimes elastic limit and plastic limit are most important in analysis of structures in civil and mechanical engnieering. If deformation model of material is simplified as plastic limit analysis principle, we can get some theoretical solution with less mathematical computation effort.

Complete solution with Maximum principal stress criterion, Tresca criterion and maximum-minimum solution with Mises criterion to simply supported circular plate under portion uniform load (See **Fig.1**) are given in Ref[1]. As we know, both Maximum principal stress criterion and Tresca criterion consider the effect of only one or two principal stresses, thus they have obvious defect, and Mises criterion is not convenient to be used because of its nonlinear formula. In recent years, Prof. Yu Mao-hong suggested Twin Shear Stress Unified Strength Theory[4)~6)]. This theory has linear formula, concise physical conception and it considers the effects of all

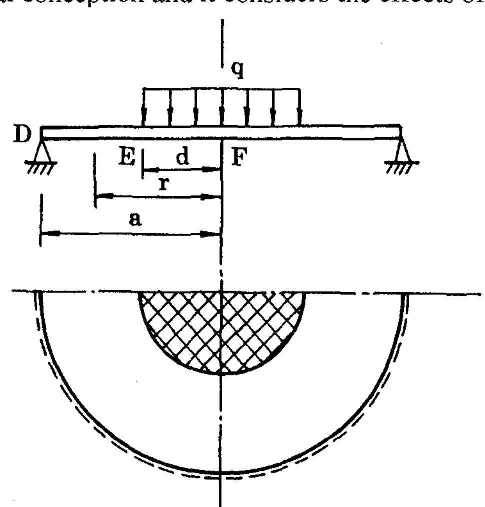


Fig. 1 Simply surpported circular plate

the three principal stresses. Unified Yield Criterion used in this paper is a part of Unified Strength theory. The plastic limit solution to simply supported circular plate derived with this criterion is suitable for a wide range of materials and engineering structures.

2. Twin Shear Stress Unified Strength Theory

Unified Strength Theory is a new system of strength theory, which can be used for most kinds of isotropic materials. Based on orthogonal octahedron of twin shear element model, this theory holds that material fails as a certain function of the two bigger principal shear stresses and their corresponding normal stresses reach a limit value. The mathematical expression of Unified Strength Theory is

$$\tau_{13} + b\tau_{12} + \beta(\sigma_{13} + b\sigma_{12}) = C \quad \text{when } \tau_{12} + \beta\sigma_{12} \geq \tau_{23} + \beta\sigma_{23} \qquad (1a)$$

$$\tau_{13} + b\tau_{23} + \beta(\sigma_{13} + b\sigma_{23}) = C \quad \text{when } \tau_{12} + \beta\sigma_{12} \leq \tau_{23} + \beta\sigma_{23} \qquad (1b)$$

Here τ_{13}, τ_{12} and τ_{23} are principal shear stresses, σ_{13}, σ_{12} and σ_{23} are normal stresses correlated to the three principal shear stresses, and
$\tau_{13} = 1/2(\sigma_1 - \sigma_3), \tau_{12} = 1/2(\sigma_1 - \sigma_2)$,
$\tau_{23} = 1/2(\sigma_2 - \sigma_3), \sigma_{13} = 1/2(\sigma_1 + \sigma_3)$,
$\sigma_{12} = 1/2(\sigma_1 + \sigma_2), \sigma_{23} = 1/2(\sigma_2 + \sigma_3)$.
σ_1, σ_2 and σ_3 are principal stresses. C, β and b are material strength parameters. β is normal stress influential coefficient, and $0 \leq \beta \leq 1$, b is weighted coefficient that reflects the influence of intermediate principal shear stress and relevant normal stress.The magnitude of these parameters can be evaluated by experimental results of triaxial test, i. e..

Rewriting (1a), (1b) in terms of principal stresses[3),4),5)], we get

$$\sigma_1 - \frac{\alpha}{1+b}(b\sigma_2 + \sigma_3) = \sigma_t \quad \text{when } \sigma_2 \leq \frac{\sigma_1 + \alpha\sigma_3}{1+\alpha} \qquad (2a)$$

$$\frac{1}{1+b}(\sigma_1 + b\sigma_2) - \alpha\sigma_3 = \sigma_t \quad \text{when } \sigma_2 \geq \frac{\sigma_1 + \alpha\sigma_3}{1+\alpha} \qquad (2b)$$

where $\alpha = \sigma_t / \sigma_c$, σ_t is uniaxial tensile strength, σ_c is uniaxial compressive strength, b reflects the effect of intermediate principal stress σ_2 on material strength. When b varies from 0 to 1, a family of convex yield criteria that are suitable for different kinds of materials are deduced.

When $\sigma_t = \sigma_c$ $(\alpha = 1)$, the unified strength criterion becomes unified yield criterion, which is suitable for metal materials. Formulae (1a), (1b) become

$$\tau_{13} + b\tau_{12} = C \text{ when } \tau_{12} \geq \tau_{23} \qquad (3a)$$

$$\tau_{13} + b\tau_{23} = C \text{ when } \tau_{12} \leq \tau_{23} \qquad (3b)$$

Similarly, formulae (2a), (2b) can be rewritten as follows,

$$\sigma_1 - \frac{1}{1+b}(b\sigma_2 + \sigma_3) = \sigma_t \quad \text{when } \sigma_2 \leq \frac{\sigma_1 + \sigma_3}{2} \qquad (4a)$$

$$\frac{1}{1+b}(\sigma_1 + b\sigma_2) - \sigma_3 = \sigma_t \quad \text{when } \sigma_2 \geq \frac{\sigma_1 + \sigma_3}{2} \qquad (4b)$$

The following criteria are special cases of Unified Strength Theory with specific values of b,

① $\sigma_t = \sigma_c$ $(\alpha = 1)$, b=0, Tresca Criterion, can be adapted for the materials with $\tau_s = 0.5\sigma_s$, where τ_s and σ_s are respectively shear yield stress and uniaxial yield strength,

② $\sigma_t = \sigma_c$ $(\alpha = 1)$, b=1, Twin Shear Stress Yield Criterion (suggested by Yu Mao-hong in 1961[2),3)]), adapted for that $\tau_s = \frac{2}{3}\sigma_s$,

③ $\sigma_t = \sigma_c$ $(\alpha = 1)$, b=0.5, a criterion in proximity to Mises Criterion, adapted for that $\tau_s = 0.6\sigma_s$,

④ $\sigma_t \neq \sigma_c$ $(0 < \alpha < 1)$, b=0, Mohr-Coulomb Criterion, which is often-used in engnieering,

⑤ $\sigma_t \neq \sigma_c$ $(0 < \alpha < 1)$, b=1, Twin Shear Stress Strength Criterion (suggested by Yu Mao-hong in 1985[4),5),6)]), adapted for materials of rock and soil.

3. Basic Equations of circular plate

When the plate is in plastic limit stress state, providing that the effect of σ_z, τ_{rz} on yield is neglectable and the values of σ_r, σ_θ are invariable along the thickness except that the sign of σ_r, σ_θ on different sides of the middle plate are different, M_r, M_θ may be expressed as $M_r = \sigma_r h^2, M_\theta = \sigma_\theta h^2$. Here h is thickness of the plate.

Because of axialsymmetry, equilibrium equation of circular plate is

$$\frac{d}{dr}(M_r r) - M_\theta = -\int_0^r q(r) r dr \qquad (5)$$

Where M_θ is circumferential bending moment, M_r is radial bending moment , and q(r) is the lateral distributed loading per unit area. Because σ_r, σ_θ and σ_z are the three principal stresses and $\sigma_z = 0$, the yield function of the plate can be expressed by M_r and M_θ. When Twin Shear Stress Unified Yield Criterion expressed by generalized stresses M_r, M_θ is used, eqs. (4a),(4b) can be expressed in term of generalized stresses M_r, M_θ as follows (**Fig. 2**),

$$\begin{aligned} &MAX(|M_r - \frac{b}{1+b}M_\theta|, |M_r - \frac{1}{1+b}M_\theta|, \\ &\frac{1}{1+b}|bM_r + M_\theta|, |\frac{1}{1+b}M_r - M_\theta|, \\ &\frac{1}{1+b}|M_r + bM_\theta|, |\frac{b}{1+b}M_r - M_\theta|) = M_p \qquad (6) \end{aligned}$$

Where $M_p = \sigma_t h^2$, and M_p is the limit bending moment of the plate.

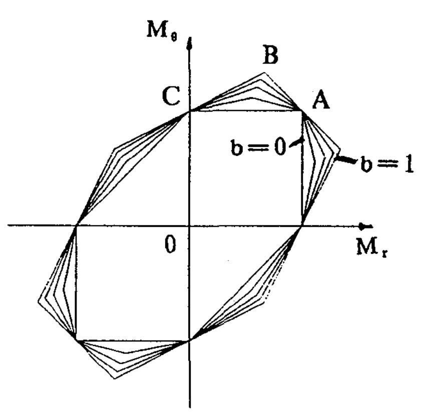


Fig. 2 Generalized Unified Yield Criterion

The center point of plate satisfies $M_r|_{r=0} = M_\theta|_{r=a}$, see point A in **Fig.2**, and simply supported boundary satisfies $M_r|_{r=a} = 0$ (point C in **Fig.2**). Stress states of all points in the plate are located on part AB and BC. From eq.6, part AB, BC in **Fig.2** can be expressed as follows,

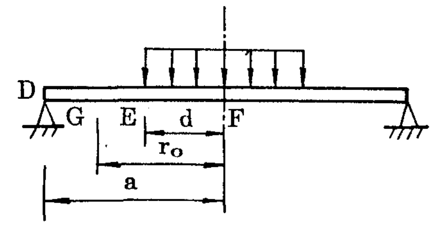
$$AB: \frac{b}{1+b}M_r + \frac{1}{1+b}M_\theta = M_p \qquad (7a)$$

$$CB: M_\theta - \frac{b}{1+b}M_r = M_p \qquad (7b)$$

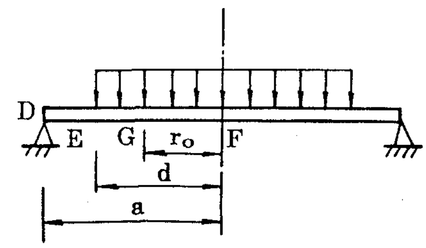
Providing that the moment of point G in the plate is located at point B in **Fig.2**, there are two possibilities, i.e.,

a) Point G is in line segment DE (See **Fig.3**(a)),

b) Point G is in line segment EF (See **Fig.3**(b)).



(a) Point G is in part DE



(b) Point G is in part EF

Fig.3 Point G in the plate

Boundary conditions and continuum conditions are,

1) $M_r|_{r=0} = 0$ is not infinite at point F,
2) $M_r|_{r=a} = 0$ at point D,
3) $M_r|_{r=d} = 0$ is continous at point E,
4) $M_r|_{r=r_0} = 0$ is continous at point G,
5) $M_\theta|_{r=r_0} = 2M_r|_{r=r_0}$ at point G according to stress condition of point B in **Fig.2**.

a) When point G is in line segment DE (**Fig.3**(a)), equilibrium equations of EF, GE and DG are follows,

$$EF: \begin{cases} r\frac{dM_r}{dr} = (1+b)(M_p - M_r) - \frac{q}{2}r^2 \\ M_\theta = (1+b)M_p - bM_r \end{cases} \qquad (8a)$$

$$GE: \begin{cases} r\frac{dM_r}{dr} = (1+b)(M_p - M_r) - \frac{q}{2}d^2 \\ M_\theta = (1+b)M_p - bM_r \end{cases} \qquad (8b)$$

$$DG: \begin{cases} r\frac{dM_r}{dr} = M_p - \frac{1}{1+b}M_r - \frac{q}{2}d^2 \\ M_\theta = M_p - \frac{b}{1+b}M_r \end{cases} \qquad (8c)$$

Here q is the plastic limit distributed loading, and q satisfies

$$q = \frac{2(1+b)(3+b)M_p}{(2+b)\left[(3+b) - 2\left(\frac{d}{r_0}\right)^{1+b}\right]d^2} \qquad (9)$$

b) When point G is in line segment EF (**Fig.3**(b)), corresponding equilibrium equations are,

$$\mathrm{GF}:\begin{cases} r\dfrac{dM_r}{dr}=(1+b)(M_p-M_r)-\dfrac{q}{2}r^2 \\ M_\theta=(1+b)M_p-bM_r \end{cases} \tag{10a}$$

$$\mathrm{EG}:\begin{cases} r\dfrac{dM_r}{dr}=M_p-\dfrac{1}{1+b}M_r-\dfrac{q}{2}r^2 \\ M_\theta=M_p-\dfrac{b}{1+b}M_r \end{cases} \tag{10b}$$

$$\mathrm{DE}:\begin{cases} r\dfrac{dM_r}{dr}=M_p-\dfrac{1}{1+b}M_r-\dfrac{q}{2}d^2 \\ M_\theta=M_p-\dfrac{b}{1+b}M_r \end{cases} \tag{10c}$$

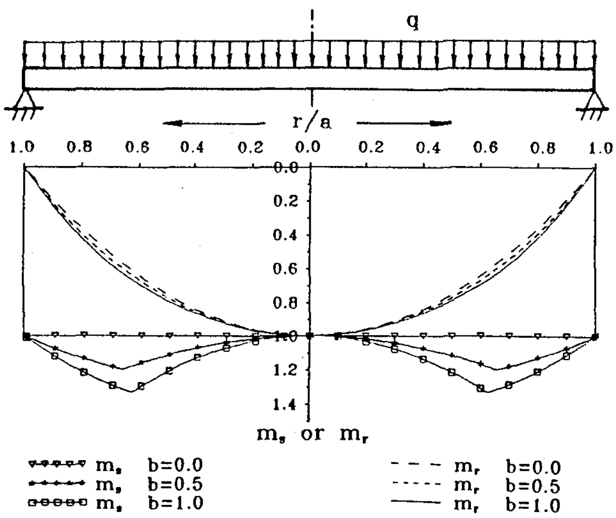
q satisfies

$$q=\frac{6+2b}{2+b}\frac{M_p}{r_0^2} \tag{11}$$

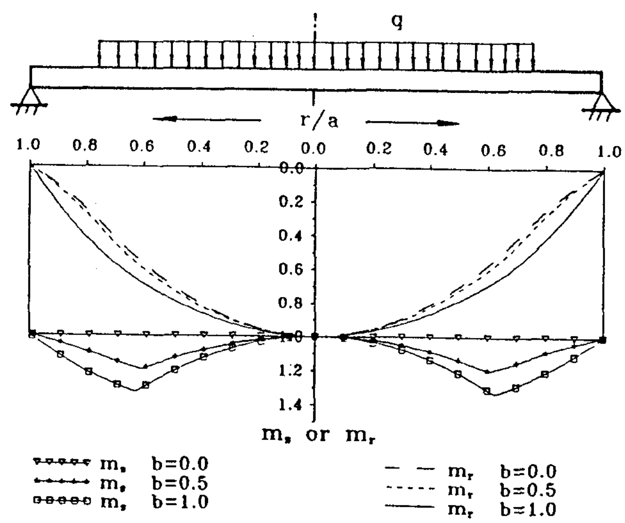
4. Field of internal moments

The field of internal moments can be derived from eqs. (8a) ~ (8c), (10a) ~ (10c) together with boundary and continuum conditions.

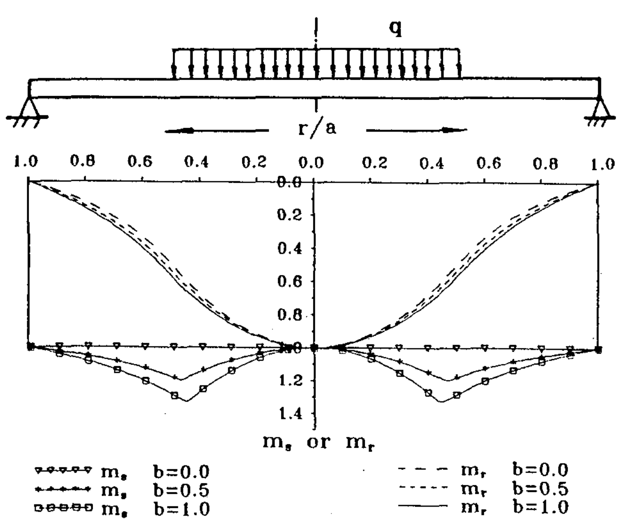
Assuming that $d_0=r_0=d$ when point G and point E in **Fig.3**(b) overlap, then fields of moment of the two cases are identical. The value of d_0 is obtained as $d_0=a/2^{1+b}$. When $d\le d_0$, point G is in the line segment DE and r_0 is determined solely according to eqs. (8a)~(8c) in the interval of (0,d], and we can get the limit loading and moment fields from the first case . Otherwise, when $d>d_0$, point G is in the line segement DE and eqs. (10a) ~ (10c) has only one solution of r_0 in the interval of (d,a], thus we can get



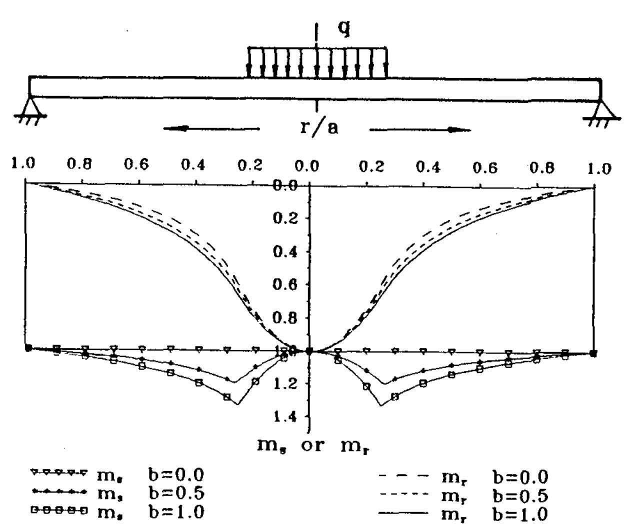
(a) d=a



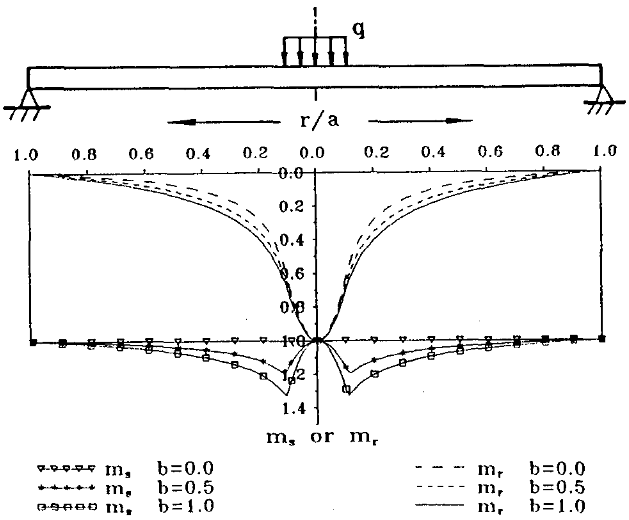
(b) d=0.75a



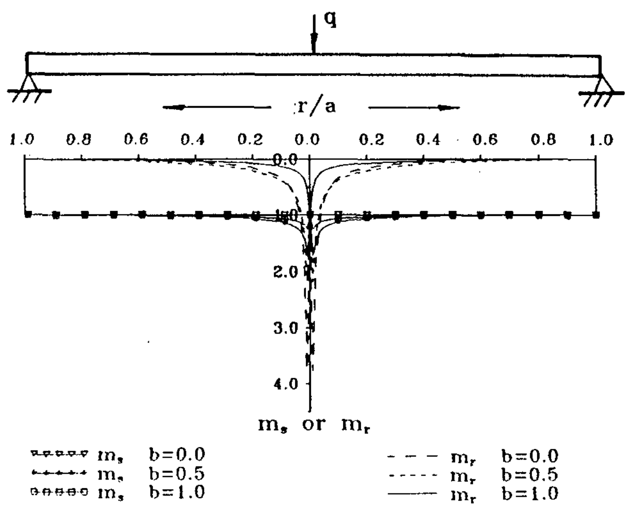
(c) d=0.5a



(d) d=0.25a



(e) d=0.1a



(f) d=0.00001a

Fig.4 Internal moment fields with different loading radius d, $(m_r = M_r / M_p, m_s = M_\theta / M_p)$

Table 1 Plastic limit loads q with values of d and b

qa^2 / M_p	d=1.0a	d=0.75a	d=0.5a	d=0.25a	d=0.1a	d=0.00001a
b=0.0 Tresca	6.0000	7.1111	12.000	38.400	214.29	2×10^{10}
b=0.5 Mises	6.4887	7.6666	12.886	40.901	224.70	2×10^{10}
b=1.0 Yu Mao-hong	6.8392	8.0638	13.509	42.669	232.69	2×10^{10}

the limit loading and moment fields from the second case. Moment fields with six different values of loading radius d (d=a, 0.75a, 0.5a, 0.25a, 0.1a, 0.00001a) are shown in **Fig.4**. We can see from **Fig.4** that moment field is dependent on material parameter b. For the same point of the plate, moments and plastic limit load increase with the increase of parameter b. **Table 1** gives out plastic limit loads q with specific values of d and b.

According to the hypothesis of Maximum principal stress condition or Tresca condition, M_0 equals M_p in the whole circular plate in spite of the change of loading radius d. That is unreasonable obviously. This case is similar to the special case of Unified Yield Criterion as b=0. When $0 < b \le 1, M_\theta$ varies with radius r and loading radius d. The varying tendency of M_0 is more reasonable than that when b=0.

When $d \to 0$, the problem is similar to the case of circular plate under concentrated load at the center. M_0 got from $0 < b \le 1$ is singular at the center. Thus Unified Yield Criterion can also reflect the moment singularity of the center of circular plate under concentrated load better.

5. Fields of Velocity

According to associated flow rule, there exist

$$\dot{k}_r = \dot{\lambda}\frac{\partial F}{\partial M_r},\ \dot{k}_\theta = \dot{\lambda}\frac{\partial F}{\partial M_\theta} \tag{12}$$

The relations of curvature rate and rate of deflection $\dot{w}$ are as follows

$$\dot{k}_r = -\frac{d^2\dot{w}}{dr^2},\ \dot{k}_\theta = -\frac{1}{r}\frac{d\dot{w}}{dr} \tag{13}$$

From eq.(7a), curvature rates in line segment AB in **Fig.2** become

$$\dot{k}_r = \frac{b}{1+b}\dot{\lambda},\ \dot{k}_\theta = \frac{1}{1+b}\dot{\lambda} \tag{14}$$

Substituting $\dot{k}_r, \dot{k}_\theta$ in eq.(13), we get the differential equation as follows,

$$\frac{1}{1+b}\frac{d^2\dot{w}}{dr^2} = \frac{b}{1+b}\frac{1}{r}\frac{d\dot{w}}{dr} \tag{15}$$

From eq.(15), the following formula is derived,

$$\dot{w}_1 = C_1\frac{r^{1+b}}{1+b} + C_2,\ 0 \le r \le r_0 \tag{16}$$

Using eq.(7b),curvature rates in line segment CB in **Fig.2** are obtained as

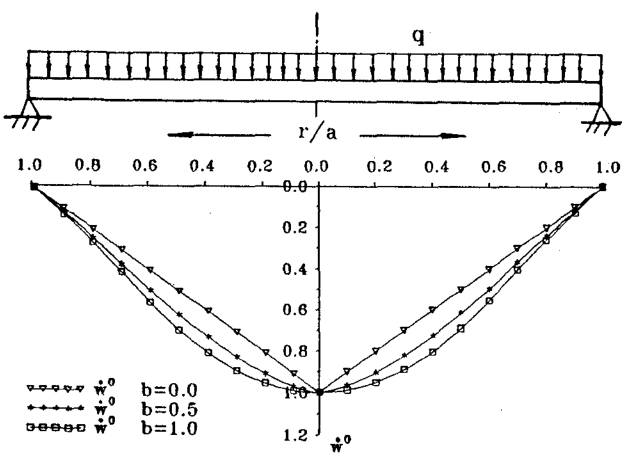
$$\dot{k}_r = -\frac{1}{1+b}\dot{\lambda},\ \dot{k}_\theta = \dot{\lambda}$$

Substituting it in eq.(13), we get another differential equation,

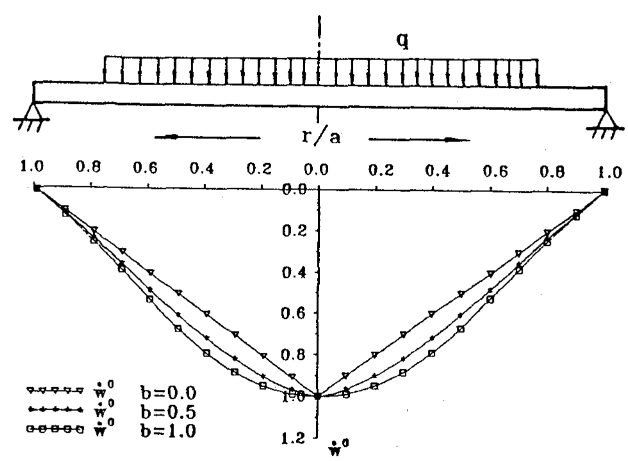
$$\frac{d^2\dot{w}}{dr^2} = -\frac{b}{1+b}\frac{1}{r}\frac{d\dot{w}}{dr} \tag{17}$$

From eq.(17), we get another formula of $\dot{w}$ as follows,

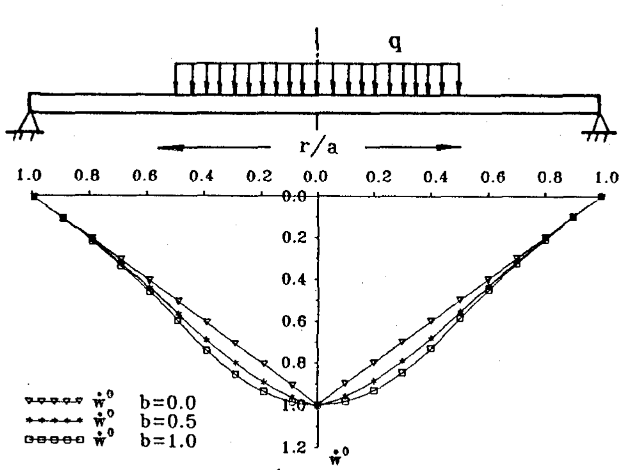
$$\dot{w}_2 = C_3(1+b)r^{\frac{1}{1+b}} + C_4,\ r_0 \le r \le a \tag{18}$$



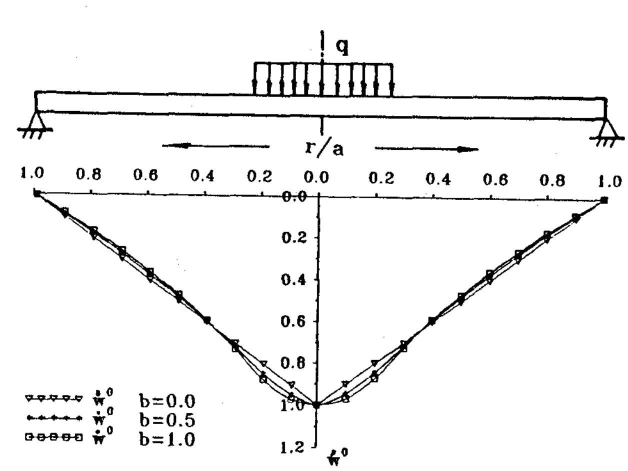
(a) d=a



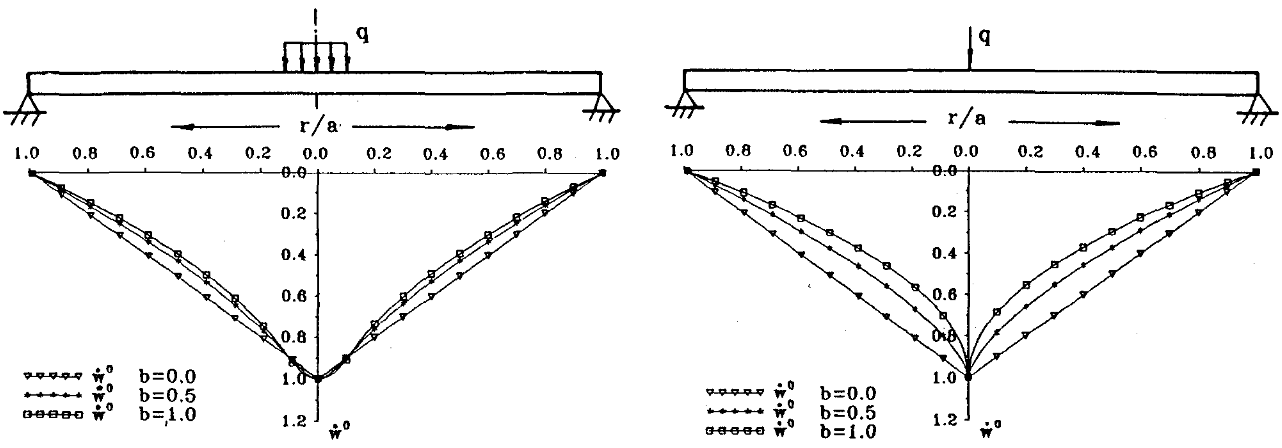
(b) d=0.75a



(c) d=0.5a



(d) d=0.25a



(e) d=0.1a

(f) d=0.00001a

Fig.5 Verocity fields with different loading radius d ($\dot{w}^0 = \dot{w} / \dot{w}_0$)

According to the following boundary conditions and continuum conditions,

$$\dot{w}_1|_{r=0} = \dot{w}_0, \dot{w}_1|_{r=r_0} = \dot{w}_2|_{r=r_0},$$
$$\frac{d\dot{w}_1}{dr}|_{r=r_0} = \frac{d\dot{w}_2}{dr}|_{r=r_0}, \dot{w}_2|_{r=a} = 0$$

We obtain the four parameters C_1, C_2, C_3, C_4. Substituting these parameters to eqs. (16), (18), we get the formulae of velocity field of circular plate as follows,

$$\dot{w} = \dot{w}_0 - \frac{\left(\frac{r_0}{a}\right)^{-\frac{2b+b^2}{1+b}} \dot{w}_0}{(1+b)^2 - (2b+b^2)\left(\frac{r_0}{a}\right)^{\frac{1}{1+b}}} \left(\frac{r}{a}\right)^{1+b}$$
$$\text{when } 0 \le r \le r_0 \qquad (19a)$$

$$\dot{w} = \frac{(1+b)^2 \dot{w}_0}{(1+b)^2 - (2b+b^2)\left(\frac{r_0}{a}\right)^{\frac{1}{1+b}}} \left[1 - \left(\frac{r}{a}\right)^{\frac{1}{1+b}}\right]$$
$$\text{when } r_0 \le r \le a \qquad (19b)$$

Because plastic limit load satisfies either equilibium conditions or limit conditions and compatible velocity field of deflection tolerated by mechanism motion can be got, the solution of plastic limit load given by this paper is complete solution. Velocity curves against six different values of loading radius d are planned in **Fig.5**. Obviously, velocity field with b=0 is not related to loading radius d, and it is not smooth at plate center. But velocity field got from $0 < b \le 1$ is related to the value of d, and it is smooth at the center. So it is more reasonable to illustrate the velocity field of limit state by using Unified Yield Criterion than by using Maximum principal stress criterion and Tresca criterion.

The results with $0 < b \le 1$ can also reflect the singularity of velovity field of circular plate under concentrated load at the center.

6. Comparison with the Existing Solutions

1).When b=0, we get the following same moment field, velocity field and plastic limit load from the two different cases,

$$\begin{cases} M_r = M_p\left[1 - \frac{ar^2}{d^2(3a-2d)}\right] \\ M_\theta = M_p \end{cases} \quad 0 \le r \le d \qquad (20a)$$

$$\begin{cases} M_r = M_p\left[1 - \frac{3r-2d}{3a-2d}\right] \\ M_\theta = M_p, \end{cases} \quad d \le r \le a \qquad (20b)$$

$$\dot{w} = \dot{w}_0(1 - \frac{r}{a}), \qquad (21)$$

$$q = \frac{6aM_p}{d^2(3a-2d)} \qquad (22)$$

They are identical to the solutions in Ref.1) by using Maximum principal stress criterion and Tresca criterion.

2).When d=a, the whole plate is under uniform load. **Fig.6** shows the relations of plastic limit load to parameter b.

From this fig., we can see that solutions of plastic limit loads in view of Maximum principal stress criterion, Tresca criterion (b=0), Twin Shear Stress Criterion (b=1)[7] are all special cases of solutions given in this paper. Solution by using Mises criterion can be approximated by that of Unified Yield Criterion of b=0.5. The maximum difference ratio of plastic limit loads caused by parameter b is 14%.

3).Let $P_l = \pi d^2 q$, i.e. P_l is total load of limit state. When $d \to 0$, that is the case of circular plate under concentrated load at the center. From the solution of the first case, it is deduced that $\lim_{d\to 0} P_l \equiv 2\pi M_p$ in spite of the variation of b. In Ref.[1], the value of P_l derived with Maximum principal stress criterion, Tresca criterion and Mises criterion is the same as $2\pi M_p$.

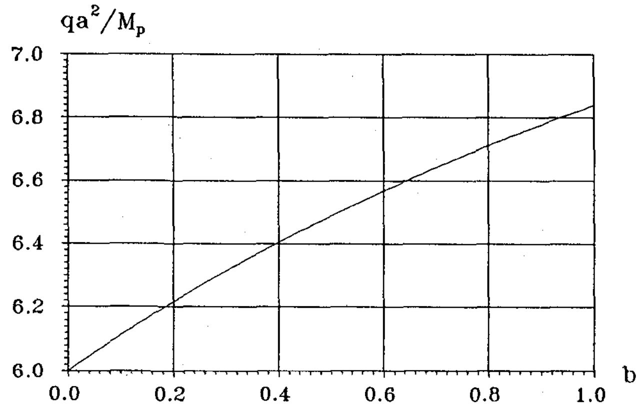


Fig.6 Plastic limit load of plate (b from 0 to 1)

7. Conclusions

1). In this paper, plastic limit analysis of circular plate under portion uniform load and concentrated load is for the first time studied in view of Twin Shear Stress Unified Yield Criterion. Complete solutions in unified form are derived. Existing solutions are all its special cases or in its linear proximity.

2). Maximum principal stress criterion, Tresca criterion can not reflect the influence of uniform load active radius on fields of M_θ and $\dot{w}$,and they can not reflect the singularity of M_θ and $\dot{w}$ at the center of circular plate under concentrated load, also velocity field at the center got with b=0 is not smooth. Application of Unified Yield Criterion can reasonably make up all the aforementioned shortcomings.

3). Unified Yield Criterion is suitable for many kinds of materials. With the variation of b, a series of different yield criteria are deduced. Accordingly, a series of different moment fields, velocity fields and plastic limit loads are derived.

4). In Unified Yield Criterion, the different value of b affects plastic limit load a lot, for example, in terms of Maximum principal stress criterion and Tresca criterion, i.e. b=0, limit load comes as the smallest, while in terms of Twin Shear Stress Criterion, i.e. b=1, limit load is the biggest. The maximum difference ratio between these limit loads is 14%. So it is of great significance to choose a proper yield criterion.

5). Unified Yeild Criterion is special case of Unified Strength Criterion as $\sigma_t = \sigma_c$. It is only suitable for metal materials. In later papers, members or structures of geomaterials e,g., concrete, rock and soil, will be analysized with Unified Strength Criterion.

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