Unified Elasto-Plastic Solution to Rotating Disc and Cylinder

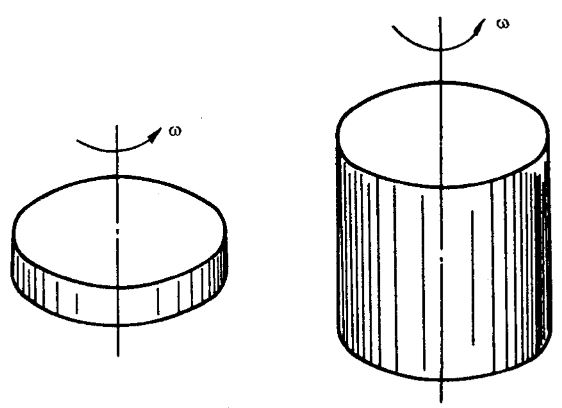
Ma Guo-wei*, Yu Mao-hong**,
Shoji Iwasaki***, Yutaka Miyamoto****, Hideaki Deto*****

*M. of Eng.,Dept. of Civil Eng., Xian Jiaotong University., Xian,710049,China
**Professor, Dept. of Civil Eng., Xian Jiaotong University., Xian,710049,China
***Dr. of Eng., Dept. of Civil and Environmental Eng., Iwate University,Ueda Morioka,Japan
****Professor, Dept. of Civil and Environmental Eng., Iwate University, Ueda Morioka, Japan
*****Dept. of Civil and Environmental Eng., Iwate University,Ueda Morioka,Japan

In this paper, Twin Shear Stress Unified Yield Criterion is employed for elasto-plastic analysis of rotating disc and cylinder. The existing solutions in view of Tresca Criterion, Mises Criterion and Twin Shear Stress Criterion are special cases of or in proximity to the results of this paper. This solution is suitable for a wide range of materials. From the analytic solution and illustrations in this paper, we can see that it is very important to choose reasonable yield criterion for structure design.
***Key Words**: rotating disc, rotating cylinder, elasto-plastic analysis, unified yield criterion.*

1. Introduction

Rotating disc and rotating cylinder, as shown in **Fig.1**(a) and (b), are often-used engineering members. For example, they are often used as the vane wheel and the rotating axle of propeller in civil engineering and mechanical engineering. When disc or cylinder rotates at angular velocity ω with increasing magnitude about axis, which is perpendicular to its plane and passes through the center, the stresses and displacements caused by centrifugal force are axisymmetric, i.e. σ_r, σ_θ and radial displacement u_r are only related to radius r. Rotating disc is in the generalized plane stress state, and rotating cylinder is in the generalized plane strain state. Ref.[1] employed Tresca Criterion for elasto-plastic analysis of rotating disc. It pointed out that analytic formula for elasto-plastic field of disc can not be obtained with Mises Criterion because of the nonlinearity of its formula, and Tresca Criterion has apparent shortcoming because it only considers the effect of two pricinpal stresses. In recent years, Yu[2)~5)] set up a new system of strength theory, which is called Twin Shear Stress Unified Strength Theory. The criterion applied in this paper, which is called Twin Shear Stress Unified Yield Criterion, is a part of it. Twin Shear Stress Yield Criterion considers effect of all the three principal stresses, it has clear physics conception and linear formula. Also, Tresca Criterion is special case of the unified yield criterion,



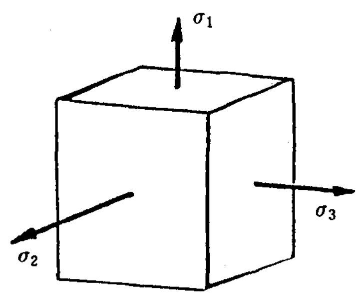
(a) Rotating disc (b) Rotating cylinder

Fig.1 Model

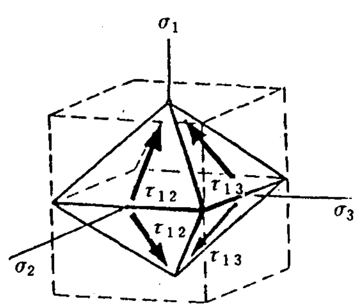
and Mises Criterion can be proximated by it. Solutions deduced with Twin Shear Stress Unified Yield Criterion are suitable for many kinds of metal materials and engineering structures.

2. Twin Shear Stress Unified Yield Criterion

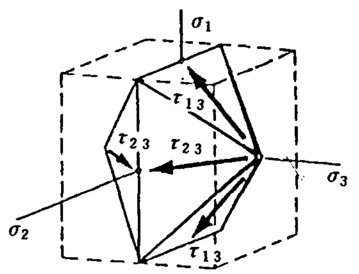
A physical interpretation of Twin Shear Stress Unified Yield Criterion may be established as follows. It is clear that there are three principal shear stresses τ_{13}, τ_{12} and τ_{23} in the three-dimensional principal stress state as shown in **Fig.2**(a). Here σ_1, σ_2, σ_3 are



(a) The three-dimensional principal stress state



(b) τ_{13}, τ_{12} adted on the orthogonal octahedron



(c) τ_{13}, τ_{12} adted on the orthogonal octahedron

Fig.2 Twin shear elements

principal stresses, and $\tau_{13} = 1/2(\sigma_1 - \sigma_3)$, $\tau_{12} = 1/2(\sigma_1 - \sigma_2)$, $\tau_{23} = 1/2(\sigma_2 - \sigma_3)$. Among τ_{13}, τ_{12}, τ_{23}, only two principal shear stresses are independent variables, and the maximum principal shear stress is equal to the sum of the other two , i. e. $\tau_{13} = \tau_{12} + \tau_{23}$. According to the twin shear idea, two twin shear element models of orthogonal octahedron are obtained as shown in **Fig.2**(b) and (c). Considering the two large principal shear stresses and their different effects on the yield of materials, formulae of the Twin Shear Stress Unified Yield Criterion are given as follows,

$$\tau_{13} + b\tau_{12} = C \quad \text{when} \quad \tau_{12} \ge \tau_{23} \qquad (1a)$$
$$\tau_{13} + b\tau_{23} = C \quad \text{when} \quad \tau_{12} \le \tau_{23} \qquad (1b)$$

Twin Shear Stres Unified Yield Criterion assumes that the yielding of materials begins when the sum of the largest principal shear stress and weighted intermediate principal shear stress $b\tau_{12}$ (or $b\tau_{23}$) resachs a magnitude C. C is material strength parameter. b is weighted coefficient that represents the effect of intermediate shear stress on the yield of materials and $0 \le b \le 1$. The magnitude of these constants can be determined by experimental results of triaxial test, i. e..

Formulae (1a), (1b) can be rewritten in term of principal stresses as follows,

$$\sigma_1 - \frac{1}{1+b}(b\sigma_2 + \sigma_3) = \sigma_s \qquad (2a)$$
$$\text{when} \quad \sigma_2 \le \frac{1}{2}(\sigma_1 + \sigma_3)$$
$$\frac{1}{1+b}(\sigma_1 + b\sigma_2) - \sigma_3 = \sigma_s \qquad (2b)$$
$$\text{when} \quad \sigma_2 \ge \frac{1}{2}(\sigma_1 + \sigma_3)$$

here σ_s is uniaxial yield strength of material.

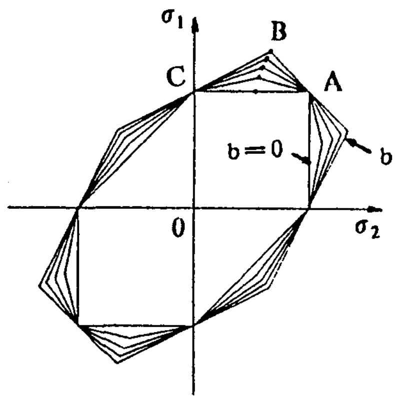
Fig.3(a) is limit trajectory of Twin Shear Stress Unified Yield Criterion in plane stress state,and **Fig.3**(b) is projection in π plane of limit surface of Twin Shear Stress Yield Criterion in three dimensional stress state. When b varies from 0 to 1,a family of convex yield criteria that are suitable for different kinds of metal materials are deduced.

The follwing criteria are special cases of Twin Shear Stress Unified Yield Criterion.

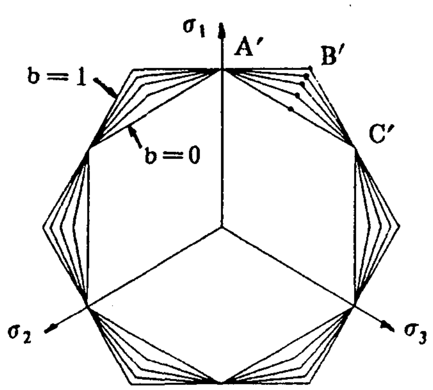
① b=1,Twin Shear Strress Yield Criterion, (suggested by Yu[2)]), can be adapted for the materials with shear yield stress $\tau_s = 2/3\sigma_s$, ② $b=\frac{1}{2}$, $b=\frac{1}{1+\sqrt{3}}$ Weighted twin shear stress yield criterions, adapted for that $\tau_s=0.6\sigma_s$, $\tau_s=0.577\sigma_s$

③ b=0, Tresca Criterion (single shear stress yield criterion), adapted for that $\tau_s=0.5\sigma_s$

The curve of Mises criterion in plane stress state is an ellipse and limit surface of it in three dimensional stress state is a cylinder. It interposes between the two curves of the unified yield criterion of b=0 and b=1. Criterions of ② can be approximated linearly to the Mises Criterion.



(a) Curves in plane stress state



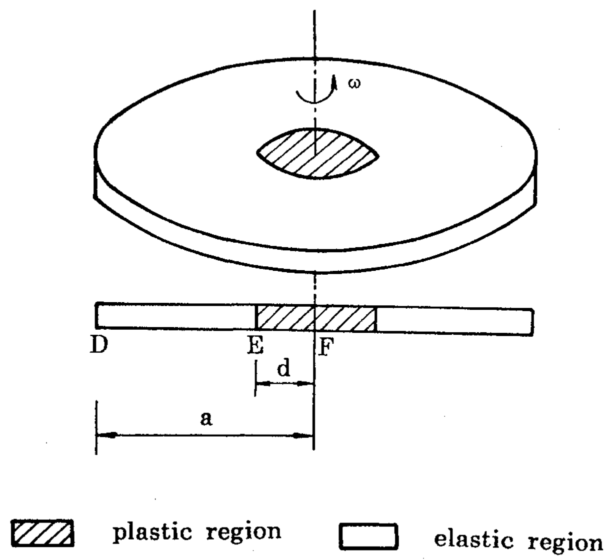
(b) Projection trajectories in three dimensional stress state

Fig.3 Twin Shear Stress Unified Yield Criterion

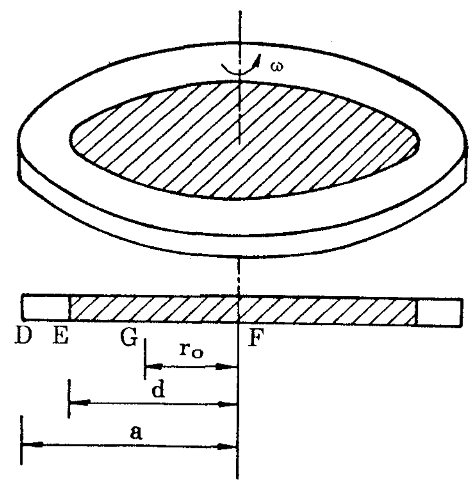
3. Basic Equations of Rotating Disc

Assuming that radius, thickness, material density of disc are respectively a, t, ρ, the stress state of each point satisfies $\sigma_1=\sigma_\theta$, $\sigma_2=\sigma_r$, $\sigma_3=\sigma_z=0$, when disc rotates at constant angular velocity ω. Equilibrium equation of rotating disc is

$$\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\theta}{r}+\rho\omega^2 r=0 \qquad (3)$$



(a) Radius of plastic zone is smaller



(b) Radius of plastic zone is bigger

Fig.4 Radius of plastic zone

With increase of rotating speed, yield of the disc starts from the center point (r=0). When disc is in elastic limit state, stresses at the center point satisfy $\sigma_r|_{r=0}=\sigma_\theta|_{r=0}=\sigma_s$. It is seen from **Fig.3**(a) that Tresca Criterion, Mises Criterion and Twin Shear Stress Unified Yield Criterion overlap at the same point A in this circumstance, thus elastic limit rotating speed ω_e deduced from these three criteria respectiviverly are the same as follows,

$$\omega_e=\frac{1}{a}\sqrt{\frac{8\sigma_s}{(3+\upsilon)\rho}} \qquad (4)$$

here υ is possion ratio.

When rotating disc is in .elasto-plastic state, stresses satisfy $\sigma_\theta \ge \sigma_r \ge \sigma_z = 0$, stress state in plastic region is on the two sides AB and BC in **Fig.3**(a). There are two possible cases of plasticity, i. e.,

(1) when radius of plastic zone is small, as shown in **Fig.4**(a), stress state of whole plastic zone is on side AB in **Fig.3**(a), stresses in elastic region only satisfy equation (3);

(2) when radius of plastic zone is bigger than a specific value of r_0, as shown in **Fig.4**(b), stress state of plastic zone is on sides AB and BC. Here G in **Fig.4**(b) is supposed to be corresponding to point B in **Fig.3**(a). Stresses in elastic region only satisfy eqaution (3), too.

In the special case that point E and point G in **Fig.4**(b) overlap, the first case and the second case are identical. That is the demarcating state of the two cases. Equations of line segments AB, BC in **Fig.3**(a) respectively are

$$\text{AB:}\quad \frac{b}{1+b}\sigma_r + \frac{1}{1+b}\sigma_\theta = \sigma_s \tag{5a}$$

$$\text{BC:}\quad \sigma_\theta - \frac{b}{1+b}\sigma_r = \sigma_s \tag{5b}$$

Boundary conditions and continuum conditions corresponding to the first case are

a) $\sigma_r|_{r=a} = 0$ at point D,

b) $\sigma_r|_{r=d}$ and $\sigma_\theta|_{r=d}$ are continuous at point E,

c) $\sigma_r|_{r=0}$ is a definite value at point F.

Boundary conditions and continuum conditions corresponding to the second case are

d) $\sigma_r|_{r=a} = 0$ at point D,

e) $\sigma_r|_{r=d}$ and $\sigma_\theta|_{r=d}$ are continuous at point E,

f) $\sigma_r|_{r=0}$ is a definite value at point F,

g) $\sigma_r|_{r=r_0}$ and $\sigma_\theta|_{r=r_0}$ are continuous at point G,

h) $\sigma_\theta|_{r=r_0} = 2\sigma_r|_{r=r_0}$ at point G.

4. Elasto-Plastic Stress Field of Rotating Disc

1) In the first case (**Fig.4**(a)), stresses in elastic region, plastic region deduced from equilibrium equation (3), limit condition (5a) and conditions a)~c) are

$$\begin{cases} \sigma_r = \sigma_s - \dfrac{\rho\omega^2}{3+b} r^2 \\ \sigma_\theta = \sigma_s + \dfrac{b\rho\omega^2}{3+b} r^2 \end{cases} \quad \text{when}\quad 0 \le r \le d \tag{6}$$

$$\begin{cases} \sigma_r = \sigma_s - \rho\omega^2 (C_1 d^2 + C_2 \dfrac{d^4}{r^2} + C_3 r^2) \\ \sigma_\theta = \sigma_s - \rho\omega^2 (C_1 d^2 - C_2 \dfrac{d^4}{r^2} + C_4 r^2) \end{cases} \tag{7}$$

$$\text{when}\quad d \le r \le a$$

where

$$C_1 = \frac{1-b}{6+2b} - \frac{1+\upsilon}{4},\ C_2 = \frac{1+b}{6+2b} - \frac{1-\upsilon}{8},$$
$$C_3 = \frac{3+\upsilon}{8},\ C_4 = \frac{1+3\upsilon}{8}$$

When the values of parameter b and Possion ratio υ are given, C_1, C_2, C_3 and C_4 are determined, so we can get the stresses in elastic region from formula (7). And ω in (6), (7) satisfies,

$$\frac{\rho a^2}{\omega^2 \sigma_s} = C_2\left(\frac{d}{a}\right)^4 + C_1\left(\frac{d}{a}\right)^2 + C_3 \tag{8}$$

2) In the second case (**Fig.4**(b)), stresses deduced from equilibrium equation (3), limit condition (5a), (5b) and conditions d)~f) are

$$\begin{cases} \dfrac{\sigma_r}{\sigma_s} = 1 - \dfrac{1}{2+b}\left(\dfrac{r}{r_0}\right)^2 \\ \dfrac{\sigma_\theta}{\sigma_s} = 1 + \dfrac{b}{2+b}\left(\dfrac{r}{r_0}\right)^2 \end{cases} \quad \text{when}\ 0 \le r \le r_0 \tag{9}$$

$$\begin{cases} \dfrac{\sigma_r}{\sigma_s} = C_5 - C_6\left(\dfrac{r_0}{r}\right)^{\frac{1}{1+b}} - C_7\left(\dfrac{r}{r_0}\right)^2 \\ \dfrac{\sigma_\theta}{\sigma_s} = C_5 - \dfrac{bC_6}{1+b}\left(\dfrac{r_0}{r}\right)^{\frac{1}{1+b}} - \dfrac{bC_7}{1+b}\left(\dfrac{r}{r_0}\right)^2 \end{cases} \tag{10}$$

$$\text{when}\quad r_0 \le r \le d$$

$$\begin{cases} \dfrac{\sigma_r}{\sigma_s} = C_5 - C_8\left(\dfrac{r}{r_0}\right)^2 - C_9\left(\dfrac{r_0}{d}\right)^{\frac{1}{1+b}} - C_{12}\left(\dfrac{d}{r_0}\right)^2 \\ \quad -C_{10}\left(\dfrac{d}{r}\right)^2\left(\dfrac{r_0}{d}\right)^{\frac{1}{1+b}} + C_{11}\left(\dfrac{d}{r_0}\right)^2\left(\dfrac{d}{r}\right)^2 \\ \dfrac{\sigma_\theta}{\sigma_s} = C_5 - C_{13}\left(\dfrac{r}{r_0}\right)^2 - C_9\left(\dfrac{r_0}{d}\right)^{\frac{1}{1+b}} + C_{12}\left(\dfrac{d}{r_0}\right)^2 \\ \quad +C_{10}\left(\dfrac{d}{r}\right)^2\left(\dfrac{r_0}{d}\right)^{\frac{1}{1+b}} - C_{11}\left(\dfrac{d}{r_0}\right)^2\left(\dfrac{d}{r}\right)^2 \end{cases} \tag{11}$$

$$\text{when}\quad d \le r \le a$$

And ω can be obtained from the following furmula,

$$\omega = \sqrt{\frac{3+b}{2+b}\frac{\sigma_s}{\rho}\frac{1}{r_0}} \tag{12}$$

r_0 satisfies

$$f(r_0) = \frac{3+b}{2+b}C_{11}\left(\frac{d}{a}\right)^4 - C_{14}\left(\frac{r_0}{d}\right)^{\frac{1}{1+b}}\left(\frac{d}{a}\right)^2\left(\frac{r_0}{a}\right)^2 + C_{15}\left(\frac{r_0}{a}\right)^2 + \frac{2+b}{3+b}C_1\left(\frac{d}{a}\right)^2 - C_{16}\left(\frac{r_0}{d}\right)^{\frac{1}{1+b}}\left(\frac{r_0}{a}\right)^2 - C_3 \tag{13}$$

here

$$C_5 = 1+b,\ C_6 = \frac{2b(1+b)}{3+2b},$$
$$C_7 = \frac{(1+b)(3+b)}{(3+2b)(2+b)},\ C_8 = \frac{3+\upsilon}{8}\frac{2+b}{3+b},$$
$$C_9 = \frac{b(1+2b)}{3+2b},\ C_{10} = \frac{b}{3+2b},$$
$$C_{11} = \left(\frac{1-\upsilon}{8} - \frac{1}{6+4b}\right)\frac{2+b}{3+b},$$
$$C_{12} = \left(\frac{1+\upsilon}{4} - \frac{1+2b}{6+4b}\right)\frac{3+b}{2+b},$$
$$C_{13} = \frac{1+3\upsilon}{8}\frac{2+b}{3+b},\ C_{14} = \frac{b(2+b)}{(3+2b)(3+b)},$$
$$C_{15} = \frac{(1+b)(2+b)}{3+b},\ C_{16} = \frac{b(1+2b)(2+b)}{(3+2b)(3+b)}$$

5. Solution Procedure and Results

Solution procedure of stress field contains the following two steps,

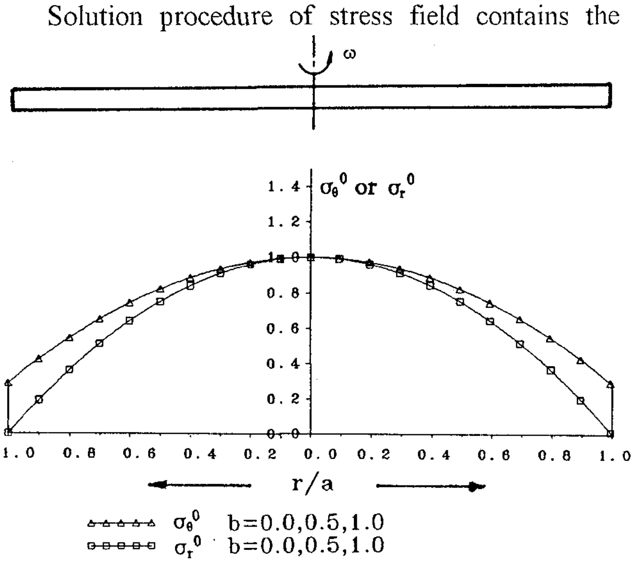
1) determination of the radius of plastic zone d_0 of the special state demarcating the two different cases. In this demarcating state, points E, G overlap, and $d_0 = d = r_0$.

Assuming $f_0(d_0) = f(r)|_{r=r_0}$, d_0 satisfies

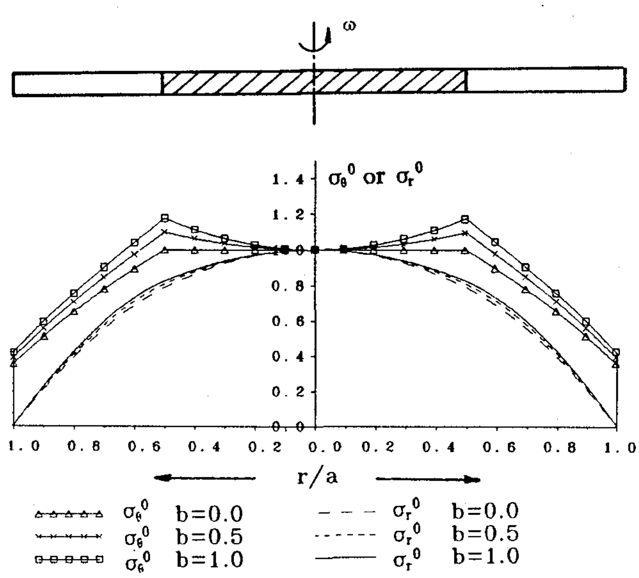
$$f_0(d_0) = \left(\frac{1-\upsilon}{8} - \frac{1+b}{6+2b}\right)\left(\frac{d_0}{a}\right)^4 + \left[\frac{1+\upsilon}{4} + \frac{3(1+b)}{2(3+b)}\right]\left(\frac{d_0}{a}\right)^2 - \frac{3+\upsilon}{8} = 0 \tag{14}$$

2) when radius of plastic zone d is specific, and $d \le d_0$, limit rotating speed ω and σ_r, σ_θ can be deduced from formulae (6), (7). When $d_0 \le d \le a$, r_0 are deduced from formula (11), then substituting r_0 in formulae (9), (10), σ_r, σ_θ and elasto-plastic limit rotating speed ω can be derived from formulae (9), (10).

The deduction process presented above can converge quickly, and then the elasto-plastic limit rotating velocity and centrifugal stress field are solely obtained. **Fig.5** are stress fields with different radius of plastic zone d. **Fig.6** shows the relation of angular velocity to plastic zone d. **Fig.7** is geometrical representation of plastic limit angular velocity ω_p of circular disc against material parameter b. In elasto-plastic state, stresses of each point of the disc are variable with parameter b. When b=0 (Tresca criterion), the derived stress is the smallest, and when b=1 (corresponding to Twin



(a) d=0.0



(b) d=0.5a

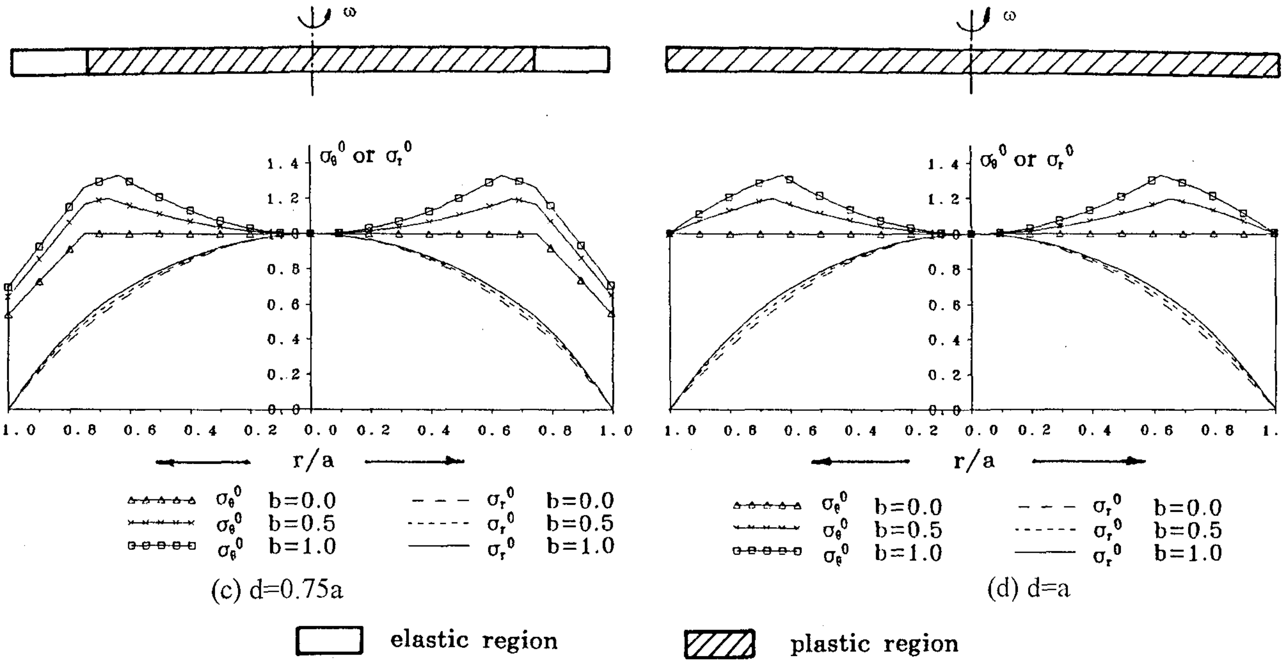


Fig.5 Stress fields with different plastic regions, $\upsilon = 0.5, \sigma_\theta^0 = \sigma_\theta/\sigma_s$, $\sigma_r^0 = \sigma_r/\sigma_s$ (b=0.0, Tresca criterion; b=0.5, approximity of Mises criterion; b=1.0, Twin Shear Stress criterion)

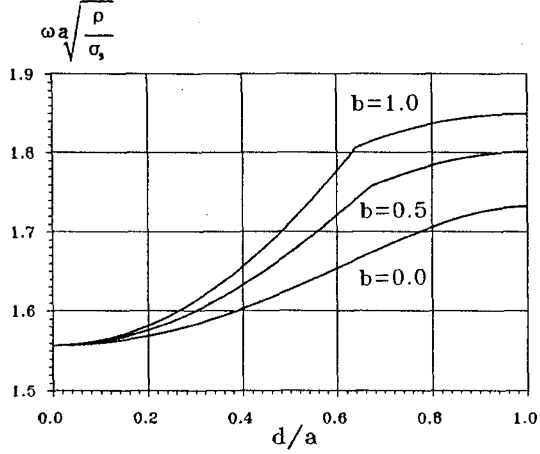


Fig.6 Relation of angular velocity to the radius of plastic zone

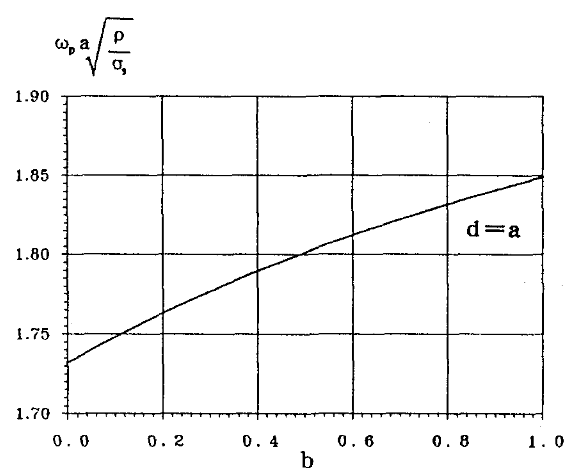


Fig.7 Relation of b to plastic limit angular velocity ω_p

Shear Stress Yield Criterion), the deduced stress is the biggest[6),7)]. When b=0, there exists $\sigma_\theta \equiv \sigma_s$ in the plastic zone of the derived elasto-plastic stress field, but when $b \neq 0$, σ_θ in the plastic part of elasto-plastic stress field derived from Twin Shear Stress Unified Yield Criterion and Mises Criterion are bigger than σ_θ. The bigger the extent of plastic zone, the greater the effect of value of b on the stress field.

6. Plastic Limit Analysis of Rotating Cylinder

Elastic solution of non-compressive perfect elasto-plastic rotating cylinder is as follows[1)],

$$\begin{cases} \sigma_r = \sigma_0 = \frac{1}{2}\rho\omega^2\left(a^2 - r^2\right) \\ \sigma_z = \frac{1}{2}\rho\omega^2\left(\frac{1}{2}a^2 - r^2\right) \end{cases} \tag{15}$$

It is seen from formula (15) that there exists $\sigma_1 = \sigma_2 = \sigma_r = \sigma_0, \sigma_3 = \sigma_z$. Stress state is situated at point A' in **Fig.3**(b). Stress state deduced from Tresca Criterion, Mises Criterion and Twin Shear Stress Criterion respectively overlap at this point. Formula of this stress state is

$$\sigma_\theta - \sigma_z = \sigma_s \tag{16}$$

Substituting formula (16) in (15), we get

$$\omega_p = \omega_e = \frac{2}{a}\sqrt{\frac{\sigma_s}{\rho}} \tag{17}$$

So elastic limit rotating velocity equals plastic limit rotating velocity for circular cylinder, i.e., when $\omega = \omega_e$, each point in the cylinder yields simultaneously, and the limit velocity deduced in view of Tresca Criterion, Mises Criterion and Twin Shear Stress Yield Criterion respectively is the same.

Substituting ω_e in formula (15), we can get the similar centrifugal stress field as follows,

$$\begin{cases} \sigma_r = \sigma_\theta = 2\sigma_s\left(1 - \dfrac{r^2}{a^2}\right) \\ \sigma_z = 2\sigma_s\left(\dfrac{1}{2} - \dfrac{r^2}{a^2}\right) \end{cases} \tag{18}$$

7. Conclusions

(1) In this paper, Elasto-plastic stress field and elasto-plastic rotating speed of rotating disc and rotating cylinder are studied for the first time by using Twin Shear Stress Unified Yield Criterion. Existing solutions are all the special cases of or in proximity to the results of this paper.

(2) A series of different yield criteria can be derived by choosing different value of b in Twin Shear Stress Unified Yield Criterion. b has no effect on elastic limit rotating speed of rotating disc and rotating cylinder, but it affects plastic limit speed a lot.

(3) The difference ratio of maximum plastic limit rotating speed, derived with Twin Shear Stress Criterion (b=1) to minimum plastic limit rotating speed deduced with Tresca Criterion (b=0) is 14%. So in view of economy, we should choose the yield criterion carefully in design.

(4) Twin Shear Stress Unified Yield Criterion used in this paper, is special case of Twin Shear Stress Unified Strength Criterion. It is only suitable for metal materials. In future, members or structures consisting of concrete, rock and soil, will be analysized with Twin Shear Stress Unified Strength Criterion. Thus a new system of unified strength theory, which has concise physical concept and simple mathematical formula, may be applied to a wide range of material nonlinear analysis by finite element method.

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