# THE CCS METHOD FOR TWO-DIMENSIONAL ADVECTION AND DIFFUSION EQUATION

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## 1. INTRODUCTION

In the past several decades, the numerical solutions of advection and diffusion equation have received considerable attention and found extensive applications in engineering. The diverse numerical methods for solving the equation had been developed. Tests of these methods found in the literature had revealed that in the case of small Peclet number, satisfactory solution can be obtained by most of methods. However, when advective term dominates the equation, first order methods such as FTCS and common first order upwind method will suffer from considerable numerical diffusion; and second order methods such as Lax-Wendroff method and QUICK method can reduce the diffusion to some extent, but frequently introduce excessive unphysical oscillation.

In this paper, the combined cubic spline method (the CCS method) for two-dimensional advection and diffusion problems is developed by introducing a weight factor $\theta$ in the treatment of time derivative term. The performance of this method is demonstrated by a test example.

## 2. THE CCS METHOD

The governing equation for two-dimensional advection and diffusion problem is

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}=Dx\frac{\partial^2 f}{\partial x^2}+Dy\frac{\partial^2 f}{\partial y^2} \tag{1}$$

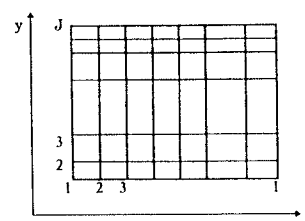


Fig.1 Sketch of mesh for 2-D CCS scheme

If the time derivatives are expressed by a combined algorithm, and the spatial derivatives are approximated by a series of cubic splines passing through grid points $\{x_1,y_j\},\{x_2,y_j\},\ldots\ldots,\{x_I,y_j\}$, j=1,J for x direction and $\{x_i,y_1\},\{x_i,y_2\},\ldots\ldots,\{x_i,y_J\}$, i=1,I for y direction(see Fig.1), the governing equation (1) is transformed to

$$\begin{aligned}\frac{f_{i,j}^{n+1}-f_{i,j}^{n}}{\Delta t}&=\theta\left(Dx_{i,j}^{n}Mx_{i,j}^{n}+Dy_{i,j}^{n}My_{i,j}^{n}-u_{i,j}^{n}Nx_{i,j}^{n}-v_{i,j}^{n}Ny_{i,j}^{n}\right)+\\&(1-\theta)\left(Dx_{i,j}^{n+1}Mx_{i,j}^{n+1}+Dy_{i,j}^{n+1}My_{i,j}^{n+1}-u_{i,j}^{n+1}Nx_{i,j}^{n+1}-v_{i,j}^{n+1}Ny_{i,j}^{n+1}\right)\end{aligned} \tag{2}$$

in which $Nx_{i,j}^n=\left.\frac{\partial f}{\partial x}\right|_{i,j}^{n}$ ; $Mx_{i,j}^n=\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j}^{n}$; $Ny_{i,j}^n=\left.\frac{\partial f}{\partial y}\right|_{i,j}^{n}$ ; $My_{i,j}^n=\left.\frac{\partial^2 f}{\partial y^2}\right|_{i,j}^{n}$; $\theta$ is a weight coefficient, for explicit scheme $\theta=1$, for fully implicit scheme $\theta=0$, for the scheme similar to Crank-Nicolson $\theta=0.5$.

For the cubic spline interpolation passing through points $\{x_1,y_j\},\{x_2,y_j\},\ldots\ldots,\{x_I,y_j\}$, we can obtain[1]

$$\frac{\Delta x_{i-1}}{6}Mx_{i-1,j}^n+\frac{\Delta x_{i-1}+\Delta x_i}{3}Mx_{i,j}^n+\frac{\Delta x_i}{6}Mx_{i+1,j}^n=\frac{f_{i+1,j}^n-f_{i,j}^n}{\Delta x_i}-\frac{f_{i,j}^n-f_{i-1,j}^n}{\Delta x_{i-1}}\quad (i=2,I-1) \tag{3}$$

where $\Delta x_i=x_{i+1}-x_i$. $Mx_{i,j}^n(i=2,I-1)$ can be determined by Eq.(3) and boundary conditions. $Nx_{i,j}^n(i=1,I)$ can be calculated from

$$Nx_{1,j}^{n} = -\left(\frac{Mx_{1,j}^{n}}{3} + \frac{Mx_{2,j}^{n}}{6}\right)\Delta x_1 + \frac{f_{2,j}^{n} - f_{1,j}^{n}}{\Delta x_1} \qquad (4)$$

$$Nx_{i,j}^{n} = \left(\frac{Mx_{i,j}^{n}}{3} + \frac{M_{i-1,j}^{n}}{6}\right)\Delta x_{i-1} + \frac{f_{i,j}^{n} - f_{i-1,j}^{n}}{\Delta x_{i-1}} \qquad (i=2,I) \qquad (5)$$

Similarly, we obtain $My_{i,j}^{n}$ and $Ny_{i,j}^{n}$ by the cubic spline interpolation passing through grid points $\{x_i, y_1\}, \{x_i, y_2\}, \ldots\ldots, \{x_i, y_J\}$.

Finally, $f_{i,j}^{n+1}(i = 1, I; j = 1, J)$ are obtained from Eq.(2).

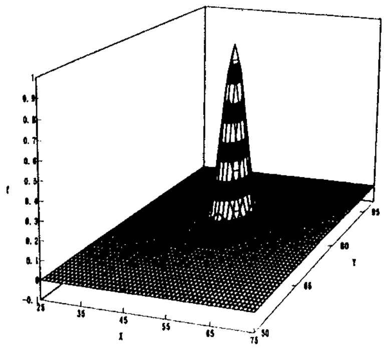

Fig.2 Initial distribution

### 3. A COMPUTATIONAL EXAMPLE AND THE PERFORMANCES OF THE CCS METHOD

We consider a pure advection problem with initial distribution of a Gaussian hill centered at $x_c = 500, y_c = 750$, for the computational domain of a square with 100×100 cells and △x=△y=10(Fig.2). The flow field is given by $V_x = -\omega(y - y_0)$ and $V_y = \omega(x - x_0)$, where $x_0 = 500, y_0 = 500, \omega = 2\pi / 628$.

Since it is a pure advection problem, the distribution should remain constant throughout the rotation. Therefore we can measure the accuracy of a method by its ability to transport the concentration distribution without deformation. The result after a revolution(t=628 s) by the CCS method with $\theta$ =0.5 is shown in Fig.3. For the purpose of comparison, the result by time-splitting 2-D CIP method[2] is shown in Fig.4. It is found that the CCS method has a much smaller undershooting. The peak value is reduced by 4% for the CCS method and 20% for the time-splitting CIP method, respectively. It is also revealed that the CCS method is almost free from phase error and less numerical oscillation.

Further studies show that, in most cases, the accuracy of the CCS method is strongly influenced by the value of $\theta$. It is found that when $\theta$ =0.45 ~0.50 , the CCS method perform best, while a too small value of $\theta$ may lead to excessive diffusion, and in contrast a too large value may lead to unphysical oscillation. On the whole, the CCS method, due to its remarkable accuracy and simplicity along with the ability to be easily extended to multi-dimensional problems, has been found to be advantageous in solving the advection and diffusion equation.

### REFERENCE


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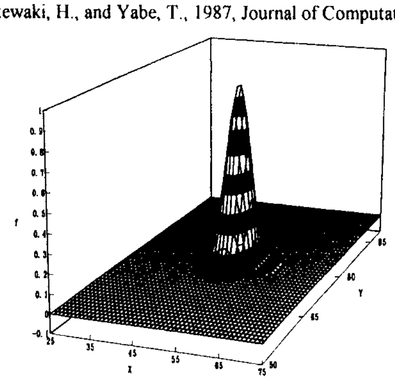

Fig.3 The CCS scheme

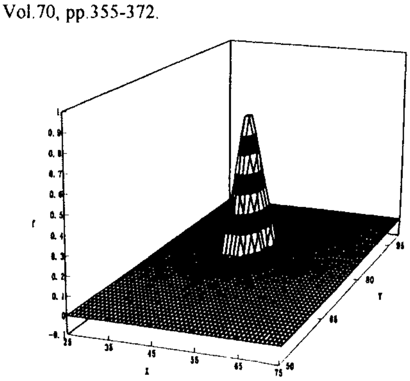

Fig.4 The time-splitting CIP scheme