EFFICIENT AND ACCURATE NUMERICAL SCHEME FOR ONE-DIMENSIONAL INFILTRATION

九州工業大学工学部 学生員 Le Dinh HONG
正 員 秋山 壽一郎
正 員 浦 勝

1. INTRODUCTION: From the viewpoint of computational efficiency, accuracy of mass conservation, and flexible applicability to all practical situations, Richards' equation in the 'conservative' form is solved by the finite-difference method, incorporating the Newton-Raphson scheme. Additionally, an 'updating' coefficient is introduced to further enhance the convergence rate.

2. THEORY: The one-dimensional Richards' equation in the 'conservative' form can be rewritten as

$$\partial\theta/\partial t - \partial[K(\partial\Psi/\partial z - 1)]/\partial z = 0 \quad \ldots\ldots (1)$$

where $\Psi(z,t)$ is the pressure head; $\theta(\Psi)$ the volumetric moisture content; $K(\Psi)$ the hydraulic conductivity; t the time; and z the depth oriented positively downward.

The finite-difference form of Eq. (1) written for cell m is

$$\omega\Delta t/\Delta z^2\,[-K^{n+1}_{m+1/2}(\Psi^{n+1}_{m+1} - \Psi^{n+1}_{m} - \Delta z) + K^{n+1}_{m-1/2}(\Psi^{n+1}_{m} - \Psi^{n+1}_{m-1} - \Delta z)]$$

$$+\ \theta^{n+1}_{m} - \theta^{n}_{m} + (1-\omega)\Delta t/\Delta z\,[q^{n}_{m+1/2} - q^{n}_{m-1/2}] = 0 \quad .. (2)$$

where ω is a time-weighting coefficient, $q_{m+1/2}$ and $q_{m-1/2}$ the average fluxes across the interfaces $m+1/2$ and $m-1/2$, respectively; n previous time level; $n+1$ current time level.

Denoting the left-hand side of Eq. (2) as a function R, Taylor series expansion of Eq. (2) about an assumed solution yields

$$R^{n+1,r+1} = R^{n+1,r} + \Sigma\,\partial R/\partial\Psi_j|^{n+1,r}\delta\Psi_j = 0 \quad \ldots.. (3)$$

where $j = m-1$, m and $m+1$; r denotes the iteration level; $\delta\Psi = \Psi^{n+1,r+1} - \Psi^{n+1,r}$ are the unknowns to progress from the known values at iteration r to the next unknown values at iteration r+1 at the same time level.

Using the arithmetic average for the internodal hydraulic conductivities, the linearized form of Eq.(3) written for a particular cell m (Fig. 1) has the following form:

$$E^{n+1,r}_{m}\delta\Psi_{m-1} + F^{n+1,r}_{m}\delta\Psi_{m} + G^{n+1,r}_{m}\delta\Psi_{m+1} + H^{n+1,r}_{m} = 0 \quad (4)$$

To further enhance the convergence rate of the Newton-Raphson iterative scheme, an 'updating' coefficient η will be introduced at the start of each new time step as follows: let $n + 1$ be replaced by n, and at the first iteration level ($r = 0$) the value of $\Psi_m^{n+1,0}$ is set by $\Psi_m^{n} + \eta(\Psi_m^{n} - \Psi_m^{n-1})$.

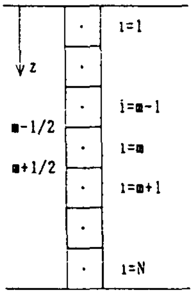

Fig. 1. Finite-difference grid.

3. ILLUSTRATIVE EXAMPLES: The calculated parameters used by Cooley[1] and Celia et al.[2] are summarized in Table 1.

(a). Example 1 (1-D Vertical Infiltration into Moderately Dry Soil)

Fig. 2 presents the profiles of moisture content normalized by the saturated moisture content obtained by Cooley[1] and our simulation. From time 1.2 hr the theoretical downward velocity of the established front is 12.75 cm/hr. The computed velocity given by Cooley is 12.43 cm/hr, i.e., 2.53% lower than the theoretical velocity. The computed velocity by our simulation is 12.84 cm/hr, i.e., 0.65 % higher than the theoretical velocity. Fig. 3 presents the number of iterations per time step required in the present model and the effect of the updating coefficient η upon the convergence rate. It must be noted that Cooley's model required at least 53 iterations per time step.

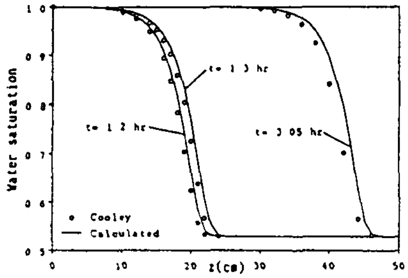

Fig. 2. Comparison of water saturation profiles computed by Cooley and our simulation.

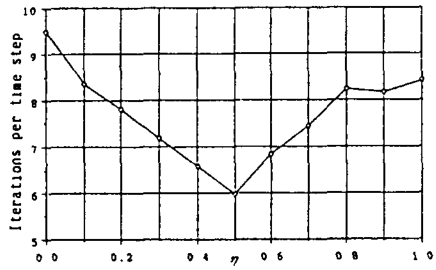

Fig. 3. Effect of η on the convergence rate.

(b). Example 2 (1-D Vertical Infiltration into Dry Soil)

Fig. 4 shows good agreement in the pressure head profiles between Celia et al.'s[2] and our result. In Fig. 5, the number of iterations required per time step during the first 20 time steps is plotted. In our simulations, a minimum of two iterations per time step was imposed in order to obtain a stable numerical solution. Fig. 6 presents the effect of the updating coefficient η upon the convergence rate (the results are normalized by the number of iterations calculated with coefficient $\eta = 0$). It can be seen that the coefficient η improves substantially the convergence rate for large Δt.

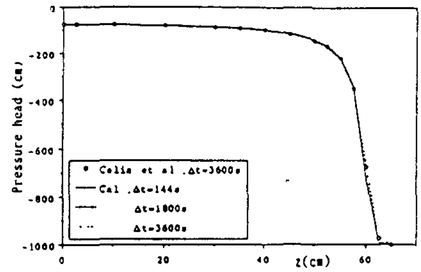

Fig. 4. Comparison of pressure head profiles computed by Celia et al. and our simulation.

4. CONCLUSIONS

(a). The conservative form of Richards' equation is used to ensure the mass balance and to circumvent the difficulty of determining an appropriate value of the specific capacity encountered by using the 'Ψ-based' form[3,4].

(b). The Newton-Raphson scheme for solving the conservative form of Richards' equation is more efficient in computation than the Newton-like, i.e., modified-Picard schemes.

(c). The effect of the updating coefficient η on the convergence rate may be influenced by several factors. However, $\eta = 0.5$ can be used to at least enhance the convergence rate of the proposed scheme.

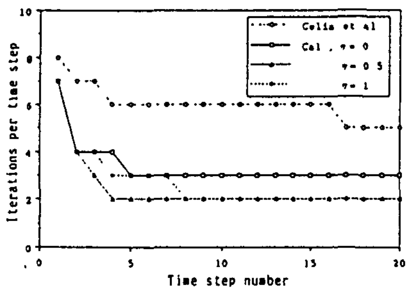

Fig. 5. Number of iterations per time step for Δt = 100 s required by Celia et al.'s method and our simulation.

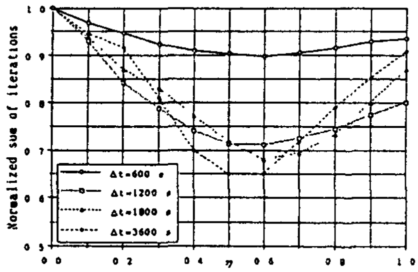

Fig. 6. Relation between the normalized sum of iterations and the updating coefficient η.

Table 1. Calculation conditions.

Example 1: Cooley[1] Richards' Eq. in Ψ-based form Newton-like iterative scheme	Example 2: Celia et al.[2] Richards' Eq. in conservative form Newton-like iterative scheme
$\theta = \theta_s(5.4/\lvert\Psi\rvert)^{0.2}$ $\Psi \leq -5.4$ cm $K = K_s(5.4/\lvert\Psi\rvert)^{2.6}$ $\Psi \leq -5.4$ cm $\theta = \theta_s = 0.52$ $\Psi \geq -5.4$ cm $K = K_s = 3.125$ cm/hr $\Psi \geq -5.4$ cm	$\theta(\Psi) = \theta_r + \dfrac{\theta_s - \theta_r}{[1 + (\alpha\lvert\Psi\rvert)^n]^m}$ $K(\Psi) = K_s \dfrac{\{1 - (\alpha\lvert\Psi\rvert)^{n-1}[1 + (\alpha\lvert\Psi\rvert)^n]^{-m}\}^2}{[1 + (\alpha\lvert\Psi\rvert)^n]^{m/2}}$
$\Psi(z,t=0) = -130.54$ cm $\Psi(z=0,t>0) = -5.4$ cm $\Psi(z=49,t>0) = -130.54$ cm Length of column L = 49 cm $\Delta z = 1$ cm, $\Delta t = 0.1$ hr Convergence tolerance $\varepsilon = 0.001$(cm) Time-weighting coefficient $\omega = 1$	$\alpha = 0.0335$, $n = 2$, $m = 1 - 1/n$ $\theta_s = 0.368$, $\theta_r = 0.102$, $K_s = 0.00922$ cm/s $\Psi(z,t=0) = -1000$ cm $\Psi(z=0,t>0) = -75$ cm $\Psi(z=70,t>0) = -1000$ cm Length of column L = 70 cm $\Delta z = 2.5$ cm, $\Delta t = 100$ s, 3600 s Convergence tolerance $\varepsilon = 10^{-[illegible]}$(sec^{-1}) Time-weighting coefficient $\omega = 1$

REFERENCES

1) Cooley, R. L., Water Resour. Res., 19(5), 1271-1285, 1983.
2) Celia, M. A. et al., Water Resour. Res., 26(7), 1483-1496, 1990.
3) Huyakorn, P. S. et al., Water Resour. Res., 20(8), 1099-1115, 1984.
4) Kaluarachchi, J. J., and J. C. Parker, Water Resour. Res., 25(1), 43-54, 1989.