A Comparative Analysis of Network Equilibrium Solutions

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1. Introduction

In designing a road network, the traffic engineer often uses the celebrated Highway Capacity Manual as a practical guide in helping to determine the capacities of streets and intersections. Depending on the level of service to be provided, the capacities are determined and the designed network tested by checking whether the estimated traffic volumes exceed capacities of any road segments. If so, more capacities can be provided by designing improved facilities. It is clear, then, that inequalities such as link capacity constraints are significant for the planning process of network. In this paper, alternative solutions to capacity constraint network are investigated in order to support network design or network capacity evaluation from the point of view of network equilibrium.

2. Capacity Constraint Network Equilibrium

In a capacity constraint network with origin- destination demand and link performance function, the traffic equilibrium assignment involves determining the vector of link flows, $\mathbf{x}$, and a vector of queues $\mathbf{d}$, satisfying the flow conservation conditions and bound conditions of link flows. This problem can be described as a user equilibrium problem incorporating capacity restriction, and the link flow pattern can be obtained by solving the equivalent mathematical problem as follows:

$$\text{minimize} \quad z(\mathbf{x}) = \int_0^{x_a} t_a(x, c_a)\,dx$$

$$\text{subject to} \ \sum_a f_r^w = q_w \quad \forall w$$

$$x_a = \sum_w \sum_r f_r^w \delta_{ar}^w \quad \forall a$$

$$x_a \le C_a$$

where

W the set of OD pairs in the network

A the set of links

$t_a(x_a, c_a)$ travel time on arc, $a \in A$

f_r^w flow value on route r

δ_{ar}^w 1 if link a on route r, 0 otherwise

q_w the demand for OD-pair $w \in W$

c_a the capacity of link $a \in A$

It is not so easy to solve such the capacity constraint assignment problem, and the standard Frank-Wolfe method is not applicable because of the addition of capacity constraints. There are still, however, alternative solutions for the capacity constraint network. The first is the constrained optimization in the sense of mathematical programming, such as Augmented Lagrange method, Successive Quadratic Programming, etc. It is a mathematical device by explicit inclusion of capacity constraints and their dual variables is explained to be queue delays associated with respective constraints (Inoue, 1986; Bell and Iida, 1997). Consistent with the assumption that each driver traveling from an origin to a destination will have perfect knowledge of the travel costs and delays via routes and will choose the route in a user-optimized fashion, at optimum the following first-order condition are satisfied for every OD-pair $w \in W$ and every path:

$$\left.\begin{array}{l} \sum_a (t_a + d_a)\delta_{ar}^w = u_w \ \text{ if } \ f_r^w > 0 \\ \sum_a (t_a + d_a)\delta_{ar}^w \ge u_w \ \text{ if } \ f_r^w = 0 \end{array}\right\} \ \forall r, w$$

The optimality condition reveals an extended Wardrop's principle in the capacity constraint network.

The second is a convex combination method with modifying the line search in Frank-Wolfe algorithm, proposed by Daganzo (1977). It can be considered to be a kind of trade-off between conventional assignment and capacity restriction. The standard user-optimized formulation does not need to be rectified and alternatively the restriction is implicitly accommodated into the algorithm procedure, i.e.

$$\underset{\alpha}{\text{minimize}} \ z(x^{(k)} + \alpha(y^{(k)} - x^{(k)}))$$

$$\text{subject to} \quad 0 \le \alpha \le \alpha_{\max}$$

$$\alpha_{\max} = \min_a \{(c_a - x_a^{(k)}) / (y_a^{(k)} - x_a^{(k)}) \,|\, x_a^{(k)} < y_a^{(k)}\}$$

At iterations, the optimal move size in the direction of $\mathbf{y}^{(k)} - \mathbf{x}^{(k)}$, has always an upper bound $\alpha_{\max}$, which forces the vehicular flow on each link not to excess the its throughput capacity. Depending on the manneristic difference in integrating capacity restriction, in the sequential context these two types of capacity constraint assignment are denoted as ECA and ICA, which mean explicit capacity assignment and implicit assignment, respectively.

3. Comparing Assignment Methods

In the previous researches different performance functions are embraced in capacity constraint network models, namely, BPR function and Davidson function, etc. The link performance function, which relates the travel time on each link to the flow traversing the link are applied, is characteristically asymptotic to a certain level of flow known as capacity of the link under consideration. The performance function is undefined for higher values of flow, since such flows cannot be observed. When the flow approaches capacity, the queues at the intersection will start growing, clogging upstream intersections and finally causing traffic to come to a halt.

Based on the inclusion of different types of link performance function, a broad classification of user equilibrium is provided in Figure 1 for understanding the positioning of ECA and ICA. It is no use incorporating the Davidson function into a capacity constraint UE, because both them have considered the link capacity constraints in different ways. From now on attention will be paid on

Standard UE	BPR	......Frank-Wolfe Algorithm
	Davidson	......**ICA** ......Daganzo Algorithm
Capacity Constraint UE	BPR	......**ECA** ......*Mathematical Programming*
	Davidson	...... no necessity

Figure 1: Classification of User Equilibrium

BPR-based ECA and Davidson-Based ICA methods.

The unawareness of higher flow is adopted by the ECA in which the running time below a given flow level is simulated by BPR function while average time spent queuing at link exit is predicted endogenously by the extent of flow beyond the level. The peculiarity of predicting queuing times manifests the ECA model is superior to the primitive user equilibrium in which the travel cost is always governed by BPR function, even in the level of flow in excess of the capacity. However, the ECA model is not so much ideal, because it accommodates a larger amount of a given trip table irrespective of the realistic world of network throughput, as like the primitive UE. There might be the possibility that the solution to capacity constraint network still exists, even if the network flow nearly approaches the cut capacity (shown in the right of Figure 2).

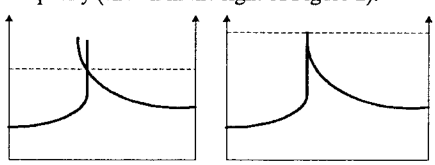

Figure 2: ECA equilibrium for network with one OD-pair with two paths

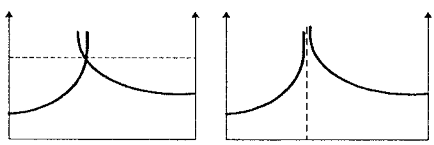

Figure 3: ICA equilibrium for network with one OD-pair and two paths

In contrast to ECA, the ICA method adopts Davidson function, which is asymptotic to a link capacity, and interpolation technique, to ensure none of the capacity constraints is violated, so that link capacity constraints do not have to appear in the formulation of the equilibrium model. Although having computational convenience mentioned before, the ICA method cannot forecast delay spent at link exit, and the capacity of each link is never actually touched because the cost of using the link becomes prohibitive. This property might lead to no solution to network equilibrium problems as long as the network is overloaded beyond its throughput (shown in the right of Figure 3), and therefore implies a preliminary condition for the ICA applications. While the advantage of the ICA method has been utilized (Daganzo, 1977; Akamatsu et al, 1995; Yang et al, 2000), there has been no interest in the potential presumption for the method, not even a recognition that such a problem may exist.

Another consciousness of the Davidson-based ICA solution is that some users of the network may incur an unnecessarily high cost, allowing a majority of the users to have a greatly reduced cost. If this reduces the company's total shipping cost, then this would certainly be optimal. In the urban transportation context, this type of cost reduction does not occur as a matter of course. Each individual user of the network chooses his own path, and he wishes merely to minimize his own travel time.

To conclude this section, it is worth mentioning that link capacities could be implemented in the concept of link costs, which can effectively implies capacity constraints. Seeing that both ECA and ICA have the unrealistic extremities respectively, some kind of transaction between the two methods becomes a meaningful subject for capacity constraint networks.

4. Numerical Example

An example is executed with hypothetical data set to verify the observation of the properties in capacitated network models. The network with three OD pairs is shown in Figure 4. The values of input parameters applied to both BPE and Davidson function are listed in Tables 1, and Table 2 shows a trip table in the network. Table 3 illustrates link flow, delay, cost predicted by BPR-based ECA and Davidson-based ICA.

The ECA is solved by SQP and foretell, as expected before, flow, running travel time, delay. The traffic volume on links 1, 3 reach their capacities, and corresponding overload delays take place.

Table1: Link Performance for BPR orDavidson

link	t_a^0	c_a
1	10	600
2	17	500
3	9	800
4	60	400

The mini-cut separating the origin from destination indicates that the maximum flow through the centroid pair is 1200, which identifies the minimum cut capacity. Though the demand traversing the network is below the maximum flow of node pair, the congestion still arises at the exits of links 1 and 3, due to network users only minimizing their own travel times. In other words, the cut capacity gives the upper bound of network flow, rather than network capacity in transportation network comprising route choice behaviors. When the trip table scaled by a multiplier more than 1.0 becomes the input of the ECA, as long as the OD demand does not reach the maximum flow, it is discovered that there still exist mathematical solutions and delays produced, which responds to the vertical queues having been used in many traffic studies.

Table 2 Trip Table

2	3	D / O
600	400	1
	600	2

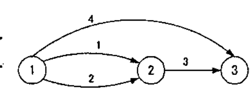

Figure 4: Network

Table 3: Equilibrium Solutions

	BPR-based ECA		Davison-based ICA	
link	Flow	delay	Flow	Cost
1	600	5.6	452.4	40.6
2	200	0	290.9	40.6
3	800	33.2	743.3	126.9
4	200	0	256.7	167.5

On the contrary, the solution of the ICA reports relatively lower flows in links 1, 3, which is congested in that of the ECA. It is explained by the fact that the ICA method compels some flows from more often concentrated links toward rarely used links 2, 4, to enable none of the link capacity is violated. This compulsion results in that the extra cost is added to the users traveling on the links that are not pursued in realty by a majority of travelers. Similarly, the ICA is executed with a larger OD demand. As expected, no solution exists once when the inputted demand excesses a critical value.

5. Conclusions

The properties of network equilibrium assignment with explicit or implicit constraints of link capacities are compared and investigated with theoretic analysis and pedagogical example. The insight of such properties will be used in the evaluation of maximum flow of network or network capacity reliability.