SOLUTION OF THE EQUATION FOR FLOW IN LAKES BY OPERATOR SPLITTING METHOD

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INTRODUCTION

For the mathematical modelling of transport processes in lakes reasonably accurate and efficient models are necesarry. Models used in our previous research (Ivetic et.al.,1986) were sufficiently accurate but, being based on the explicit algorithms, they were not so efficient. Time periods of large scale motions as well as of the changes of the boundary conditions (wind shear stress and solar heating) are much longer than periods of the surface gravity waves whose celerity limits the length of time step in an explicit scheme. A new, implicitly formulated, numerical method based on the operator splitting technique which retains good accuracy has been developed.

Basic equations for the flow in a lake are :

a) momentum equations for horizontal directions (i=1,2):

$$\frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_j}(u_i u_j) = 2\varepsilon_{ijk}u_j\Omega_k-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu_t\frac{\partial^2 u_i}{\partial x_j \partial x_j} \qquad j=1,2,3 \qquad (1)$$

where (ν_t) is turbulent eddy viscosity defined according to Smagorinsky's assumption of local turbulence equilibrium. In the vertical direction hydrostatic law is assumed. For the case of homogeneous density only equation of continuity is necessary to close the system.

$$\frac{\partial u_j}{\partial x_j} = 0 \qquad (2)$$

The case with thermal stratification, where density deficit equation is encoutered, is treated in (Ivetic et al.,1987). In the same Report theretical background and detailed outline of the method are given.

NUMERICAL METHOD

Numerical method is based on the control volume approach. Spatial derivatives are discretized on the staggered grid. For convective terms, second order upwinding scheme is used while, momentum diffusion terms are centred in space. Time advancing is performed in several fractional steps and for each step the most suitable solution technique was searched. More details about this powerful technique, called operator splitting, or the method of fractional steps, and mostly used by Russian and (recently) by some French authors, could be found in Yanenko(1971) and Marchuk(1975).

I step

This step is performed explicitly by Adams-Bashford method which is second order accurate in time. All convective terms and momentum diffusion in horizon-

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tal directions are included.

$$\frac{u_i^*-u_i^n}{dt}=\frac{3}{2}f_i^n-\frac{1}{2}f_i^{n-1} \qquad (i=1,2)$$

$$f_i^n=\frac{\partial}{\partial x_1}(\nu_t\frac{\partial u_i^n}{\partial x_1})+\frac{\partial}{\partial x_2}(\nu_t\frac{\partial u_i^n}{\partial x_2})-\left(\frac{\partial(u_1^n u_i^n)}{\partial x_1}+\frac{\partial(u_2^n u_i^n)}{\partial x_2}+\frac{\partial(u_3^n u_i^n)}{\partial x_3}\right) \qquad (3)$$

where superscripts (n-1), (n), and (*) denote previous time level, current time level and first fractional step, respectively.

II step

In this step vertical momentum diffusion and Coriolis' term are solved implicitly by Crank-Nicolson method which is also second order accurate in time.

$$\frac{u_1^{**}-u_1^*}{dt}=\frac{1}{2}\frac{\partial}{\partial x_3}[\nu_t^n(\frac{\partial u_1^{**}}{\partial x_3}+\frac{\partial u_1^*}{\partial x_3})]+fu_2^*;\ \frac{u_2^{**}-u_2^*}{dt}=\frac{1}{2}\frac{\partial}{\partial x_3}[\nu_t^n(\frac{\partial u_2^{**}}{\partial x_3}+\frac{\partial u_2^*}{\partial x_3})]-fu_1^{**} \qquad (4)$$

where superscript (**) denotes flow field after second step.

III step

In this step, continuity equation integrated over column, from the bottom (z_F) to the water level (HH), and pressure terms from eq.(1) are solved:

$$\frac{u_1^{n+1}-u_1^{**}}{dt}=-g\frac{\partial HH}{\partial x_1};\ \frac{u_2^{n+1}-u_2^{**}}{dt}=-g\frac{\partial HH}{\partial x_2};\ \frac{\partial HH}{\partial t}=-\int_{z_f}^{HH}(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2})dx_3 \qquad (5)$$

Momentum equations are first discretized, then integrated over columns and introduced into continuity equation. Right-hand side of the continuity equation is centered in time (Crank-Nicolson). This gives equations for every column.

$$\frac{HH^{n+1}-HH^n}{dt}-\frac{dtg}{2}(HH^{n+1}-z_F)\nabla^2 HH^{n+1}=-\frac{1}{2}(\frac{\Delta(u_1^{**}+u_1^n)}{\Delta x_1}+\frac{\Delta(u_2^{**}+u_2^n)}{\Delta x_2}) \qquad (6)$$

The resulting system of equations has to be solved for water levels in the center of each column. Equations (6) are solved by additional operator splitting in coordinate directions x1 and x2. Our intention was to solve the system (6) in one attempt but it became apparently necessary to employ line iterations. It is not so big loss of efficiency because of the significantly less number of the equations . For Courant number equals 1 no iterations are necessary and problems up to Cr = 4 do not require more than 10 iterations. In the latter case it corresponds to the increase of computational time by 15-20 %. Flow field on time level (n+1) is obtained explicitly from equations (5). For explicit scheme used in our previous research about the same computation time was necessary.

Appropriate boundary conditions were applied on every step and pecularities related to the method of fractional steps have been taken into account.

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