On the Crack Problem of 2-D Orthotropic Materials based on Boundary Element Method

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1 Introduction

It is well known that microcracks in brittle materials, e.g., concrete, rocks and ceramics, often control overall deformation and failure mechanisms because the disturbed microcracks in such materials not only lead to macrocrack initiation and final failure, but also induce progressive material damage. So, the study on the crack problems is of significant importance in both mechanical and civil engineering. The purpose of this paper is to study the boundary element method for the crack problem in orthotropic materials. As an initial attempt, only the homogenous and two-dimensional problem was considered in this paper.

2 Basic equations and crack problem

In the orthotropic elastic materials, the strain-displacement relation, the equlibrium equation and the constitutive equation are given as:

$$\begin{gathered}\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \\ \sigma_{ij,j} + X_i = 0 \\ \sigma_{ij} = c_{ij}^{kl}\epsilon_{kl} \quad \text{or} \quad \epsilon_{ij} = s_{ij}^{kl}\sigma_{kl}\end{gathered} \tag{1}$$

where ϵ_{ij}, u_i, σ_{ij} and X_i are strain, displacement, stress and body force, respectively. c_{ij}^{kl} and s_{ij}^{kl} are the elastic and compliance tensors, respectively.

Based on the complex elastic potentials [1], the displacement and the stress fields of orthotropic materials can be expressed as:

$$\begin{aligned} D &= u_1 + iu_2 \\ &= \delta_1\phi'(z_1) + \rho_1\bar{\phi}'(\bar{z}_1) + \delta_2\chi'(z_2) + \rho_2\bar{\chi}'(\bar{z}_2) \end{aligned} \tag{2}$$

$$\begin{aligned} \Phi &= \sigma_{11} - \sigma_{22} + 2i\sigma_{12} \\ &= -4[\gamma_1^2\phi''(z_1) + \bar{\phi}''(\bar{z}_1) + \gamma_2^2\chi''(z_2) + \bar{\chi}''(\bar{z}_2)] \end{aligned} \tag{3}$$

$$\begin{aligned} \Theta &= \sigma_{11} + \sigma_{22} \\ &= 4[\gamma_1\phi''(z_1) + \bar{\gamma}_1\bar{\phi}''(\bar{z}_1) + \gamma_2\chi''(z_2) + \bar{\gamma}_2\bar{\chi}''(\bar{z}_2)] \end{aligned} \tag{4}$$

were γ_1, γ_2 are the characteristic roots while δ_λ, ρ_λ $(\lambda = 1, 2)$ are parameters associated with the material constants which are given as:

$$\begin{aligned} \delta_1 &= (1+\gamma_1)\beta_2 - (1-\gamma_1)\beta_1 \\ \delta_2 &= (1+\gamma_2)\beta_1 - (1-\gamma_2)\beta_2 \end{aligned}$$

$$\begin{aligned} \bar{\rho}_1 &= (1+\gamma_1)\beta_2 + (1-\gamma_1)\beta_1 \\ \bar{\rho}_2 &= (1+\gamma_2)\beta_1 + (1-\gamma_2)\beta_2 \end{aligned}$$

were $\beta_\lambda = s_{22}^{11} - s_{22}^{22}\alpha_\lambda^2$, and α_λ^2 are the two roots of the following equation:

$$s_{22}^{22}\alpha^4 - 2(s_{22}^{11} + 2s_{12}^{12})\alpha^2 + s_{11}^{11} = 0 \tag{5}$$

Also, γ_λ and α_λ are related with each other by the following relations:

$$\gamma_\lambda = \frac{\alpha_\lambda - 1}{\alpha_\lambda + 1}, \quad \alpha_\lambda = \frac{1+\gamma_\lambda}{1-\gamma_\lambda} \quad (\lambda = 1, 2) \tag{6}$$

Considering the existence of cracks, the boundary value problems can be described as:

$$\begin{aligned} c_{ij}^{kl}u_{k,lj} + X_i &= 0 && (\text{in } B) \\ u_i &= \hat{u}_i && (\text{on } \partial B_1) \\ s_i = \sigma_{ij}n_j &= \hat{s}_i && (\text{on } \partial B_2) \\ s_i = \sigma_{ij}n_j &= 0 && (\text{on } S_1, S_2, ..., S_M) \end{aligned} \tag{7}$$

Here, s_i is the traction vector on the boundaries. $S_1, S_2, ..., S_M$ are the crack face boundaries.

3 Boundary element method

From Somigliana formula, the solutions to the crack problem above can be expressed by:

$$\begin{aligned} C_{ij}u_j(\boldsymbol{x}) = \overset{\circ}{u}_i(\boldsymbol{x}) &+ \int_{\partial B} G_{ij}(\boldsymbol{x},\boldsymbol{y})s_j(\boldsymbol{y})\,ds_y \\ &- \int_{\partial B} S_{ij}(\boldsymbol{x},\boldsymbol{y})u_j(\boldsymbol{y})\,ds_y \\ &- \sum_{k=1}^{M}\int_{S_k} S_{ij}(\boldsymbol{x},\boldsymbol{y})[u_j](\boldsymbol{y})\,ds_y \end{aligned} \tag{8}$$

$$\begin{aligned} \sigma_{ij}(\boldsymbol{x}) = &\int_{\partial B} \Sigma_{ijk}G_{kl}(\boldsymbol{x},\boldsymbol{y})s_l(\boldsymbol{y})\,ds_y \\ &- \int_{\partial B} \Sigma_{ijk}S_{kl}(\boldsymbol{x},\boldsymbol{y})u_l(\boldsymbol{y})\,ds_y \\ &- \sum_{k=1}^{M}\int_{S_k} \Sigma_{ijk}S_{kl}(\boldsymbol{x},\boldsymbol{y})[u_l](\boldsymbol{y})\,ds_y \end{aligned} \tag{9}$$

The kernels G_{ij} and S_{ij} are the fundamental solution and the associated fundamental solution, respectively. In order to solve these integral equations numerically, based on the boundary conditions, they can be reduced to the following system:

$$C_{ij}u_j(\boldsymbol{x}) = \overset{\circ}{u}_i(\boldsymbol{x}) + \sum_I^{N_B}[A_{ij}^I(\boldsymbol{x})s_j^I - B_{ij}^I(\boldsymbol{x})u_j^I]$$

$$- \sum_{K=1}^{M}\sum_{I}^{N_K} B_{ij}^I(\boldsymbol{x})[u_j]^I \tag{10}$$

$$0 = \sum_I^{N_B}[C_{ij}^I(\boldsymbol{x})s_j^I - D_{ij}^I(\boldsymbol{x})u_j^I]$$

$$- \sum_{K=1}^{M}\sum_{I}^{N_K} D_{ij}^I(\boldsymbol{x})[u_j]^I \tag{11}$$

$$A_{ij}^I(\boldsymbol{x}) = \int_{E_I} G_{ij}(\boldsymbol{x},\boldsymbol{y})f_I(\boldsymbol{y})\,ds_y$$

$$B_{ij}^I(\boldsymbol{x}) = \frac{\delta_{ij}}{2}f_I(\boldsymbol{y}) + \int_{E_I} G_{ij}(\boldsymbol{x},\boldsymbol{y})f_I(\boldsymbol{y})\,ds_y$$

$$C_{ij}^I(\boldsymbol{x}) = -\frac{\delta_{ij}}{2}f_I(\boldsymbol{y}) + \int_{E_I} n_l\Sigma_{lik}G_{kj}(\boldsymbol{x},\boldsymbol{y})f_I(\boldsymbol{y})\,ds_y$$

$$D_{ij}^I(\boldsymbol{x}) = \int_{E_I} n_l\Sigma_{lik}S_{kj}(\boldsymbol{x},\boldsymbol{y})f_I(\boldsymbol{y})\,ds_y$$

In Fukui's recent work [2], the influence coefficients according to constant elements in Eqs.(10) and (11) have been derived. However, because the displacement field near the crack tip is in proportion to $\sqrt{s}$ (s is the distance from the crack tip), the constant element is not suitable for the crack tip area. So here, as did in Fukui's work [3], the basic function $f_I(\boldsymbol{x}) = \sqrt{2s/l}$ is adopted to derive the influence functions of the crack tip element where s is the distance from the crack tip while l is the length of the crack tip element (see Fig.1).

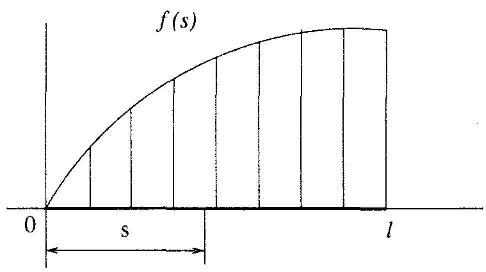

Fig.1 Crack tip element

Consider a linear element as shown in Fig.2. Then, based on Eqs.(2), (3) and (4), the displacement and stress fields due to the unit displacement of the crack tip element can be obtained as:

$$\int_E D^S(\boldsymbol{x},\boldsymbol{y})\,ds_y = (B_{1j}\boldsymbol{x} + iB_{2j}\boldsymbol{x})U_j = \delta_1 V h_1(z_1)$$

$$+\rho_1\bar{V}\bar{h}_1(\bar{z}_1) + \delta_2 W h_2(z_2) + \rho_2\bar{W}\bar{h}_2(\bar{z}_2) \tag{12}$$

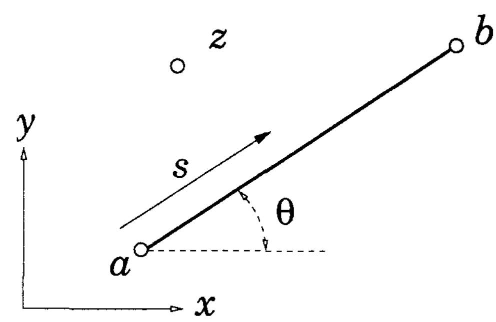

Fig.2 Linear element (a, b)

$$\int_E \Phi^S(\boldsymbol{x},\boldsymbol{y})\,ds_y = U_j\int_E(\sigma_{11}^j - \sigma_{22}^j + 2i\sigma_{12}^j)\,ds_y$$

$$= -4[\gamma_1^2 V k_1(z_1) + \bar{V}\bar{k}_1(\bar{z}_1)$$

$$+ \gamma_2^2 W k_2(z_2) + \bar{W}\bar{k}_2(\bar{z}_2)] \tag{13}$$

$$\int_E \Theta^S(\boldsymbol{x},\boldsymbol{y})\,ds_y = U_j\int_E(\sigma_{11}^j + \sigma_{22}^j)\,ds_y$$

$$= 4[\gamma_1 V k_1(z_1) + \bar{\gamma}_1\bar{V}\bar{k}_1(\bar{z}_1)$$

$$+ \gamma_2 W k_2(z_2) + \bar{\gamma}_2\bar{W}\bar{k}_2(\bar{z}_2)] \tag{14}$$

$$h_\lambda(z) = \frac{e^{-i\theta/2}}{\sqrt{2l}\pi t_\lambda}\Big\{2(b_\lambda - a_\lambda)^{1/2}$$

$$+ (z - a_\lambda)^{1/2}\log\left[\frac{(z-a_\lambda)^{1/2} - (b_\lambda - a_\lambda)^{1/2}}{(z-a_\lambda)^{1/2} + (b_\lambda - a_\lambda)^{1/2}}\right]\Big\} \tag{15}$$

$$k_\lambda(z) = \frac{e^{-i\theta/2}}{\sqrt{2l}\pi t_\lambda(z - a_\lambda)^{1/2}}\Big\{\frac{(b_\lambda - a_\lambda)^{1/2}(z-a_\lambda)^{1/2}}{b_\lambda - z}$$

$$+ \log\left[\tan\left(\frac{\pi}{4} + \frac{1}{2}\arcsin\left(\frac{b_\lambda - a_\lambda}{z - a_\lambda}\right)^{1/2}\right)\right]\Big\} \tag{16}$$

$$t_\lambda = e^{i\theta} + \gamma_\lambda e^{-i\theta},\ (\lambda = 1, 2)$$

From Eqs.(12), (13) and (14), the fundamental solutions of the crack tip element could be obtained. Then, the resulting equations (10) and (11) could be solved numerically, from which we can analyze the crack parameters further.

4 Conclusions

Based on the complex potentials, the boundary element method for the crack problem of orthotropic materials has been studied. The crack tip element was especially introduced, and the fundamental solutions of it have been derived, based on which the multi-crack problem for orthotropic materials could be solved numerically.

References

[1] Green, A.E. and W. Zerna : *Theoretical Elasticity*, 2nd ed. Oxford, (1968).

[2] Takuo Fukui, DeFa Wang and Takaaki Kobayashi: *Analysis of Two-dimensional Orthotropic Elastostatic Problem by Fast Multipole Boundary Element Method*, Journal of Boundary Element Method, **21**, pp. 85–90, (2004-12).

[3] Takuo Fukui, Tetsuro Mochida and Koichi Inoue: *On crack extension analysis by boundary element method*, Journal of Boundary Element Method, **14**, pp. 47–52, (1997-12).