MAGNETO-ELASTICITY ANALYSIS OF AN INFINITE PLANE CONTAINING AN ELLIPTICAL RIGID INCLUSION

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1. Introduction

In recent years a great number of papers are devoted to the problem of magneto-elasticity, which is concerned with the interacting effect of an external magnetic field on the deformation of an elastic body because of the extensive practical applications of magnetic material in diverse fields such as geophysics, electric power engineering, and microelectronics. However, there is still a lack in probing general analytical methods for this type of problem. The objective of this study is to develop an analytical method for the plane problem of magneto-elasticity, in which a closed form solution for an infinite plane containing an elliptical rigid inclusion in a constant primary magnetic field is obtained.

2. Analysis of Magnetic Field

The behaviors of electromagnetic solid interaction are governed by the basic laws of Maxwell and elasticity. When an electro-magnetic field is steady, the elastic field and the electromagnetic field do not contain directly interaction term [1]. The problem to be considered is specified in Fig.1, where a uniformly distributed magnetic field with direction angle δ is applied to an infinite plane containing an elliptical rigid inclusion. It is assumed that the material of the medium is soft ferromagnetic and isotropic elasticity, and the permeability of the inclusion is considered the same as vacuum (Paramagnetism). With the following conformal transformation

$$z = \omega(\zeta) = E_0\zeta + E_1/\zeta \tag{1}$$

the exterior of the inclusion in the *z*-plane is mapped to the exterior of the unit circle in the ζ-plane. Here $E_0 = (a+b)/2$, $E_1 = (a-b)/2$, and a and b denote the major and minor axes of the elliptical inclusion, respectively. For a steady magnetic field, the magnetic field intensity $\mathbf{H}(x,y)$ can be expressed as

$$H_x - iH_y = -A'(z) = -A'(\zeta)/\omega'(\zeta) \tag{2}$$

where the complex magnetic potential A(z) is

$$A(z) = A(x,y) = \Theta(x,y) + i\cdot T(x,y) \tag{3}$$

Using mapping function, it becomes

$$A(z) = A[\omega(\zeta)] \equiv A(\zeta) \tag{4}$$

The boundary condition due to the magnetic field intensity can be expressed as

$$-\left\{A(\sigma)-\overline{A(\sigma)}\right\} = 2i\int\left\{H_x\cos(n,x)+H_y\cos(n,y)\right\}ds$$
$$= 2i\int H_n(s)ds + const \tag{5}$$

where σ denotes a boundary point on the unit circle in the ζ-plane. H_n represents the magnetic field intensity normal to the boundary. Considering the huge difference of permeability between the ferromagnetic material and the inclusion, we approximately have

$$H_n = \frac{\partial\Theta(x,y)}{\partial n} = 0 \tag{6}$$

The potential A(ζ) to be determined can be decomposed to two parts as

$$A(\zeta) = A_1(\zeta) + A_2(\zeta) \tag{7}$$

where $A_2(\zeta)$ for the uniform magnetic field H_0 is obtained from Eq.(2) as

$$A_2(\zeta) = -H_0\omega(\zeta)e^{-i\delta} \tag{8}$$

Substituting Eqs.(6), (7) and (8) into (5), and using Cauchy integral on the unit circle, $A_1(\zeta)$ is

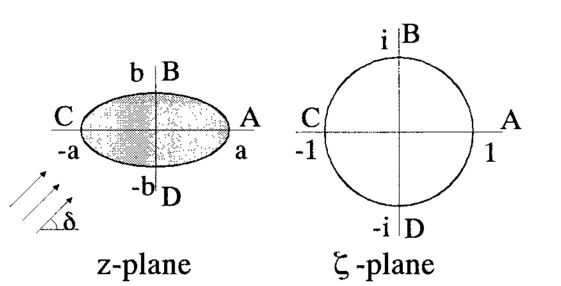

Fig.1 Infinite plane with an elliptical inclusion

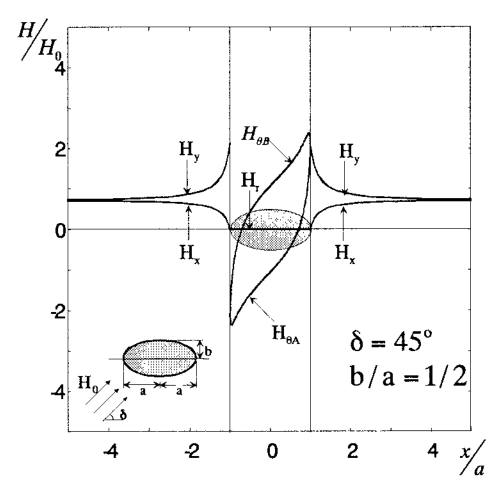

Fig.2 Distribution of magnetic field intensity along *x*-axis and inclusion boundary

obtained. The magnetic field intensity can therefore be determined as

$$\alpha(\zeta) \equiv H_x - iH_y = H_0\left(E_0 e^{-i\delta} - \overline{E_0} e^{i\delta}/\zeta^2\right)/\omega'(\zeta) \quad (9)$$

Fig.2 shows the distribution of magnetic field intensity along the x-axis and the inclusion boundary. The parameters used in the computation are chosen as $\delta = 45^\circ$ and $b/a = 1/2$.

3. Stress Analysis

Considering the effect of magnetic field, stress components can be expressed in terms of two harmonic functions $\varphi(z)$ and $\psi(z)$ [2] and the mapping function as

$$\sigma_x + \sigma_y = 4\,\mathrm{Re}[\Phi'(\zeta)/\omega'(\zeta)] \quad (10)$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\left[\overline{\omega(\zeta)}\left(\Phi'(\zeta)/\omega'(\zeta)\right)'/\omega'(\zeta) + \Psi'(\zeta)/\omega'(\zeta)\right] + \mu_e\alpha^2(\zeta)$$

The displacement boundary condition is obtained as

$$\kappa\Phi(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\Phi'(\sigma)} - \overline{\Psi(\sigma)} - \frac{1}{2}\mu_e\overline{\int\alpha^2(\sigma)\omega'(\sigma)d\sigma} = 2G(u+iv) \quad (11)$$

where $\kappa = 3 - 4\nu$ and $\kappa = (3-\nu)/(1+\nu)$ are for the plane strain and the plane stress condition, respectively. Here, we consider a rigid inclusion ($u = v = 0$) without loss of generality. Similarly, stress function $\Phi(\zeta)$ is obtained using Cauchy integral to Eq.(11) as

$$\Phi(\zeta) = \frac{\mu_e}{2\kappa}\cdot H_0^2\overline{E_0}e^{2i\delta}\frac{1}{\zeta} + \text{const.} \quad (12)$$

Function $\Psi(\zeta)$ is obtained from the condition of analytical continuity as

$$\Psi(\zeta) = \kappa\overline{\Phi}(1/\zeta) - \frac{\overline{\omega}(1/\zeta)}{\omega'(\zeta)}\Phi'(\zeta) - \frac{\mu_e}{2}\int\alpha^2(\zeta)\cdot\omega'(\zeta)d\zeta \quad (13)$$

Knowing the stress functions in (12) and (13), stress components can be computed by (10). Figs.3 and 4 show the stress distributions along the x-axis and the inclusion boundary (σ_r and σ_θ are the normal and circumferential stress components) with the magnetic field applied at 45 degrees.

4. Stress Intensity Factors

In the case of $b = 0$, the elliptical inclusion is reduced to a rigid line as shown in Fig. 5. The stress intensity factors K_I and K_{II} for the respective modes I and II can be computed by the following expression [3]:

$$K_I - iK_{II} = \sqrt{\pi}e^{-i\frac{\lambda}{2}}\Phi'(\zeta_0)/\sqrt{\omega''(\zeta_0)} \quad (14)$$

where λ is the angle between the direction of the rigid line and the x-axis, ζ_0 is the point on the unit circle corresponding to the inclusion tip. The stress intensity factors are normalized as

$$F_I - iF_{II} = (K_I - iK_{II})/(H_0^2\cdot\mu_e\sqrt{\pi a}) \quad (15)$$

Fig.5 shows the variation of F_I at the tip z_a (z_c) with changing direction δ from zero to 180 degrees for different values of κ.

5. References

[1] G. Paria, Magneto-elasticity and magneto-thermo-elasticity, Advances in applied mechanics, vol. 10, Academic Press, (1967), 73-112.

[2] N.I. Muskhelishvili, Some Basic Problems of the Mathematical Theory of Elasticity. Noordhoff Ltd., 1963.

[3] N. Hasebe, and T. Takeuchi, Stress analysis of a semi-infinite plate with thin rigid body, *Int. J. Engng. Sci.*, (1985), vol. 23, 531-539.

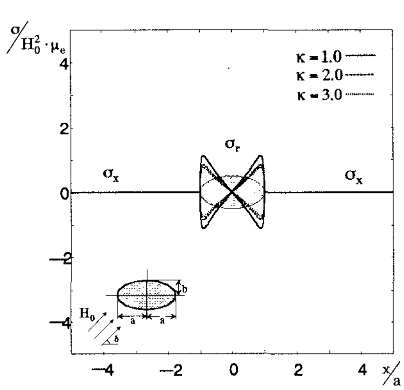

Fig.3 (σ_x, σ_r) along x-axis and inclusion boundary (b/a=1/2)

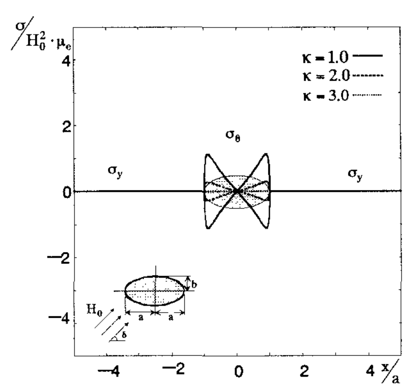

Fig. 4 $(\sigma_y, \sigma_\theta)$ along x-axis and inclusion boundary (b/a=1/2)

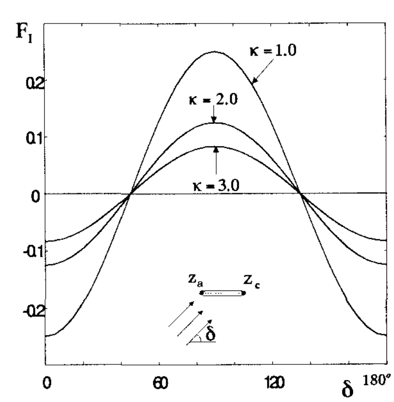

Fig.5 Stress intensity factors versus δ