Green's function of mixed boundary value problem under point heat source

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1. Introduction

As a fundamental solution, the Green's functions of heat source in an infinite plane with an inhomogeneity have found extensive applications in thermal stress analysis. Therefore, this kind of problems has received considerable attention in the pasted thirty years. Parkus [1] studied the Green's function of a point heat source in a semi-infinite plane. Fukui et al., [2] derived the solution of a heat source embedded in an infinite plane containing a circular hole whose boundary is assumed isothermal. Fukui et al., [3] treated the problem of an infinite plane with a circular inclusion having different material properties from those of the surrounding matrix, in which heat flux generated from a heat source, passes through the inclusion and flows into a heat sink. Zhang and Hasebe [4] derived the Green's function of a heat source accompanying with an adiabatic crack. Recently Yoshikawa and Hasebe [5] studied the Green's functions of a pair of heat source and sink in an infinite plane with an elliptical hole or a rigid elliptical inclusion, in which the hole and inclusion boundaries are assumed either adiabatic or isothermal. In another development [6], these authors derived the solution of an arbitrarily shaped hole under the heat source and sink by employing a special conformal mapping technique. In [7], they considered the Green's function of a rigid arbitrarily shaped inclusion under a pair of heat source and sink.

This paper is aimed at deriving the Green's function for a heat source embedded in an infinite plane containing a debonded rigid arbitrarily shaped inclusion whose boundary is assumed either adiabatic or isothermal.

2. Formulation

We consider now a thermoelastic problem of a debonded rigid arbitrary shaped inclusion undergoes a point heat source with intensity M and a heat sink with intensity M locates at infinity. One debonding is assumed to occur on the interface between the rigid inclusion and the elastic matrix. L and S denote the segments of debonded and the bonded boundaries, respectively, while α and β represent the coordinates of both ends of S.

Using a rational mapping function [8]

$$z=\omega(\zeta)=E_0\zeta+\sum_{k=1}^{n}\frac{E_k}{\zeta_k-\zeta}+E_{-1} \tag{1}$$

where E_0, E_k, E_{-1} and ζ_k are constants, an infinite region outside an arbitrarily shaped hole in the z-plane can be mapped onto exterior of a unit circle in the ζ-plane. The temperature function can be given as [6]

$$\Upsilon(\zeta)=-\frac{\mathrm{M}}{2\pi\mathrm{k}}\left\{\log(\zeta-\zeta_a)+\Gamma\log\left(\frac{\zeta-\zeta_a'}{\zeta}\right)\right\}+\text{const.} \tag{2}$$

where $\zeta_a'\equiv 1/\overline{\zeta_a}$, ζ_a represents the coordinate of the point of heat source on the ζ-plane, k denotes the conductivity of the material, and the value of the constant term can be determined by the temperature at a standard point. By employing the complex stress functions $\varphi(\zeta)$ and $\psi(\zeta)$, the boundary conditions on the unit circle can be written as

$$\varphi(\sigma)\{1-\delta(\sigma)(1+\kappa)\}+\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)}+\overline{\psi(\sigma)}=f(\sigma) \tag{3}$$

where

$$f(\sigma)=\begin{cases}0 & \text{On } L\\ 2\mathrm{G}\alpha'\int\Upsilon(\sigma)\omega'(\sigma)d\sigma\equiv H(\sigma) & \text{On } S\end{cases} \tag{4}$$

and $\delta(\sigma)=0$ for σ on the segment L, whereas $\delta(\sigma)=1$ for σ on the segment S. $\alpha'=\alpha(1+\nu)$, $\kappa=3-4\nu$ for plane strain, while $\alpha'=\alpha$, $\kappa=(3-\nu)/(1+\nu)$ for plane stress. ν, α and G represent the Poisson's ratio, the linear thermal expansion coefficient and the shear modulus of the material, respectively.

The stress functions can be broken down into two parts

$$\varphi(\zeta)=\varphi_1(\zeta)+\varphi_2(\zeta) \tag{5a}$$

$$\psi(\zeta)=\psi_1(\zeta)+\psi_2(\zeta) \tag{5b}$$

The first components denote the Green's function of a traction-free arbitrarily shaped hole in an infinite plane under the heat source, which can be given [6]

$$\varphi_1(\zeta)=\varphi_{11}(\zeta)+\varphi_{12}(\zeta) \tag{6a}$$

$$\psi_1(\zeta)=-\overline{\varphi_1(1/\overline{\zeta})}-\frac{\overline{\omega(1/\overline{\zeta})}}{\omega'(\zeta)}\varphi_1'(\zeta) \tag{6b}$$

where

$$\varphi_{11}(\zeta)=\frac{\alpha\mathrm{MGR}}{4\pi\mathrm{k}}\left[\{\omega(\zeta)-\omega(\zeta_a)\}\{\log(\zeta-\zeta_a)-1\}+A\log(\zeta)\right], \tag{6c}$$

$$\varphi_{12}(\zeta)=\frac{\alpha\mathrm{MGR}}{4\pi\mathrm{k}}\left[\{\omega(\zeta)-\omega(\zeta_a)\}\log\left(\frac{\zeta}{\zeta-\zeta_a'}\right)-\sum_{k=1}^{n}\frac{E_kB_k}{\zeta_k-\zeta}\right]-\sum_{k=1}^{n}\frac{E_k}{\omega'(\zeta_k')}\frac{g_{1k}}{\zeta_k-\zeta} \tag{6d}$$

$$A=-E_0\Gamma\zeta_a'+\sum_{k=1}^{n}\frac{E_k}{\zeta_k-\zeta_a},\qquad \zeta_k'=1/\overline{\zeta_k}, \tag{6e}$$

$$B_k=\log\frac{\zeta_a}{\zeta_a'}-\overline{\log\frac{\zeta_k'}{\zeta_k'-\zeta_a'}}-2+\frac{1}{\omega'(\zeta_k')}\left\{\overline{A}\zeta_k+\overline{E_0}+\sum_{j=1}^{n}\frac{\overline{E_k}}{(\overline{\zeta_j}-\overline{\zeta_k'})(\overline{\zeta_j}-\overline{\zeta_a})}\right\} \tag{6f}$$

$$g_{1k}=\overline{\varphi_{12}'(\zeta_k')} \tag{6g}$$

where R=(1+ν)/(1−ν) for plane strain and R=(1+ν) for plane stress. The value of the unknown constant g_{1k} can be determined by solving a system of linear algebraic equations derived from (6g). The second functions of (5) are unknown functions that must be single-valued and holomorphic outside the unit circle.

Consider now the boundary conditions on the unit circle to derive the unknown function $\varphi_2(\zeta)$. Introducing a Plemelj function

$$\chi(\zeta)=(\zeta-\alpha)^m(\zeta-\beta)^{1-m} \tag{7}$$

where

$$m = 0.5 - i\ln\kappa/2\pi \tag{8}$$

Substituting (5) and (6) into (3), a relation is obtained

$$\varphi_2(\sigma)\{1-\delta(\sigma)(1+\kappa)\} + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi_2'(\sigma)} + \overline{\psi_2(\sigma)} = f(\sigma) + \varphi_1(\sigma)\delta(\sigma)(1+\kappa) \tag{9}$$

Multiplying both sides of (9) with a factor $d\sigma/[2\pi i(\sigma-\zeta)\chi^{-}(\sigma)]$, and carrying out the Cauchy integration along the unit circle in clockwise direction, a closed form solution of $\varphi_2(\zeta)$ can be derived

$$\varphi_2(\zeta) = \chi(\zeta)\left[\frac{1}{2\pi i}\sum_{i=1}^{N}\{\int_S \frac{H(\sigma)+(1+\kappa)\varphi_1(\sigma)}{\chi^{-}(\sigma)(\sigma-\zeta)}d\sigma\} - \sum_{k=1}^{n}\frac{E_k}{\chi(\zeta_k)\overline{\omega'(\zeta_k')}}\frac{g_{2k}}{\zeta_k-\zeta}\right] \tag{10}$$

where

$$g_{2k} = \overline{\varphi_2'(\zeta_k')} \tag{11}$$

whose value can be determined by solving this system of linear algebraic equations. Substituting (4) and (6) into (10), the second stress function can be rewritten as

$$\begin{aligned}\varphi_2(\zeta) = \frac{\alpha \mathrm{MGR}}{4\pi \mathrm{k}}\Bigg[&\{\omega(\zeta_a)+\Gamma\omega(\zeta_a')-(1+\Gamma)\omega(\zeta)\}\log(\frac{\zeta}{\zeta-\zeta_a'}) \\ &-\sum_{k=1}^{n}C1_k\log(\frac{\zeta}{\zeta-\zeta_k}) + \sum_{k=1}^{n}\frac{E_k(B_k+1)}{\zeta_k-\zeta} + \chi(\zeta)\Bigg\{\int_0^{\zeta_a'}\frac{\omega(\zeta_a)+\Gamma\omega(\zeta_a')}{\chi(\sigma)(\sigma-\zeta)}d\sigma \\ &-\sum_{k=1}^{n}\int_0^{\zeta_k}\frac{C_{1k}}{\chi(\sigma)(\sigma-\zeta)}d\sigma - \int_0^{\zeta_a'}\frac{(1+\Gamma)\omega(\sigma)}{\chi(\sigma)(\sigma-\zeta)}d\sigma + \sum_{k=1}^{n}\frac{C_{2k}}{\zeta_k-\zeta}\Bigg\}\Bigg] \\ &+\sum_{k=1}^{n}\frac{E_k g_{1k}}{\overline{\omega'(\zeta_k')}(\zeta_k-\zeta)} - \chi(\zeta)\sum_{k=1}^{n}\frac{E_k(g_{1k}+g_{2k})}{\chi(\zeta_k)\overline{\omega'(\zeta_k')}(\zeta_k-\zeta)}\end{aligned} \tag{12a}$$

where

$$C_{1k} = E_k(\frac{1}{\zeta_k-\zeta_a} + \frac{\Gamma}{\zeta_k-\zeta_a'} - \frac{\Gamma}{\zeta_k}) \tag{12b}$$

$$C_{2k} = \frac{E_k}{\chi(\zeta_k)}\left\{(1+\Gamma)\log\frac{\zeta_k}{\zeta_k-\zeta_a'} - B_k - 1\right\} \tag{12c}$$

Another stress function $\psi_2(\zeta)$ can be derived by analytic continuation on the traction-free boundary as

$$\psi_2(\zeta) = -\overline{\varphi_2(1/\bar{\zeta})} - \frac{\overline{\omega(1/\bar{\zeta})}}{\omega'(\zeta)}\varphi_2'(\zeta) \tag{13}$$

3. Stress distribution

Numerical example of stress distribution is considered for the problem of the heat source accompanying with a rectangular rigid inclusion. One debonding is assumed to generate symmetrically on the interface between the inclusion and the matrix. The Poisson ratio is taken to be 0.3, and the plane strain state is considered. For the case when the heat source locates at point (2*a*, 0) on the *x*-axis, dimensionless stress distribution along the *x*-axis and inclusion boundary (for the half-plane *y*>0) under adiabatic condition is shown in Fig.1. It can be seen that the normal and tangential stresses on the debonded boundary are zero, which indicate that the traction-free condition is satisfied. All the stress components have singularities at debonding tip, and have concentrations at the corners of the inclusion. The stress components σ_x and σ_y have singularities at the point of heat source. The tangential stress on the *x*-axis is zero due to the symmetry.

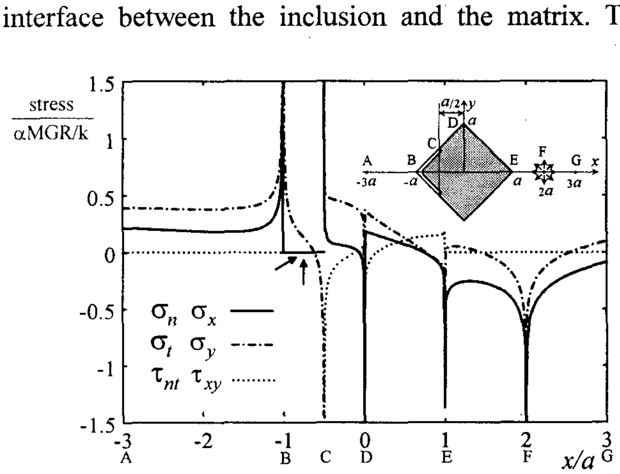

Figure 1. Stress distribution along the inclusion boundary and the *x*-axis.

4. Conclusion

A closed form solution, the Green's function, of a heat source located at any point in an infinite plane containing an either adiabatic or isothermal debonded arbitrarily shaped rigid inclusion, is obtained.

The basic point in the derivation procedure of the Green's functions for mixed boundary value problem is the use of the Cauchy integration method that is usually employed to solve stress or displacement boundary value problems.

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