One dimensional nonlinear model of sand densification by homogenization method

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1, Introduction

There are many methods to estimate the overall properties of media with heterogeneous microstructures, which were proposed by such as Voigt, Reuss, Taylor; H. Horii & Nemat-Nasser, M. Hori etc. One of the most powerful methods for periodic microstructures is the homogenization theory, firstly used by Sanchez-Palencia (1974) in seepage problem. This method requires that the medium is periodic or well random. It has been shown that structures of a unit cell do not affect the effective properties noticeably and volume fraction is an important quantity for linear elastic media. But this is not true for nonlinear materials specially when they are in the intense interaction among the constituents. Homogenization theory is advantageous when the interaction is intensified during nonlinear zone of material properties. In authors' knowledge few publications on application of homogenization theory to nonlinear properties of each constituent appear.

This paper extends the homogenization method to the case including nonlinear constitutive laws of each constituent. Firstly, we give out the generalized description of material nonlinearity problem, then homogenization process is applied. Finally we discuss one dimensional case of the densification problem.

2, Statement of the problem

Governing equations $\frac{\partial \sigma_{ij}^{\varepsilon}}{\partial x_j} + f_i^{\varepsilon}(x) = 0 \quad \varepsilon_{ij}(u^{\varepsilon}) = \frac{1}{2}\left(\frac{\partial u_i^{\varepsilon}}{\partial x_j} + \frac{\partial u_j^{\varepsilon}}{\partial x_i}\right) \quad \sigma_{ij}^{\varepsilon} = E_{ijkl}^{\varepsilon}(x, \varepsilon_{rs}^{\varepsilon})\varepsilon_{kl}(u^{\varepsilon})$ (1,2,3)

Boundary condition: $\sigma_{ij}^{\varepsilon} n_j = F_i(x)$ on Γ_F and $u_i^{\varepsilon} = \bar{u}_i(x)$ on Γ_o.

Where $\varepsilon = \frac{x}{y}$ denotes Y-periodicity and $\varepsilon \ll 1$. y is the fast variable and x is the slow variable. $E_{ijkl}^{\varepsilon}(x, \varepsilon_{rs}^{\varepsilon}) = E_{jikl}^{\varepsilon}(x, \varepsilon_{rs}^{\varepsilon}) = E_{ijlk}^{\varepsilon}(x, \varepsilon_{rs}^{\varepsilon}) = E_{jilk}^{\varepsilon}(x, \varepsilon_{rs}^{\varepsilon})$ is the function of strain history or stress history, generally expressing a linear or nonlinear constitutive law (Sometimes in incremental form with matrix $[D_{ep}]$ for elastoplastic constitutive law, at this time all quantities are replaced by their increments).

3, Asymptotic expansion

Fundamental assumption for homogenization theory: Displacement can be expanded as series of ε :

$u_i^{\varepsilon}(x) = u_i(x,y) = u_i^0(x,y) + \varepsilon u_i^1(x,y) + \varepsilon^2 u_i^2(x,y) + \cdots \qquad u_i^{\alpha}(x,y) = u_i^{\alpha}(x, y+Y),\ \alpha = 0, 1, 2.$ (4)

Y: The minimum periodicity of the microstructure.

$\frac{d}{dx_i} = \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}$ (5) and let $\sigma_{ij}^{\varepsilon}(x) = \frac{1}{\varepsilon}\sigma_{ij}^0(x,y) + \sigma_{ij}^1(x,y) + \varepsilon\sigma_{ij}^2(x,y) + \cdots$ (6)

And by use of averaging technique in a periodic cell unit Y: $\langle \Phi \rangle = \frac{1}{|Y|}\int_Y \Phi dy$ (7)

Let the coefficients of ε^{α} power zero (because a polynomial of ε is obtained) and both macroscopic problem and cell (microscopic) problem are derived out. Firstly, $u_i^0(x,y) = u_i^0(x)$ and $\sigma_{ij}^0(x,y) = 0$ from ε^{-2} power. $u_i^0(x)$ depends only on the macroscopic scale x.

The Macroscopic problem

$$\frac{\partial \langle \sigma_{ij}^1 \rangle}{\partial x_j} + \langle f_i(x,y) \rangle = 0 \qquad \langle \varepsilon_{ij} \rangle = \varepsilon_{ij}^0 = \frac{1}{2}\left(\frac{\partial u_i^0}{\partial x_j} + \frac{\partial u_j^0}{\partial x_i}\right) \qquad \langle \sigma_{ij}^1 \rangle = E_{ijkl}^h(x, \varepsilon_{rs}^0)\varepsilon_{kl}^0$$

$$\langle \sigma_{ij}^1 \rangle n_j = F_i(x) \quad on \quad \Gamma_F \qquad\qquad u_i^0 = \bar{u}_i(x) \quad on \quad \Gamma_0 \tag{8}$$

Where $E_{ijkl}^h(x, \varepsilon_{rs}^0) = \frac{1}{|Y|}\int_Y [E_{ijkl}(x, \varepsilon_{rs}^0) - E_{ijpm}(x, \varepsilon_{rs}^0)\frac{\partial W_p^{kl}}{\partial y_m}]dy$ (9) and W_p^{kl} is given by the solution of cell problem and it expresses the effect of interaction among the constituents of a cell unit on overall properties. This is specially effective for materials with high defect density. If $E^h_{ijkl}(x,\varepsilon^0_{rs})$ is known equation set (8) express a usual boundary value problem. It can be easily solved by analytical method or numerical method usually used.

The Cell problem

From ε^{-1} power cell problem $\frac{\partial \sigma^1_{ij}}{\partial y_j}=0$ is obtained.

$$\frac{\partial}{\partial y_l}\{E^{\varepsilon}_{klij}(x,\varepsilon^0_{rs})[(\frac{\partial u^0_i}{\partial x_j}+\frac{\partial u^0_j}{\partial x_i})+(\frac{\partial u^1_i}{\partial y_j}+\frac{\partial u^1_j}{\partial y_i})]\}=0 \quad (10)$$ with the Y-periodicity boundary condition.

Its solution is, given $\frac{\partial u^0_i}{\partial x_j}$ for each cell problem, $u^1_i(x,y)=-W^{kl}_i(y)\frac{\partial u^0_k}{\partial x_l}+c(x)$ (11)

$$u_i(x,y)=u^o_i(x)+\bar{u}^1_i(y) \qquad \bar{u}^1_i(y)=-\varepsilon w^{kl}_i(y)\frac{\partial u^0_k}{\partial x_l} \qquad (12)$$

Where $\bar{u}^1_i(y)$ is the fluctuation of displacement relative to averaged displacement $u^o_i(x)$ (or rigid displacement) for a cell unit.

$$\sigma_{ij}(x,y)=\sigma^1_{ij}(x,y)=\frac{1}{2}E^{\varepsilon}_{ijkl}(x,\varepsilon^{\varepsilon}_{rs})[(\frac{\partial u^0_k}{\partial x_l}+\frac{\partial u^0_l}{\partial x_k})+(\frac{\partial u^1_k}{\partial y_l}+\frac{\partial u^1_l}{\partial y_k})] \qquad (13)$$

Eq.(13) is independent of parameter ε, giving a equivalent constitutive relation. That is, considering a sample as a cell unit, strain $\frac{\partial u^0_i}{\partial x_j}$ vs $\sigma^h_{ij}=<\sigma^1_{ij}(x,y)>$ is easily determined if all microconstitutive laws of each constituent are known. The $\sigma^h_{ij}\sim\frac{\partial u^0_i}{\partial x_j}$ is consistent to that tested in Lab. What is the difference from the usual one is that the interaction of microstructures is emphasized in this proposal..

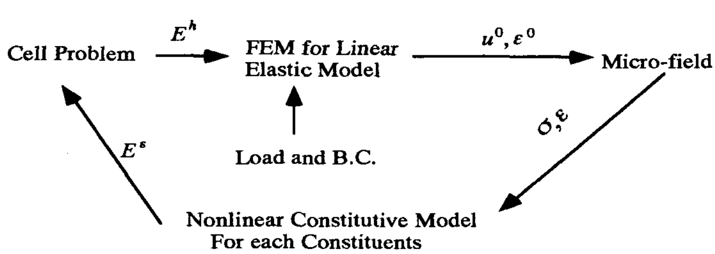
4, Numerical procedure

Equation set (8) is equivalent to following virtual work principle:

$$\int_{\Omega}<\sigma^1_{ij}>\delta\varepsilon^0_{ij}dV=\int_{\Omega}f_i\delta u^0_i dV+\int_{\Gamma_F}F_i\delta u^0_i d\Gamma \qquad (14)$$

u^0 can be solved from equation (12) if $\{E^h\}$ is known. δ denotes variational.
Equation (10) is equivalent to following virtual work principle:

$$\int_Y E^{\varepsilon}_{klij}(x,\varepsilon^0_{rs})\frac{\partial W^{pq}_i(y)}{\partial y_j}\frac{\partial V_k}{\partial y_l}dy=\int_Y E^{\varepsilon}_{klpq}(x,\varepsilon^0_{rs})\frac{\partial V_k}{\partial y_l}dy$$ (15) for any $V_k=V_k(y)$ with Y-periodicity.



5 Conclusions

(1), A new boundary value problem is derived out based on homogenization theory and a equivalent constitutive relation is obtained if behaviors of all constituents are known.
(2) The equivalent constitutive law emphasizes the interaction among the elements of microstructures. What is the bridge between this method and Eshelby problem or Tanaka-Mori problem needs further theoretical work.

6, References (Omitted)