Linear Stability Analysis of Sand Dunes Incorporating the Pressure Gradient

○ MORALES, Randulph Paulo[1], IZUMI, Norihiro[2]
[1] Graduate Student, Graduate School of Engineering, Hokkaido University
[2] Professor, Graduate School of Engineering, Hokkaido University

1 INTRODUCTION

Fluvial dunes are found whenever a fluid in the low flow regime interacts with an erodible bed. As the fluid flow changes from a low flow regime to an upper flow regime, dunes disappear and antidunes start to form. The presence of this repetitive, wave-like features suggest that there is an inherent instability in the bed making it impossible for a horizontal bed to stay horizontal. The formation of different bed features is of great interest because of its importance in the prediction of bed resistance on river channels. Dune formations provide big flow resistance, whereas antidunes provide little flow resistance.

Linear stability analysis has been applied to investigate the formation of dunes. Fredsøe (1974) derived a bedload transport formula including the effect of gravity on an inclined bed. Fredsøe's formulation is performed on the basis of the work of Meyer-Peter & Müller (1948). Kovacs & Parker (1994) also proposed a vectorial bedload transport formula on the basis of the Bagnold hypothesis and the work of Ashida & Michiue (1972). Although numerous studies were performed regarding the formation of fluvial dunes, the analyses failed to include the effect of the resistant force due to the pressure gradient exerted on a sediment particle.

In this study, a bedload transport formula incorporating the pressure gradient is derived. The effect of pressure gradient on dunes is then investigated in terms of linear stability analysis.

2 FORMULATION

2.1 GOVERNING EQUATIONS

The coordinate system (x, y) and the conceptual diagram of the flow are presented in **Fig. 1**. The curves $y = R(x) + D(x)$ and $y = R(x)$ refer to the upper and lower boundaries of the flow domain, respectively. The term D is the local flow depth, and R is the reference point where the velocity in the logarithmic velocity distribution vanishes.

The flow in an open channel is described by the Reynolds-averaged two-dimensional Navier-Stokes equations. A quasi-steady assumption, in which the flow is assumed to adapt instantaneously to changes in bed elevation, is employed in this model. The nondimensional Navier-Stokes equations are written as

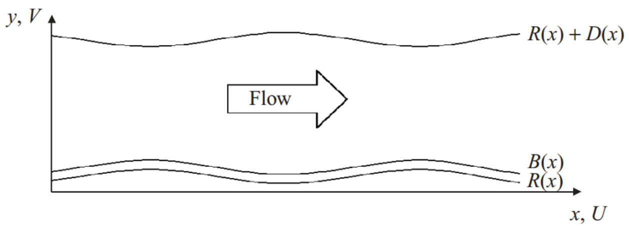

Fig. 1. Conceptual diagram of flow and the coordinate system

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{\partial P}{\partial x} + 1 + \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} \quad (1)$$

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = -\frac{\partial P}{\partial y} - S^{-1} + \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} \quad (2)$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \quad (3)$$

where x and y are the coordinates in the streamwise and the depth direction, respectively. The variables U and V are the velocity components in the x and y direction, while P and S is the pressure and the average bed slope, respectively. The Reynolds stress tensor, $T_{ij}(i, j = x, y)$ is expressed by the mixing length turbulent model.

$$(T_{xx}, T_{yy}) = 2\nu_T\left(\frac{\partial U}{\partial x}, \frac{\partial V}{\partial y}\right) \quad (4a)$$

$$(T_{xy}) = \nu_T\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) \quad (4b)$$

$$\nu_T = l^2\left|\frac{\partial U}{\partial y}\right| \quad (4c)$$

$$l = \kappa(y - Z)\left(\frac{D + R - y}{D + R}\right)^{\frac{1}{2}} \quad (4d)$$

where ν_T is the eddy viscosity, l is the mixing length, and κ is the Kármán constant $(= 0.4)$.

The above equations are nondimensionalized using the following expressions

$$(U^*, V^*) = U^*_{f0}(U, V) \quad (5a)$$

$$(x^*, y^*, d^*_s) = D^*_0(x, y, d_s) \quad (5b)$$

$$(P^*, T^*_{ij}) = \rho U^{*2}_{f0}(P, T_{ij}) \quad (5c)$$

where the superscript * denotes the dimensional variables. The variables d^*_s and λ_p denotes the sediment diameter and porosity, respectively, while U^*_{f0} and D^*_0 refers to the friction velocity and the flow depth in the base state condition.

The stream function is introduced as

$$(U, V) = \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x}\right) \quad (6)$$

Equations (1) and (2) is then rewritten in terms of Ψ. The term P is then eliminated from the rewritten equations.

$$\frac{\partial \Psi}{\partial y}\frac{\partial \nabla^2\Psi}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \nabla^2\Psi}{\partial y} - 4\frac{\partial^2}{\partial x \partial y}\left(\nu_T\frac{\partial^2\Psi}{\partial x \partial y}\right) + \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\left[\nu_T\left(\frac{\partial^2\Psi}{\partial y^2} - \frac{\partial^2\Psi}{\partial x^2}\right)\right] = 0 \quad (7)$$

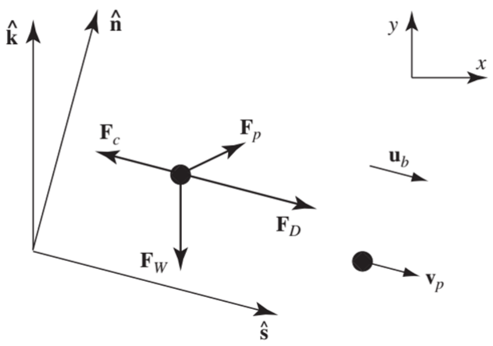

Fig. 2. Force balance on a sediment particle

In order to facilitate the application of the boundary conditions at the water surface and the riverbed, coordinate transformation is introduced.

$$(\xi, \eta) = \left(x, \frac{y - R(x)}{D(x)} \right) \tag{8}$$

2.2 BOUNDARY CONDITIONS

The boundary conditions at the water surface and the bottom are

$$u \cdot e_{ns} = 0 \qquad \text{at } \eta = 1 \tag{9a}$$

$$e_{ns} \cdot \mathsf{T} \cdot e_{ns} = 0 \qquad e_{ts} \cdot \mathsf{T} \cdot e_{ns} = 0 \qquad \text{at } \eta = 1 \tag{9b,c}$$

$$u \cdot e_{nb} = 0 \qquad u \cdot e_{tb} = 0 \qquad \text{at } \eta = 0 \tag{9d,e}$$

where $\boldsymbol{u}$ is the velocity vector $(= (u, v))$. The unit vectors tangential and normal to the water surface are $\boldsymbol{e}_{ts}$ and $\boldsymbol{e}_{ns}$. On the other hand, the unit vectors tangential and normal to the bed surface are $\boldsymbol{e}_{tr}$ and $\boldsymbol{e}_{nr}$. The term T denotes the sress tensor expressed as

$$\mathsf{T} = \begin{bmatrix} -P + T_{xx} & T_{xy} \\ T_{xy} & -P + T_{yy} \end{bmatrix} \tag{10}$$

2.3 BEDLOAD FORMULATION

2.3.1 *FORCE BALANCE ON A SEDIMENT PARTICLE*

A fully nonlinear and vectorial bedload formulation, which is applicable for an arbitrary bed slope in the streamwise and transverse direction was formulated by Kovacs & Parker (1994). The formulation was based on the work of Ashida and Michiue (1972) and the Bagnold hypothesis. In Kovacs and Parker's formulation, the pressure distribution in the flow is approximated to be hydrostatic in calculating the particle's immersed weight. In this paper, the effect of pressure gradient is introduced into the formulation.

In order to derive the bedload transport equation including the effect of pressure gradient, a force balance on a sediment particle is performed. **Fig. 2** shows the different forces acting on a sediment particle moving in the bedload layer.

The vector $\hat{k}$ is the unit vector in the upward direction, while $\hat{n}(= e_{nr})$, and $\hat{s}(= e_{tr})$ refers to the unit vector normal and tangential to the bed surface. The variable u_b is the flow velocity in the vicinity of the bed and v_p is the velocity vector of the sediment particle. The gravity force, drag force, and the dynamic Coulomb resistive force are represented by F_W, F_D, and F_C, respectively. The resistant force due to the fluid pressure is introduced as F_P. The equations of the forces acting on the particle are written as

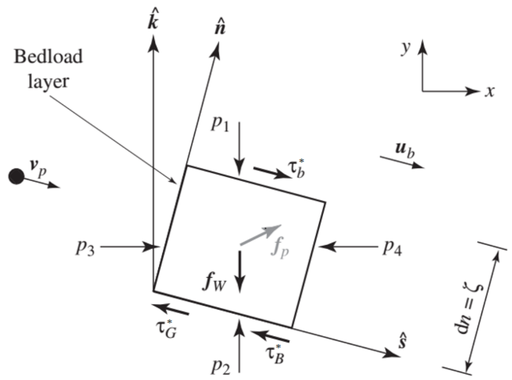

Fig. 3. Force balance on the bedload layer per unit area

$$F_W = -\frac{4}{3} \rho R_s g \pi \left(\frac{1}{2} d_s^* \right)^3 \hat{k} \tag{11}$$

$$F_D = \frac{1}{2} \rho C_D \pi \left(\frac{1}{2} d_s^* \right)^2 \mid u_r \mid u_r \tag{12}$$

$$F_P = -\frac{4}{3} \pi \left(\frac{1}{2} d_s^* \right)^3 \left(\frac{\partial P^*}{\partial x}, \frac{\partial P^*}{\partial y} \right) \tag{13}$$

$$F_C = -(F_W + F_P) \cdot (-\hat{n}) \mu_c \hat{t}_{vp} \tag{14}$$

where R_s is the submerged specific gravity $(R_s = 1.65)$, C_D is the drag coefficient, u_r is the fluid velocity relative to the particle velocity, v_p (i.e. $u_r = u_b - v_p$). The unit vector in the direction of the particle motion is denoted as $\hat{t}_{vp}$. The term μ_c is the dynamic Coulomb friction coefficient $(= 0.84)$.

The force balance on a sediment particle tangential to the bed is written as

$$F_D + W_g + P_t + F_c = 0 \tag{15}$$

where W_g and P_t are the tangential components of F_w and F_p, respectively. Moreover, the terms u_r, u_b, and v_p are normalized by $(R_s g d_s^*)^{1/2}$.

Substituting (11)-(14) into (15), the nondimensional equation for the particle velocity is obtained.

$$v_p = a^{\frac{1}{2}} \left[\tau_b^{\frac{1}{2}} - \tau_{c0}^{\frac{1}{2}} \left\{ \left(\hat{k} + \nabla P \right) \cdot \hat{n} + \frac{\hat{s}}{\mu_c} \right\}^{\frac{1}{2}} \right] \tag{16}$$

where τ_b and τ_{c0} are the nondimensional bed shear stress and the critical Shields shear stress for a horizontal bed, respectively. The relationship between the flow velocity, u_b, and the friction velocity, $u_f = (\tau_b / \rho)^{1/2}$, is expressed by the coefficient a.

2.3.2 *FORCE BALANCE ON THE BEDLOAD LAYER PER UNIT AREA*

The volume of particles in the bedload layer per unit area χ, shown in **Fig. 3**, is expressed as

$$\chi = \zeta \eta_s \tag{17}$$

where ζ is the thickness of the bedload layer, η_s is the volume fraction of sediment within the bedload layer.

A force balance is performed on the bedload layer per unit area. The forces acting on the bedload layer per unit area are shown in **Fig. 3**. The terms τ_b^* and τ_B^* denote the shear stress acting on the top and at the bottom of the bedload layer, respectively, while τ_G^* is the grain stress represented as the Coulumb friction stress, and p_{1-4} refers to the force due to the fluid pressure. The equations of the forces acting on the bedload layer per unit area are written as

$$f_W = -\chi \rho R_s g \hat{k} \tag{18}$$

$$f_p = -\chi \left(\frac{\partial P^*}{\partial x}, \frac{\partial P^*}{\partial y} \right) \tag{19}$$

$$\tau_G^* = (f_w + f_p) \cdot (-\hat{n}) \mu_c \hat{t}_{vp} \tag{20}$$

The force balance on a sediment particle tangential to the bed is written as

$$\tau_b^* + (f_W \cdot \hat{s})\hat{s} + (f_p \cdot \hat{s})\hat{s} = \tau_G^* + \tau_B^* \tag{21}$$

The shear stress on the top and at the bottom of the bed are normalized using $(\rho R_s g d_s^*)$. The bedload layer per unit area is also normalized using the term d_s^*. The nondimensional form of (21) is obtained as

$$\tau_b - \chi \left[\left(\hat{k} + \nabla P \right) \cdot \hat{s} \right] \hat{s} = \chi \left[\left(\hat{k} + \nabla P \right) \cdot \hat{n} \right] \mu_c \hat{t}_{vp} + \tau_B \tag{22}$$

From Kovacs and Parker (1994), the shear stress at the bottom of the bedload layer, τ_B, is equal to the critical Shields shear stress, τ_c, which is expressed as

$$\tau_B \cdot \hat{s} = \tau_c = \tau_{c0} \left(\hat{k} + \nabla P \right) \cdot \left(\hat{n} + \frac{\hat{s}}{\mu_c} \right) \tag{23}$$

Substituting (23)into (22) and rearranging the terms. The volume of particles, χ, in the bedload layer per unit area is then written as

$$\chi = \frac{\tau_b - \tau_c}{\mu_c \left(\hat{k} + \nabla P \right) \cdot \left(\hat{n} + \frac{\hat{s}}{\mu_c} \right)} \tag{24}$$

2.4 EXNER EQUATION AND BEDLOAD FORMULA

The non-dimensional Exner equation expresses the time variation of the bed elevation.

$$\frac{\partial B}{\partial t} + \frac{\partial \Phi}{\partial x} \tag{25}$$

where Φ is the nondimensional bedload sediment transport rate per unit width expressed as

$$\Phi = \frac{q_b^*}{(R_s g d_s^{*2})^{1/2} d_s^*} = \chi v_p \tag{26}$$

Substituting (16) and (24) into (26), the nondimensional bedload sediment transport rate per unit width is rewritten as

$$\Phi = \frac{a^{\frac{1}{2}}}{\mu_c \gamma} (\tau_b - \gamma \tau_{c0}) \left(\tau_b^{1/2} - \gamma^{1/2} \tau_{c0}^{1/2} \right) \tag{27a}$$

$$\gamma = \tfrac{S}{R_s} P_x \left(S - B_x + \tfrac{1}{\mu_c} \right) + \left(1 + \tfrac{S}{R_s} P_y \right) \left(1 + \tfrac{B_x - S}{\mu_c} \right) \tag{27b}$$

where B_x is the x derivative of the bedload layer. P_x and P_y are the x and y derivative of the piezometric pressure, respectively.

3 BASE STATE SOLUTION

The base state flow is the initial condition of the linear stability analysis wherein the bed is subjected under uniform flow conditions. The variables in the base state are expressed as

$$(\Psi, P, D, Z, R, B) = (\Psi_0, P_0, 1, 0, R_0, B_0) \tag{28}$$

The variables in the base state are substituted to the governing equations and the following solutions are obtained

$$P_0(\eta) = S^{-1}(1 - \eta) \tag{29}$$

$$\Psi_0(\eta) = \frac{1}{\kappa} \left[(R_0 + \eta) \ln \left(\frac{R_0 + \eta}{R_0} \right) - \eta \right] \tag{30}$$

Integrating from $\eta = 0$ to 1, the friction coefficient C is obtained.

$$C^{-1} = \frac{U_{a0}^*}{U_{f0}^*} = \frac{1}{\kappa} \left[(1 + R_0) \ln \left(\frac{1 + R_0}{R_0} \right) - 1 \right] \tag{31}$$

where U_{a0}^* is the depth-averaged velocity in the base state.

4 LINEAR STABILITY ANALYSIS

4.1 PERTURBATION EXPANSION

Perturbation expansion is imposed on the base state solution and the variables are expanded as follows

$$(\Psi, P, D, Z, R, B) = (\Psi_0, P_0, 1, 0, R_0, B_0) + A(\hat{\Psi}_1, \hat{P}_1, \hat{D}_1, \hat{R}_1, \hat{R}_1, \hat{R}_1) + c.c. \tag{32}$$

$$(\hat{\Psi}_1, \hat{P}_1, \hat{D}_1, \hat{R}_1) = (\Psi_1, P_1, D_1, R_1) \exp[\mathrm{i}(\alpha \xi - \Omega t)] \tag{33}$$

where A is the amplitude of the pertubation, $c.c.$ is the complex conjugate, α and Ω are the wave number and the angular frequency of perturbation, respectively.

At $O(A)$, equation (1) rewritten in terms of Ψ and equation (7) reduces to

$$\mathrm{i}\alpha + \mathcal{P}(\eta)\Psi_1(\eta) + \mathcal{P}^D(\eta) D_1 + \mathcal{P}^R(\eta) R_1 = 0 \tag{34}$$

$$\mathcal{L}^\Psi(\eta)\Psi_1(\eta) = \mathcal{L}_1^D + \mathcal{L}^R(\eta) R_1 = 0 \tag{35}$$

where $\mathcal{P}$ and $\mathcal{L}$ are linear operators. The boundary conditions also reduces to

$$\Psi_1(1) = 0 \qquad P_1(1) = 0 \tag{36a,b}$$

$$\Psi_1(0) = 0 \qquad \mathcal{D}\Psi_1(0) = 0 \tag{36c,d}$$

where $\mathcal{D} = \partial / \partial \eta$ The stream function Ψ_1 is expanded with the use of Chebyshev polynomials

$$\Psi_1 = \sum_{n=1}^{N} a_n T_n(\zeta) \tag{37}$$

The term T_n is the nth order of the Chebyshev polynomial, and ζ is the variable of the Chebyshev polynomial defined in the range $[-1, 1]$. The physical domain $0 \leq \eta \leq 1$ is transformed into the domain of the Chebyshev polynomial using $-1 \leq \zeta \leq 1$ using the transformation expression

$$\zeta = \frac{2 \ln[(\eta + R_0)/R_0]}{\ln[(1 + R_0)]/R_0} - 1 \tag{38}$$

Equation (38) is evaluated at the Gauss-Lobatto points, expressed as

$$\zeta_j = \cos(j\pi/N) \qquad (j = 1, 2, \ldots, N-2) \tag{39}$$

Arranging the results of the formulation above, a linear system of algebraic equations is obtained

$$\mathsf{L} \cdot \mathsf{a} = l R_1 \tag{40a}$$

where

$$\mathsf{a} = \left[a_0, \quad a_1, \quad a_2, \quad \ldots, \quad , a_{N-1} \quad , a_N \quad , D_1 \right] \tag{40b}$$

$$\mathsf{L} = \begin{bmatrix} \check{\mathcal{P}}^{\Psi} T_0(1) & \ldots & \check{\mathcal{P}}^{\Psi} T_N(1) & \check{\mathcal{P}}^{D}(1) \\ T_0(1) & \ldots & T_N(1) & 0 \\ \cdot & \ldots & \cdot & \cdot \\ \cdot & \ldots & \cdot & \cdot \\ \cdot & \ldots & \cdot & \cdot \\ \check{\mathcal{L}}^{\Psi} T_0(\zeta_{N-2}) & \ldots & \check{\mathcal{L}}^{\Psi} T_N(\zeta_{N-2}) & \check{\mathcal{L}}^{D}(\zeta_{N-2}) \\ \check{\mathcal{D}} T_0(-1) & \ldots & \check{\mathcal{L}} T_N(-1) & 0 \\ T_0(-1) & \ldots & T_N(-1) & 0 \end{bmatrix} \tag{40c}$$

$$l = \left[-\check{\mathcal{P}}^{R}, \quad 0, \quad -\check{\mathcal{P}}^{R}(\zeta_1), \quad \ldots, \quad -\check{\mathcal{P}}^{R}(\zeta_{N-2}), \quad 0, \quad 0 \right] \tag{40d}$$

the ˇ denotes the linear operator with η transformed into ζ. Because L is regular, (40a) is solved to be

$$\mathsf{a} = \mathsf{L}^{-1} l R_1 \tag{41}$$

and the perturbed stream function, Ψ_1 and flow depth, D_1 are obtained in the forms containing R_1 as a factor.

4.2 PERTURBATION EXPANSION OF BEDLOAD

The bedload transport rate is expanded as

$$\Phi = \Phi_0 + A\Phi_1 \exp[\mathrm{i}(\alpha\xi - \Omega t)] \tag{42}$$

Φ can be expressed by Ψ, D, and R

$$\Phi = \Phi_{,\Psi_0} \Psi_1 \zeta_b + \Phi_{,D_0} D_1 + \Phi_{,R_0} R_1 \tag{43}$$

where ζ_b is the ζ coordinate corresponding to $\eta = B_0$; $\Phi_{,\Psi_0} = \Phi_{,\Psi}|_{\Psi=\Psi_0}$; $\Phi_{,D_0} = \Phi_{,D}|_{D=1}$; and $\Phi_{,R_0} = \Phi_{,R}|_{R=1}$. Substituting the above equations into the Exner equation, the complex frequency Ω is obtained. The growth rate Ω has a general functional form as

$$\Omega = f(\alpha, F; C, \mu_c) \tag{44}$$

The imaginary part of Ω refers to the growth rate of the perturbation.

5 RESULTS AND DISCUSSION

Fig. 4 shows the neutral curves of the growth rate Ω as a function of the wave number, α, and the Froude number, F. The dashed line represents the neutral curve using the formulation of Kovacs and Parker (1994). The solid line, on the other hand, represents the neutral curve derived by accounting the effect of the pressure gradient in Kovacs and Parker's original formulation.

Dunes form in the region where $\Omega > 0$, whereas the perturbation tends to vanish in the region where $\Omega < 0$. Moreover, the neutral curve represents the condition to which the dunes neither grow nor decay.

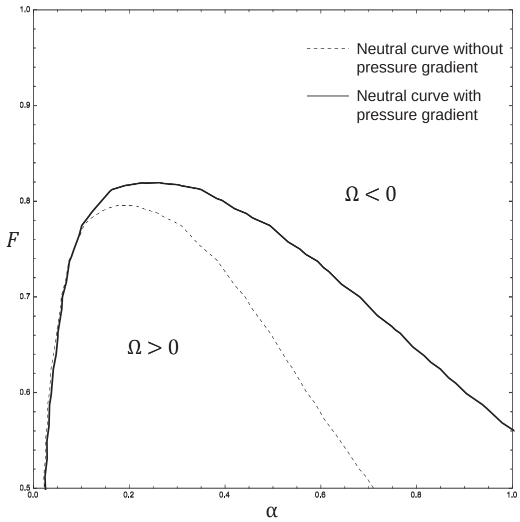

Fig. 4. Instability diagram ($C^{-1} = 20, \mu_c = 0.84$)

From the figure, accounting the effect of pressure gradient in the bedload formulation causes the unstable region to expand. The region of dune formation expanded in the range of larger wave number, α, and Froude number, F. Results indicate that the critical Froude number and its corresponding dominant wave number becomes larger when the effect of pressure gradient is taken into account on the bedload formula.

6 CONCLUSION

A bedload transport equation including the effect of pressure gradient is derived based on Kovacs and Parker's formulation. In order to account the effect of pressure gradient, a force balance on a sediment particle and the bedload per unit area was performed. A linear stability analysis was then performed utilizing the newly derived bedload transport equation. The obtained result is then compared to the instability diagram produced by using Kovacs and Parker bedload transport formula.

Results show that accounting the pressure gradient causes the region of dune formation to expand in the range of large wavenumber and Froude number. It can be considered that because of the pressure gradient, the effect of gravity in the bedload formula is reduced.

REFERENCES

1. Colombini, M. (2004). Revisiting the linear theory of sand dune formation. Journal of Fluid Mechanics vol. 502, 1-16
2. Izumi, N. & Sato, H. (2008). The Effect of Pressure Gradient on the Dune-Flat Bed Transition. JSCE: Journal of Applied Mechanics Vol. 11, 851-857.
3. Kovacs, A. & Parker, G. (1994). A new vectorial bedload formulation and its application to the time evolution of straight rivers. Journal of Fluid Mechanics Vol. 267, 153-183.
4. Yamaguchi, S. & Izumi, N. (2005). Weakly nonlinear analysis of dunes by the use of a sediment transport formula incorporating the pressure gradient. 4th IAHR Symposium on RCEM(pp. 813-820). Urbana: Taylor & Francis Group.