Weight function determinations for shear cracks in reinforced concrete beams with virtual crack extension technique

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1. INTRODUCTION

The oblique shear failure of reinforced concrete (RC) beams has long been known to be a brittle and catastrophic type of failure, while the problem of how shear failures occur in RC beams still remains, by and large, based on rather (semi-) empirical considerations.

The fracture mechanics provides a possible theoretical approach of analyzing the oblique shear crack of RC beams. Employing a nonlinear concrete bridging model to consider nonlinear characteristics of the concrete cracking behavior, the Linear Elastic Fracture Mechanics (LEFM) is applicable for studying concrete cracking. In LEFM, a weight function concept which was firstly proposed by [1] possesses very strong advantages because the stress intensity factors for any arbitrary state of loading can be determined if the weight function of a given crack geometry is evaluated from a (perhaps simple) reference state of loading owing to load-independent characteristics of the weight function. As a further step, the crack opening profiles can be estimated following a weight function based integral equation proposed in [2]

Unfortunately, until now, the number of fracture problems with a closed form analytical solution of weight function is very small. Virtual Crack Extension (VCE) technique, as suggested by [3, 4], provides an efficient finite element calculation of stress intensity factors and nodal weight functions. This technique has been employed in determining the 2-D Mode I weight functions in [5] and extended to 2-D mixed mode fracture problems in [6] through the use of symmetric mesh in the vicinity of crack tip. Obviously, the oblique shear crack of RC beams can be simply considered as 2-D mixed mode fracture problems.

For the failure mode of RC beams, one of the decisive factors is the size effect. [7, 8] concluded that the shear crack leading to failure occurs only in a beam with its shear span to depth ration from 2.5 to 8.0. Thus, weight functions for beam geometries with varying sizes are necessary, whereas the weight functions depend on the beam geometry. Therefore, this study is dedicate to presenting an approach of weight function determination for the shear cracked RC beams with varying shear span/beam depth ratio using only weight functions for beams with two shear span to beam depth ratios.

2. FORMULATION

Exploiting VCE technique in finite element method, the nodal weight functions for Mode I 2-D crack problems was represented in the displacement differentiation form according to the physical meaning of weight function. This technique was extended to mixed mode cracks, with combined tension and shear loading conditions, in [6] through the use of symmetric mesh in the crack tip neighborhood. The symmetric mesh provides the decoupling characteristic of the stress, strain, displacement and traction field parameters into Mode I and Mode II components with respect to x axis in the crack tip neighborhood as shown in Fig. 1, as a result, the stress intensity factors and nodal weight functions are separated into Mode I and Mode II components. To simulate the √r and 1/√r displacement and stress variation at the crack-tip vicinity for fracture problems, the symmetric mesh around the crack-tip is formed by assembling the degenerated quarter-point quadratic elements with √r displacement variation. The decoupled nodal weight functions for Mode I and Mode II at i's nodal location (x_i, y_i) with crack length (a) and inclination angle (β) can be represented in the displacement differentiation form as

$$h_{I(II)x}(x_i, y_i, a, \beta) = \frac{H}{2K_{I(II)}} \frac{\partial U_{I(II)x}(x_i, y_i, a, \beta)}{\partial a} \tag{1a}$$

$$h_{I(II)y}(x_i, y_i, a, \beta) = \frac{H}{2K_{I(II)}} \frac{\partial U_{I(II)y}(x_i, y_i, a, \beta)}{\partial a} \tag{1b}$$

where $h_{I(II)x}$ and $h_{I(II)y}$ are Mode I and II weight function components along x and y axes, respectively. $K_{I(II)}$ are Mode I and II stress intensity factors for Mode I and II. $U_{I(II)x}$ and $U_{I(II)y}$ are the displacement components along the x and y axes for Mode I and II deformations. H is the effective modulus.

By applying VCE technique with symmetric mesh in the crack tip neighborhood to the mixed mode fracture problems, the decoupled strain energy release rate G_I for Mode I and G_{II} for Mode II can be obtained from the decomposed displacement components $\{U_I\}$ and $\{U_{II}\}$, the changes in global stiffness $\Delta[K]$ and decomposed nodal force components Δf_I and Δf_{II} as follows:

$$G_I = -\frac{1}{2}\{U_I\}^T \frac{\partial[K]}{\partial a}\{U_I\} + \{U_I\}^T \frac{\partial\{f_I\}}{\partial a} \tag{2a}$$

$$G_{II} = -\frac{1}{2}\{U_{II}\}^T \frac{\partial[K]}{\partial a}\{U_{II}\} + \{U_{II}\}^T \frac{\partial\{f_{II}\}}{\partial a} \tag{2b}$$

The decouple nodal displacement derivatives, $\partial U_{I(II)x}(x_i, y_i, a, \beta)$ and $\partial U_{I(II)y}(x_i, y_i, a, \beta)$ for the entire structure, can be obtained through the following processes. To simplify the problem, the inclined angle β is considered as constant for a given crack. For a given β, the decoupled Mode I and Mode II displacement components can be expressed functionally as

$$\{U_{I(II)}\} = \{U_{I(II)}(x, y, a)\} \tag{3}$$

Applying the chain rule of differentiation with respect to the crack length (a) produces the following equation after rearrangement:

$$\frac{\partial\{U_{I(II)}\}}{\partial a} = \frac{d\{U_{I(II)}\}}{da} - \frac{\partial\{U_{I(II)}\}}{\partial x}\cdot\frac{dx}{da} - \frac{\partial\{U_{I(II)}\}}{\partial y}\cdot\frac{dy}{da} \tag{4}$$

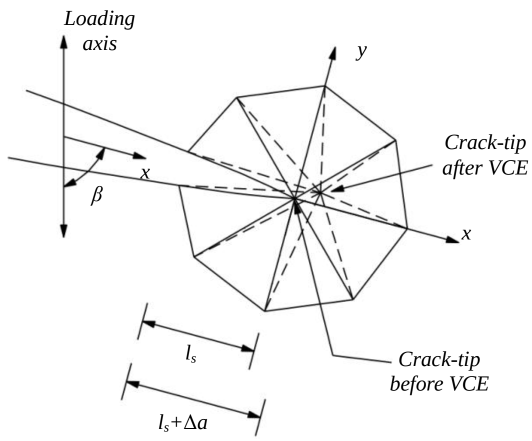

Fig. 1 Symmetric mesh in crack-tip neighborhood with respect to the global x axis

Since the stresses, strains and displacements of Mode I and Mode II are independent with each other and should satisfy the equilibrium equation and compatibility condition, we can obtain that

$$[K]\{U_{I(II)}\}-\{f_{I(II)}\}=0 \tag{5}$$

where $[K]$ is the matrix of global stiffness of original crack geometry. Taking total differentiation of Eq. (5) with respect to crack length and after rearranging, we have

$$\frac{d\{U_{I(II)}\}}{da}=[K]^{-1}\left[\frac{d\{f_{I(II)}\}}{da}-\frac{d[K]}{da}\{U_{I(II)}\}\right] \tag{6}$$

As the changes of the elemental stiffness for the entire structure and the decoupled nodal forces occur only in a few elements around the crack tip as a result of the VCE, $d[K]/da$ and $d\{f_{I(II)}\}/da$ of Eq. (6) can be expressed as

$$\frac{d[K]}{da}=\sum_{i=1}^{N_c}\frac{[k_i]_{a+\Delta a}-[k_i]_a}{\Delta a} \tag{7a}$$

$$\frac{d\{f_{I(II)}\}}{da}=\sum_{i=1}^{N_f}\frac{\{f_{I(II)}\}_{a+\Delta a}-\{f_{I(II)}\}_a}{\Delta a} \tag{7b}$$

where Δa is the VCE in direction collinear with the oblique crack. $[k_i]$ is elemental stiffness matrix. $\{f_{I(II)}\}$ are Mode I and II decoupled nodal forces. N_c is the number of elements around the crack tip. N_f is the number of crack-face elements with nodal perturbation of $f_{I(II)}$ as a result of VCE.

The last two terms of Eq. (6) serve as the correction factors of changing the total displacement derivatives to partial displacement derivatives for the oblique crack, which are null for nodes without geometric changes as a result of VCE. For the VCE, which is collinear with an oblique crack, we have $dy/da=0$.

Then, the nodal weight functions for Mode I and Mode II, with crack length (a) and inclination angle (β) at (x_i, y_i) locations, can be expressed as

$$h_{I(II)x}(x_i,y_i,a,\beta)=\frac{H}{2K_{I(II)}}\left\{\frac{d\{U_{I(II)x}\}}{da}-\frac{\partial\{U_{I(II)x}\}}{\partial x}\frac{dx}{da}\right\} \tag{8a}$$

$$h_{I(II)y}(x_i,y_i,a,\beta)=\frac{H}{2K_{I(II)}}\left\{\frac{d\{U_{I(II)y}\}}{da}-\frac{\partial\{U_{I(II)y}\}}{\partial x}\frac{dx}{da}\right\} \tag{8b}$$

and

$$\frac{\partial\{U_{I(II)}\}}{\partial x}=\frac{1}{\det[J]}\left\{\frac{\partial[N_iU_{I(II)i}]}{\partial\xi}\cdot\frac{\partial y}{\partial\eta}-\frac{\partial[N_iU_{I(II)i}]}{\partial\eta}\cdot\frac{\partial y}{\partial\xi}\right\} \tag{9}$$

where $\{U_{I(II)i}\}$ and N_i are the decoupled nodal displacements and shape function at i's nodal location. $\det[J]$ is the determinant of Jacobian Matrix between local coordinates (ξ, η) and global coordinates (x, y) for isoparametric elements.

3. RESULTS AND DISCUSSIONS

This study is dedicated to provide some references for the fracture analysis of a shear crack in an RC beam. As reported in [6], the dependence of weight function on constraint conditions for a given crack geometry can be circumvented through combining all self-equilibrium forces, which include the applied surface tractions and the reaction force induced from the selected constraint conditions, with the nodal weight functions of different constraint conditions of the same crack geometry. The constraint conditions as shown in Fig. 2 are employed in all calculations in this study because the stress state under these constraint conditions is clarity and convenient for analyzing.

3.1 Weight function for changing *h*/*w* ratios

For an RC beam, the failure mode is strongly dependent on the shear span/beam depth (h/w) ratio. [7, 8] reported that the critical inclined shear crack leading to collapse typically occurs only in beams, with $2.5<h/w<8.0$. Therefore, oblique edge crack geometries as shown in Fig. 2 with oblique angle β=45º, h_2/h_1=1.5, a/w=0.3, 0.4, 0.5, 0.6 and h/w=2.5, 5.0, 7.5 are employed for the detailed weight function calculation.

With respect to the coordinates (x', y'), Fig. 3 shows the weight function components along the left face for the crack geometries with a/w=0.3, 0.4, 0.5, 0.6 and h/w=2.5, 5.0, 7.5 in relation to the distance away from the point A in Fig. 2. For a given group of a/w and h/w ratios, $h_{Ix'}$ and $h_{IIx'}$ keep almost constant while $h_{Iy'}$ and $h_{IIy'}$ decrease with an increase of the distance away from the point A. Specially, in all $h_{Ix'}$ and $h_{IIx'}$, there is a consistent increase of absolute value with an increase of both a/w ratio and h/w ratio. For different a/w ratios, the constant values of $h_{Ix'}$ and $h_{IIx'}$ increase linearly with the increase of h/w ratio, which means the $h_{Ix'}$ and $h_{IIx'}$ for different a/w and h/w ratios can be obtained through linear interpolation of the presented results. In terms of $h_{Iy'}$ and $h_{IIy'}$, they increase with respect to the increase of a/w ratio, nevertheless stay nearly unchanged for a fixed a/w ratio and changing h/w ratio.

All the orderly trends represented in the curves of the weight function components can be interpreted theoretically as following. As shown in Fig. 4, for a 2-D crack geometry subjected to any arbitrary combined Mode I and Mode I load condition, the linear elastic stress field around a crack tip can be expressed with as simple analytical form as

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=\frac{K_I}{\sqrt{2\pi r}}\begin{Bmatrix}\cos(\theta/2)[1-\sin(\theta/2)\sin(3\theta/2)]\\ \cos(\theta/2)[1+\sin(\theta/2)\sin(3\theta/2)]\\ \sin(\theta/2)\cos(\theta/2)\cos(3\theta/2)\end{Bmatrix}+\frac{K_{II}}{\sqrt{2\pi r}}\begin{Bmatrix}-\sin(\theta/2)[2+\cos(\theta/2)\cos(3\theta/2)]\\ \sin(\theta/2)\cos(\theta/2)\cos(3\theta/2)\\ \cos(\theta/2)[1-\sin(\theta/2)\sin(3\theta/2)]\end{Bmatrix} \tag{10}$$

thus, for any point around the crack tip, θ and r are constants. Then,

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=[C]\cdot\begin{Bmatrix}K_I\\ K_{II}\end{Bmatrix} \tag{11}$$

where $[C]$ is a matrix of constants. According to the weight function concept, SIFs due to applied load is an integration of the applied load and the weight at their points of application.

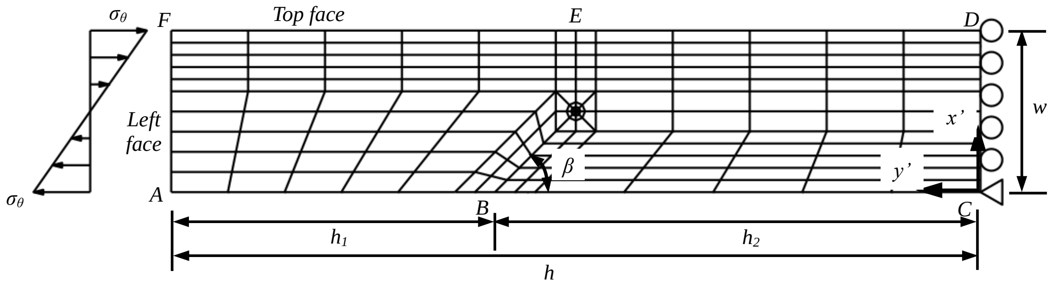

Fig. 2 Finite Element Model for a cracked beam geometry

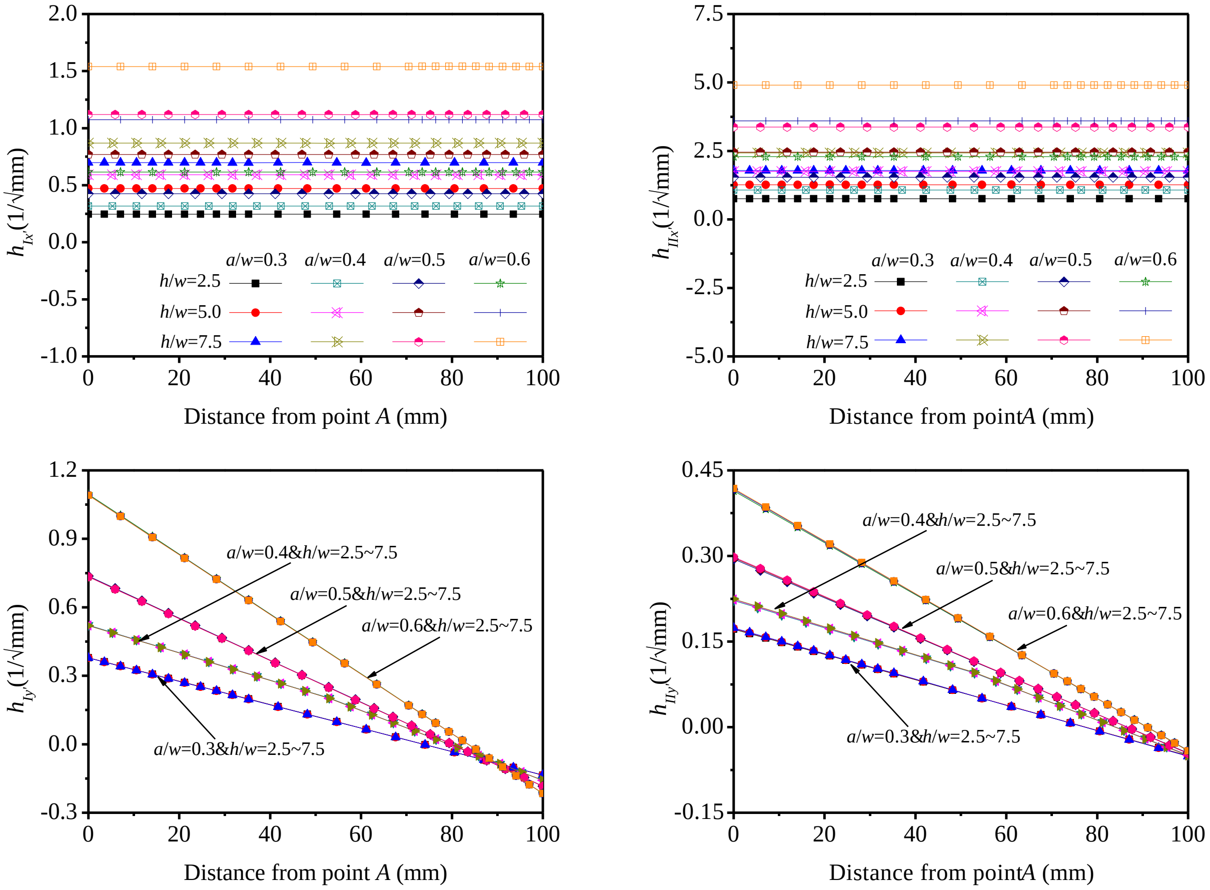

Fig. 3 Nodal weight functions left face

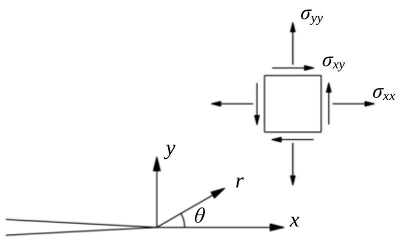

Fig. 4 Elastic stress field at crack-tip

Thus, the weight functions at a point on a crack geometry are equivalent to the SIFs if a unit concentrated load is applied on the point. Hence the $h_{Ix'}$, $h_{IIx'}$ and $h_{Iy'}$, $h_{IIy'}$ at a point on the left face can be obtained if a unit load is applied at the point along the x' and y' axes, respectively. In the elastic mechanics scale, it is easily imaginable that the stresses of any point around the crack-tip stay the same if a unit load applied along the x' axis moving from point A to F in Fig. 2 owing to the unchanged force lever from the crack-tip and vary linearly if the unit load applied along the y' axis. Similarly, following Eq. (11), the SIFs should experience the same trends manifested in the stresses. Therefore, $h_{Ix'}$, $h_{IIx'}$ and $h_{Iy'}$, $h_{IIy'}$ should be as shown in Fig. 3.

Considering that loads are generally applied on the top face of an RC beam, weight functions on the top face are presented as well in this study. Regarding the point E which is closest point away from the crack-tip on the top face as the origin and defining the direction from F to D as the positive direction, plots of the weight function components for the top face for the oblique edge crack geometries with a/w=0.3, 0.4, 0.5, 0.6 and h/w=2.5, 5.0, 7.5 are shown in Fig. 5. In all $h_{Ix'}$ and $h_{IIx'}$ curves, a consistent decrease from maximum to almost zero is observed from point F to E due to the decreasing force lever, while the $h_{Iy'}$ and $h_{IIy'}$ stay at a certain plateau in most of F-E region and drop dramatically to almost zero just adjacent E. When the load applied location passes E, the force lever turns into zero and consequently both $h_{Ix'}$, $h_{IIx'}$ and $h_{Iy'}$, $h_{IIy'}$ remain almost zero in most of E-D region. Due to the increasing content of local disturbing in the stresses around the crack-tip as the applied load approaching the crack-tip, a slight fluctuation is observed in all curves within a small region adjacent the origin point E. The characteristics shown in the weight function curves for the top face can be interpreted similarly as for the left face. Therefore, if a group of $h_{Ix'}$, $h_{IIx'}$ and $h_{Iy'}$, $h_{IIy'}$ weight function components for the top face of a crack geometry with a span (h) wider than the disturbed region is given, the weight function components for a crack geometry with a small span or a larger span can be obtained through cutting from or extending the given weight function components, respectively.

3.2 Weight function verification

Considering the load independent characteristic of weight function, the nodal weight functions of the oblique edge crack geometries determined under the pure bending loads can be applied to the evaluation of stress intensity factors and the corresponding strain energy release rates for the mixed fracture mode under remote tension loads. With the weight functions on the left face of the oblique edge crack geometries with h/w=5.0 and $0.3 \leq a/w \leq 0.6$ from pure bending loads, the strain energy release rates evaluated following weight function method and J-Integral method for the oblique edge crack elastic geometries under pure tension load conditions are listed in Table 1. The less than 1 percent discrepancies for all a/w ratios further confirm the applicability and reliability of the weight functions determination for mixed mode fracture problems through applying VCE technique with symmetric mesh around crack-tip.

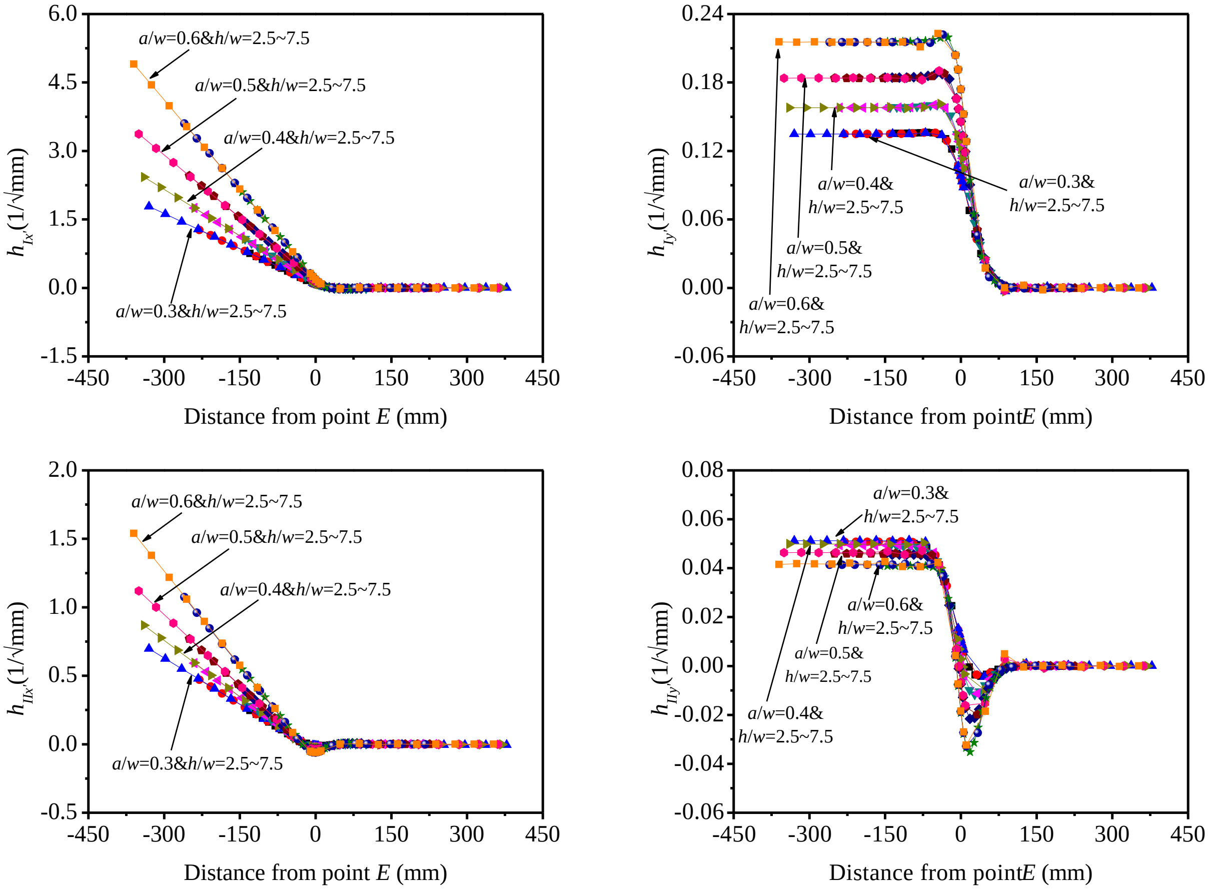

Fig. 5 Nodal weight functions on top face

Table 1 Normalized strain energy release rates

a/*w*	Weight function method	J-Integral method	Discrepancy (%)
0.3	3.990	3.976	-0.352
0.4	6.502	6.487	-0.238
0.5	11.776	11.746	-0.262
0.6	24.372	24.295	-0.319

4. CONCLUSIONS

In this study, an efficient finite element method, where a VCE technique is coupled with symmetric mesh in the crack-tip neighborhood, is used in evaluating both strain energy release rates and weight function for the oblique edge crack geometry. Specific findings and conclusions are summarized as follows:

(1) The weight functions on the left face and top face are evaluated for the crack geometries with h/w=2.5~7.5 and a/w=0.3~0.6. In LEFM, for an elastic cracked geometry under any arbitrary load conditions, the stresses of any points around the crack-tip can be related to the Mode I and Mode II stress intensity factor with a constant matrix. In addition, the stresses vary linearly for a unit load moving along the boundaries of the cracked geometry. Thus, the weight functions should vary linearly along the boundaries as is observed in all weight functions along the left face and top face for the RC beam. Therefore, the weight functions along all boundaries of an RC beam with any shear span/beam depth ratio can be evaluated using the corresponding weight functions for two RC beams with different shear span/beam depth ratios.

(2) For the same crack geometries, the weight functions obtained from pure bending load conditions are employed in evaluating the strain energy release rates due to pure tension loads according to the weight function concept. The less than 1% discrepancies between the normalized stress energy release rates from the weight function concept and J-Integral method confirms the reliability and applicability of evaluating weight functions using the VCE technique.

REFERENCES

1) Bueckner, H. F.: Field singularities and related integral representations. Methods of analysis and solutions of crack problems: Springer Netherlands. 239-314, 1973.
2) Cox, B., & Marshall, D.: The determination of crack bridging forces. International journal of fracture, 49(3), 159-176, 1991.
3) Hellen, T.: On the method of virtual crack extensions. International journal for numerical methods in engineering, 9(1), 187-207, 1975.
4) Parks, D. M.: A stiffness derivative finite element technique for determination of crack tip stress intensity factors. International journal of fracture, 10(4), 487-502, 1974.
5) George, T. S.: Stiffness derivative finite element technique to determine nodal weight functions with singularity elements. Engineering Fracture Mechanics, 19(4), 685-699, 1984.
6) George, T. S., and Chien-Tung, Y.: Weight function calculations for mixedmode fracture problems with the virtual crack extension technique. Engineering Fracture Mechanics, 21(6), 1119-1149, 1985.
7) Muttoni, A., and Ruiz, M. F.: Shear strength of members without transverse reinforcement as function of critical shear crack width. ACI Structural Journal, 105(2), 2008.
8) Zararis, P. D., and Papadakis, G. C.: Diagonal shear failure and size effect in RC beams without web reinforcement. Journal of structural engineering, 127(7), 733-742, 2001.