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平成27年度　土木学会北海道支部　論文報告集　第72号

ANALYSIS ON THE PERMANENT FORM OF SOLITARY STEP IN THE BEDROCK RIVER BED

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1. Introduction

Due to either steep gradients or meager sediment supply, the exposure of bedrock in river usually occurs, even when channel gradients only exceed about $3-10\%$[1)]. Moreover, in this kind of river bed with mixed alluvium and bedrock, a single solitary step is often observed to be formed as shown in Figure 1.

In the previous researches, the solitary step is also in the river with bed fully covered by sediment, and it can migrate upstream with permanent form in some situation caused by erosion[2)]. However, being different from the river bed without bedrock exposure, which is easily eroded by the flow, the bedrock is resistible to flow incision but more possible to be eroded by sediment abrasion[3)].

In this research, a model is applied to explain the bedrock incision mechanism. An analysis is expressed using this model to demonstrate if the solitary step forming on the bedrock can migrate upstream with a permanent profile as well.

2. Formulation

2.1 Governing equations

In this model, we employed MRSAA (Macro-Roughness-based Saltation-Abrasion-Alluviation) model[4)] to describe the bedrock erosion. The bedrock incision takes place due to abrasion of sediment transported on the bedrock. Therefore, the bedrock incision rate can be assumed to be proportional to the net amount of sediment transported on the bed. However, the bedrock is not eroded if the bed is completely covered with sediment. Therefore, the incision rate is described by

$$\frac{\partial \tilde{\eta}_b}{\partial \tilde{t}} = -\beta(1-p)\tilde{q}_a \tag{1}$$

where $\tilde{\eta}_b$ is the bed elevation of bedrock, $\tilde{t}$ is time, β is an empirical constant, $\tilde{q}_a$ is the net amount of sediment

Fig. 1 A solitary step observed in Ikeshomanai River in eastern Hokkaido

transported on the bedrock, and the tildes denote dimensional variables which will be removed to express non-dimensional equivalent. The cover factor p represents the percentage of the bed surface which is covered with sediment, and $1-p$ therefore is the rate of bedrock exposure. The net sediment transport rate $\tilde{q}_a$ is assumed to be expressed by the capacity sediment transport rate $\tilde{q}_{ac}$ times the availability of sediment on the bed. The availability of sediment on the bed should be related to the cover factor p. For this study, we assume that the availability of sediment is approximately identitied to the cover factor, the incision rate can be written in the form

$$\frac{\partial \tilde{\eta}_b}{\partial \tilde{t}} = -\beta p(1-p)\tilde{q}_{ac} \tag{2}$$

The time variation of the thickness of the alluvial layer is assumed to be expressed by the Exner type equation, with the use of the sediment transport capacity, written in the form

$$p\frac{\partial \tilde{\eta}_a}{\partial \tilde{t}} = -\frac{1}{1-\lambda}\frac{\partial p\tilde{q}_{ac}}{\partial \tilde{x}} \tag{3}$$

The cover factor is related to the thickness of alluvial layer $\tilde{\eta}_a$ by

$$p = \begin{cases} \dfrac{\tilde{\eta}_a}{\tilde{L}_{mr}} & when \quad \tilde{\eta}_a < \tilde{L}_{mr} \\ 1 & when \quad \tilde{\eta}_a > \tilde{L}_{mr} \end{cases} \tag{4}$$

$\tilde{L}_{mr}$ is the macro roughness height. If the bed is not completely covered with sediment, sediment will deposit in the interstices of the roughness.

The total bed elevation $\tilde{\eta}$ is the sum of the elevation of bedrock surface and the alluvial thickness as follows:

$$\tilde{\eta} = \tilde{\eta}_a + \tilde{\eta}_b \tag{5}$$

The flow on moderately mild slopes can be written in the St. Venant shallow water equations of the form

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} = -g\frac{\partial \tilde{h}}{\partial \tilde{x}} - g\frac{\partial \tilde{\eta}}{\partial \tilde{x}} - \frac{\tilde{\tau}_b}{\rho \tilde{h}} \tag{6}$$

$$\frac{\partial \tilde{u}\tilde{h}}{\partial \tilde{x}} = 0 \tag{7}$$

where $\tilde{u}$ is the velocity in the $\tilde{x}$ direction, $\tilde{h}$ is the flow depth, and $\tilde{\tau}_b$ is the bed shear stress, g is the gravity acceleration (=9.8 m/s^2), and ρ is the density of water (=1,000 kg/m^3). The bed shear stress vector is assumed to be written in the form

$$\tilde{\tau}_b = \rho C_f \tilde{u}^2 \tag{8}$$

where C_f is the bed friction coefficient which is usually a weak function of the flow depth divided by the roughness height. We assumed, however, that the bed friction coefficient is constant for simplicity in this study.

2.2 Equilibrium flow without step and critical erosion condition

Suppose that the bed is flat but tilted at a uniform slope S as shown in Figure 2. The sediment supply from the upstream is $\tilde{q}_{as}$, which is assumed not to be sufficient to cover the bed completely. Because the bed has the uniform slope, the velocity is constant, and the resultant sediment transport capacity is also constant. The cover factor p is then constant along the reach. Denoting these constant values of $\tilde{u}$, $\tilde{q}_{ac}$ and p by $\tilde{u}_n$, $\tilde{q}_{acn}$, and p_n respectively, we have the relation

$$p = p_n = \frac{\tilde{q}_{as}}{\tilde{q}_{ac}(\tilde{u}_n)} = \frac{\tilde{q}_{as}}{\tilde{q}_{acn}} \tag{9}$$

Because the bed is not completely covered with sediment as assumed herein, the sediment supply $\tilde{q}_{as}$ is smaller than the sediment transport capacity $\tilde{q}_{acn}$, and therefore, p is definitely less than unity.

The flow discharge per unit with $\tilde{q}_w$ is defined by

$$\tilde{u}_n \tilde{h}_n = \tilde{q}_w \tag{10}$$

where $\tilde{h}_n$ is the follow depth in the normal flow condition over a flat bed with a constant slope S. The bed shear stress is balanced with the streamwise component of gravity force, such that

$$\rho C_f \tilde{u}_n{}^2 = \rho g \tilde{h}_n S \tag{11}$$

Substituting (10) into (11), we obtain the Froude number corresponding to this normal flow condition. That is

$$Fr_n = \frac{\tilde{u}_n}{\sqrt{g\tilde{h}_n}} = \sqrt{\frac{S}{C_f}} \tag{12}$$

Even if the sediment supply $\tilde{q}_{as}$ and the flow discharge $\tilde{q}_w$ remain the same, the sediment transport capacity $\tilde{q}_{ac}$ decreases with declined slope. When the slope is smaller than some threshold slope S_t as shown in Figure 3, the velocity becomes sufficiently small for the sediment transport capacity $\tilde{q}_{ac}$ to decrease to the sediment supply $\tilde{q}_{as}$. The bed is then completely covered with sediment ($p = 1$). The velocity which can transport the amount of sediment exactly equivalent to the sediment supply is called the threshold velocity and is denoted by $\tilde{u}_t$.

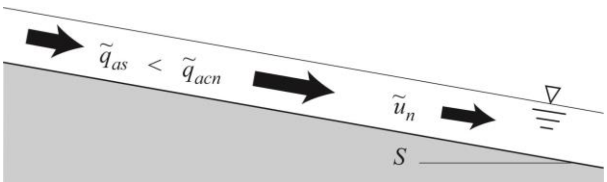

Fig. 2 Conceptual diagram of the normal flow condition without a step, and the definition of $\tilde{u}_n$

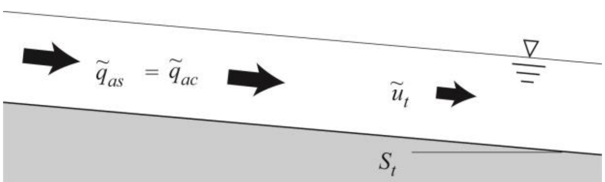

Fig. 3 Conceptual diagram of the threshold condition for bedrock incision, and the definition of $\tilde{u}_t$

The equation s of continuity and force balance are

$$\tilde{u}_t \tilde{h}_t = \tilde{q}_w,\ \rho C_f \tilde{u}_t^2 = \rho g \tilde{h}_t S_t \tag{13}$$

and

$$Fr_t = \frac{\tilde{u}_t}{\sqrt{g\tilde{h}_t}} = \sqrt{\frac{S_t}{C_f}} \tag{14}$$

2.3 Non-dimensionalization

We employed the sediment supply $\tilde{q}_{as}$ and the threshold value $\tilde{u}_t$, $\tilde{h}_t$, and S_t to normalize all the variables as shown in the following equations:

$$(\tilde{h}, \tilde{\eta}_a, \tilde{\eta}_b, \tilde{\eta}, \tilde{L}_{mr}) = \tilde{h}_t(h, \eta_a, \eta_b, \eta, L_{mr}) \tag{15a}$$

$$(\tilde{q}_a, \tilde{q}_{ac}) = \tilde{q}_{as}(q_a, q_{ac}) \tag{15b}$$

$$\tilde{x} = \frac{\tilde{h}_t}{S_t}x, \tilde{t} = \frac{\tilde{h}_t}{\beta \tilde{q}_{as}}t, \tilde{u} = \tilde{u}_t u \tag{15c-e}$$

Non-dimensional flow equations take the form

$$Fr_t^2 u \frac{\partial u}{\partial x} = -\frac{\partial h}{\partial x} - \frac{\partial \eta}{\partial x} - \frac{u^2}{h} \tag{16}$$

$$\frac{\partial uh}{\partial x} = 0 \tag{17}$$

The bed evolution equations are normalized in the form

$$\frac{\partial \eta_b}{\partial t} = -p(1-p)q_{ac} \tag{18}$$

$$\gamma p \frac{\partial \eta_a}{\partial t} = -\frac{\partial p q_{ac}}{\partial x} \tag{19}$$

$$p = \begin{cases} \dfrac{\eta_a}{L_{mr}} & when \quad 0 < \eta_a \le L_{mr} \\ 1 & when \quad \eta_a > L_{mr} \end{cases} \tag{20}$$

$$\eta = \eta_a + \eta_b \tag{21}$$

where the non-dimensional parameter γ is

$$\gamma = \frac{\beta(1-\lambda)\tilde{h}_t}{S_t} \tag{22}$$

The bedrock incision rate is usually rather small, so that β takes a considerably small value. Therefore, the value of γ is also expected to be small.

From (10), (12), (13) and (14), the non-dimensional normal flow velocity u_n is written in the form

$$u_n = \frac{\tilde{u}_n}{\tilde{u}_t} = \sqrt[3]{\frac{S}{S_t}} = S_r^{1/3} = \left(\frac{Fr_n}{Fr_t}\right)^{2/3} \tag{23}$$

where S_r is the bed slope normalized by the threshold bed slope S_t.

2.4 Quasi-steady assumption of alluvial process

We drop terms with the small parameter γ, (19) is reduced to

$$\frac{\partial p q_{ac}}{\partial x} = 0 \tag{24}$$

This means that the net sediment transport pq_{ac} is constant in space. The physical implication of this is that bedrock incision is so slow that the time variation of the bed elevation due to the alluvial process can be approximated to vanish in terms of slow time scale of the bedrock incisional process.

The dimensional form of pq_{ac} is $p\tilde{q}_{ac} = \tilde{q}_{as}$. Therefore, (24) is integrated to be

$$pq_{ac} = 1 \tag{25}$$

It means that the net sediment transport rate is constant in space even if the bed elevation is not constant. With the use of the above relation, (18) and (20) can be rewritten in the form

$$\frac{\partial \eta_b}{\partial t} = -(1 - q_{ac}^{-1}) \tag{26}$$

$$\eta_a = \frac{L_{mr}}{q_{ac}} \tag{27}$$

The time variation of the total bed elevation η is then written in the form

$$\frac{\partial \eta}{\partial t} = -L_{mr}\frac{q_{ac,u}}{q_{ac}^2}\frac{\partial u}{\partial t} - (1 - q_{ac}^{-1}) \tag{28}$$

where $q_{ac,u}$ is the partial derivative of q_{ac} with the respect to u.

3. Permanent form of a step

3.1 Flow equation on a step with a permanent form

In order to find the permanent form, we introduced the following moving coordinates:

$$x^* = x + ct, \quad t^* = t \tag{29}$$

Because of the bedrock incision, and the moving coordinate, the bed elevation η need to be adjusted in the vertical direction to achieve a permanent form. Denoting the vertical net aggradation rate by w, we obtained the following coordinate transformation:

$$\eta^* = \eta - wt \tag{30}$$

Figure 4 is geometrical relation between the incision rate in a flat portion $(1 - q_{ac}^{-1})\Delta t$, the apparent aggradation rate due to step migration $cS_r\Delta t$, and the net aggradation rate $w\Delta t$. That is

$$w = cS_r - (1 - q_{acn}^{-1}) \tag{31}$$

Applying the coordinate transformation (29)-(30) to (28), dropping the dependence on time t^* and dropping the stars for simplicity

$$-\frac{d\eta}{dx} = \frac{w}{c} + L_{mr}\frac{q_{ac,u}}{q_{ac}^2}\frac{du}{dx} + \frac{1}{c}(1 - q_{ac}^{-1}) \tag{32}$$

The flow equations (16) and (17) are invariable for the coordinate transformation.

Substituting (17) to (16) with considering $uh = 1$ to eliminate h, and then substituting the equation we get and (31) into (32), we obtained the following equation:

$$\frac{du}{dx} = \frac{S_r + c^{-1}(q_{acn}^{-1} - q_{ac}^{-1}) - u^3}{Fr_t^2 u - u^{-2} - L_{mr} q_{ac,u} q_{ac}^{-2}} \tag{33}$$

In order to go further, the bedload function has to be specified. We employed the Meyer-Peter & Müller formula of the form

$$q_{ac} = \left(\frac{u^2 - u_c^2}{1 - u_c^2}\right)^{3/2} \tag{34}$$

where u_c is the critical velocity below which bedload does not take place. Substituting (34) into (33), we obtained

$$\frac{du}{dx} = \frac{c^{-1}\left[\left(\frac{1-u_c^2}{u_n^2-u_c^2}\right)^{3/2} - \left(\frac{1-u_c^2}{u^2-u_c^2}\right)^{3/2}\right] + S_r - u^3}{Fr_t^2 u - u^{-2} - 3L_{mr}\frac{u(1-u_c^2)^{3/2}}{(u^2-u_c^2)^{5/2}}} \tag{35}$$

3.2 Boundary and regularity condition

Figure 5 is the conceptual diagram of the flow around a solitary step. We assume that the flows at infinite upstream and downstream are both in the normal flow condition as shown in figure 5. Going back to the dimensional variables, we find the following relation in the normal flow condition far upstream and downstream:

$$\rho C_f \tilde{u}_n^2 = \rho g \tilde{h}_n S \quad as \quad \tilde{x} \to \pm\infty \tag{36}$$

The origin of $\tilde{x}$ coordinate is defined somewhere around the step later. This equation can be rewritten in the non-dimensional form

$$u_n^2 = h_n S_r \quad as \quad x \to \pm\infty \tag{37}$$

Because $h = u/1$, we find

$$u_n = S_r^{1/3}, \; h_n = S_r^{-1/3} \quad as \quad x \to \pm\infty \tag{38}$$

As illustrated in figure 5, the flow is accelerated in the downstream direction and made a gradual transition from subcritical to supercritical regimes in the Froude sense upstream of the step. Therefore, the Froude critical point appears around the step. If the effect of an alluvial layer is not taken into account in (35), L_{mr} vanishes and the denominator of (35) vanishes when $u = Fr_t^{-2/3}$ which is the Froude critical point. In order to avoid the singularity at this point, the numerator should vanish as well. In this analysis, the effect of an alluvial layer is taken into account, and the denominator therefore vanishes when u is slightly larger than $Fr_t^{-2/3}$. If this value is denoted by u_1, the following equation is the definition of u_1:

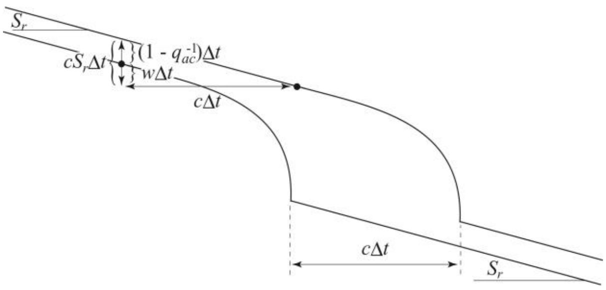

Fig. 4 Conceptual diagram of a permanent form of a step, and the definition of c and w

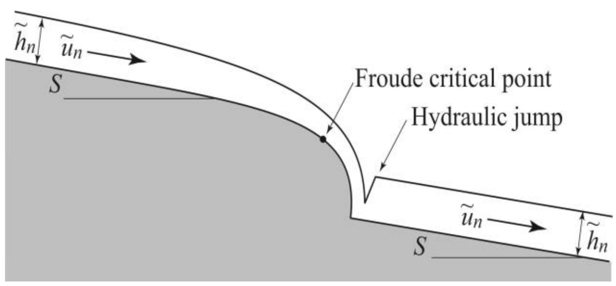

Fig. 5 Conceptual diagram of the flow around a solitary step

$$Fr_t^2 u_1 - u_1^{-2} - 3L_{mr}\frac{u_1(1-u_c^2)^{\frac{3}{2}}}{(u_1^2-u_c^2)^{\frac{5}{2}}} = 0 \qquad (39)$$

The condition for (35) not to have singularity (regularity condition) is the numerator vanishes when $u = u_1$. From this condition, the migration speed c is obtained in the form

$$c = \frac{\left(\frac{1-u_c^2}{u_n^2-u_c^2}\right)^{3/2} - \left(\frac{1-u_c^2}{u_1^2-u_c^2}\right)^{3/2}}{u_1^3 - S_r} \qquad (40)$$

If the Froude critical point is defined as the origin of x, the following boundary condition holds:

$$u = u_1 \quad when \quad x = 0 \qquad (41)$$

3.3 Numerical solution

Substituting (40) into (35), we obtain

$$\frac{du}{dx} = \frac{\frac{\left(\frac{1-u_c^2}{u_n^2-u_c^2}\right)^{\frac{3}{2}} - \left(\frac{1-u_c^2}{u^2-u_c^2}\right)^{\frac{3}{2}}}{\left(\frac{1-u_c^2}{u_n^2-u_c^2}\right)^{\frac{3}{2}} - \left(\frac{1-u_c^2}{u_1^2-u_c^2}\right)^{\frac{3}{2}}}(u_1^3 - S_r) + S_r - u^3}{Fr_t^2 u - u^{-2} - 3L_{mr}\frac{u(1-u_c^2)^{3/2}}{(u^2-u_c^2)^{5/2}}} \qquad (42)$$

The above equation is integrated to yield the velocity distributions as a function of x. The boundary conditions are the following:

$$u = s_r^{1/3} \quad as \quad x \to -\infty \qquad (43)$$

$$u = u_1 \quad as \quad x = 0 \qquad (44)$$

The problem includes four parameters: S_r, Fr_t, u_c, L_{mr}. In order for the permanent form of solitary step to exist, the four parameters cannot arbitrary given, but need to satisfy some condition. This problem is then reduced to find the domain of the four parameters. If these domains exist, through integrating equation (42) under the boundary conditions, the distribution of flow velocity and slope profile must be obtained as expected.

4. Results and discussion

First, the four parameters have their own domains due to the physical meaning that they represent, which is expressed as follow. $\tilde{u}_c$ should be smaller than $\tilde{u}_t$ for the reason that even in critical erosion condition, the sediment can still be transported from upstream. In terms of non-dimensional condition, that means $0 < u_c < 1$. The normalized uniform slope at the upstream S_r should be larger than 1 in order not to reach critical erosion condition. Considering the macro roughness height is much smaller than the flow depth in reality, we assume that $0 < L_{mr} \leq 0.1$. To ensure the existence of the two boundary conditions, the velocity at far upstream $\tilde{u}_n$ must be smaller than that at Froude critical point, $\tilde{u}_1$. In this case, we simplified the condition into $u_n = S_r^{1/3} < Fr_t^{-2/3}$.

After applying the value of each parameter into (42) within its domain as mentioned above, we found that the derivative of u with respect to x stays negative in all the conditions. It can be seen from Figure 6 and Figure 7 as examples.

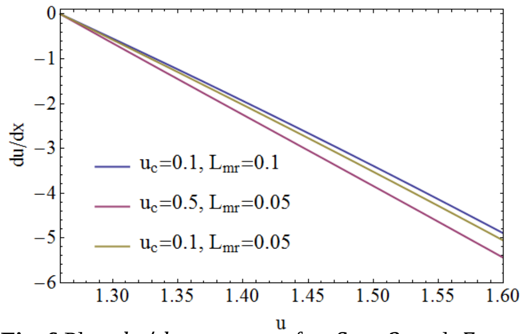

Fig. 6 Plot du/dx versus u for $S_r = 2$ and $Fr_t = 0.5$

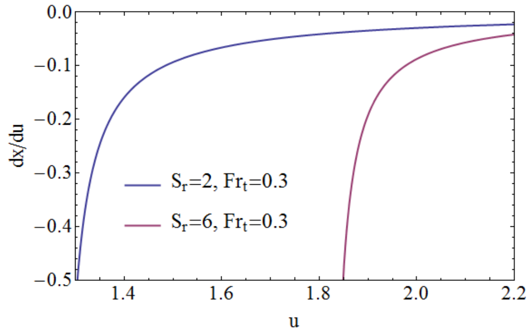

Fig. 7 Plot dx/du versus u for $u_c = 0.5$ and $L_{mr} = 0.05$

For this reason, the flow velocity cannot accelerate in the downstream direction, which contradicts to the assumption. On the other hand, the bedrock solitary step does not have a permanent form that migrates upstream.

5. Conclusion

An analysis is performed on a solitary step in the bedrock-alluvial river to demonstrate if the step can migrate upstream without changing its profile. It is found that the solitary step as assumed does not exist. Therefore, it keeps changing its shape. In this case, two situations could occur. One is that the step surface reduces its steepness, and finally disappears. The other is that it may increase its steepness and becomes a discontinuous step.

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