Linear stability analysis of the natural roll waves with the mixing length turbulent model

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ABSTRACT

This paper presents the linear stability analysis of natural roll waves. The open channel flow is analyzed by mixing length turbulent model and perturbation equations are solved in order to get the variation of Froude number vs the corresponding wave number of the roll waves. Spectral collocation method with the Chebyshev polynomials is used in solution process. The variation is compared with experimental data which were obtained from conducting experiments in open channels by Richard R. Brock(1) (1967).

1. INTRODUCTION

Roll waves can be described as large amplitude disturbances which are developed on turbulent water flows. There are two types of roll waves as natural roll waves and periodic permanent roll waves. Natural roll waves can be seen in natural flows such as ice channels, gravity currents in the laboratory, ocean and lakes Balmforth N.J and Mandre S.(2) (2004).

In this study a mathematical model is proposed to explain the variation between the Froude number and the wave number with the use of logarithmic velocity distribution. Also mixing length turbulent model is used which was proposed by Colombini (3)(2004).

2. FORMULATION

Turbulent flow in an open channel can be expressed using the Navier – Stokes equations and continuation equation of the form of non dimensional manner as follows.

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+\frac{\partial P}{\partial x}+1+\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{xy}}{\partial y}=0 \qquad (1)$$

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+\frac{\partial P}{\partial y}+\frac{1}{S}+\frac{\partial T_{yx}}{\partial x}+\frac{\partial T_{yy}}{\partial y}=0 \qquad (2)$$

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0 \qquad (3)$$

Where x and y are the stream wise and normal to the stream wise direction respectively. U and V are the components of the velocity in x and y direction respectively, S and P are the average bed slope and the pressure respectively, T_{ij} $(i, j = x, y)$ is the Reynolds stress tensor. All above equations are non-dimensionalized as

$$(U^*, V^*) = U^*_{f0}(U, V) \qquad (a)$$

$$(x^*, y^*) = H^*_0(x, y) \qquad (b)$$

$$(P^*, T^*_{ij}) = \rho U^{*2}_{f0}(P, T_{ij}) \qquad (c)$$

Where U^*_{f0} and H^*_0 are the friction velocity and the flow depth in the base state flat bed condition. The Reynolds stress tensor is expressed by using the mixing length turbulent model as follows.

$$(T_{xx}, T_{yy}) = 2\nu_T\left(\frac{\partial U}{\partial x}, \frac{\partial V}{\partial y}\right) \qquad (d)$$

$$T_{xy} = T_{yx} = \nu_T\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right) \qquad (e)$$

$$\nu_T = l^2\left|\frac{\partial U}{\partial y}\right| \qquad (f)$$

$$l = \kappa(y-z)\sqrt{\left(\frac{H-y}{H}\right)} \qquad (g)$$

Where ν_T is the eddy viscosity normalized by U^*_{f0} H^*_0, *l,z and H* are the mixing length, the bed elevation and the flow depth normalized by H^*_0 and κ is the Karman constant which is 0.4 in this analysis.

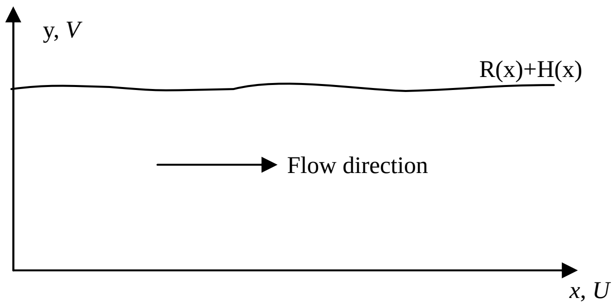

Figure 1. The conceptual diagram of flow and the coordinate system

Following variable transformation is introduced in order to describe the boundary conditions of water surface and bottom.

$$(\xi, \eta) = \left(x, \frac{y-R}{H(x)}\right) \qquad (h)$$

Where R is the reference level at which the velocity vanishes in the logarithmic velocity distribution. Then the nondimensional mixing length l can be modified as follows.

$$l = \kappa H(\eta + \frac{R-z}{H})\sqrt{(1-\frac{R}{H}-\eta)} \quad \text{(i)}$$

$\boldsymbol{U}.e_{ns} = 0$ at $\eta = 1$
$e_{ns}.\boldsymbol{T}.\ e_{ns} = 0$ at $\eta = 1$
$e_{ts}.\boldsymbol{T}.\ e_{ns} = 0$ at $\eta = 1$
$\boldsymbol{U}.e_{nb} = 0$ at $\eta = 0$
$\boldsymbol{U}.e_{tb} = 0$ at $\eta = 0$

Where U is the velocity vector, e_{ns} and e_{ts} are unit vectors normal and tangential to the water surface respectively, e_{nb} and e_{tb} are unit vectors normal and tangential to the bottom respectively. $\boldsymbol{T}$ is the stress tensor expressed as follows.

$$T = \begin{pmatrix} -P + T_{xx} & T_{xy} \\ T_{xy} & -P + T_{yy} \end{pmatrix} \quad \text{(j)}$$

3. THE ONE DIMENSIONAL BASE STATE SOLUTION

The base state solution is obtained from the above mentioned Navier Stokes equations using the flat bed normal flow conditions. The parameters are reduced to (U, V, H, z, R) = (U_0, 0, 1, 0, R_0)

The governing equations are reduced and solved using following boundary conditions to get the logarithmic velocity distribution as mentioned.

$U = 0$, $T_{xy0} = 1 - R_0$ at $\eta = 0$

$$U_0 = \frac{1}{\kappa}\ln(\frac{\eta + R_0}{R_0}) \quad (4)$$

Integration of the above logarithmic velocity distribution can be used to obtain the friction law coefficient; C as follows.

$$C^{-1} = \frac{U^*_{a0}}{U^*_{f0}} = \frac{1}{\kappa}\left[(1+R_0)\ln(\frac{1+R_0}{R_0}) - 1\right] \quad (5)$$

Where U^*_{a0} is the depth averaged velocity in the base state.

4. LINEAR STABILITY ANALYSIS

Variables are expanded as follows in order to get the perturbation solution.

$$(\varphi, P, H) = (\varphi_0, P_0, 1) + A(\hat{\varphi}_1, \hat{P}_1, \hat{H}_1) \quad (6)$$

Where A is the amplitude of the perturbation, which is a infinitesimally small. The governing equations are reduced to the following and the equations are shown using some linear operators due to lack of the space.

$$\hat{\lambda}^{\varphi}\hat{\varphi}_1 + \hat{\lambda}^H \hat{H}_1 = 0 \quad (7)$$

$$\frac{\partial \hat{P}_1}{\partial \xi} + \hat{\mu}^{\varphi}\hat{\varphi}_1 + \hat{\mu}^H \hat{H}_1'' = 0 \quad (8)$$

The perturbation is assumed to be expressed by

$$(\hat{\varphi}_1, \hat{P}_1, \hat{H}_1) = (\varphi_1, P_1, H_1)\exp\left[i(\alpha\xi - \Omega t)\right] \quad (9)$$

Where α and Ω are the wave number and the complex angular frequency of the perturbation, respectively. With the help of the Chebyshev polynomials governing equations are solved mathematically.

Figure 2 shows the contours of the growth rate of perturbation Im[Ω] in the α – F plane. Also the experimental data is overlapped in the same figure in order to do the comparison with the theoretical results. The experimental data are obtained from the experiments which were done by Richard R. Brock[1] (1967) and the series of Froude numbers are shown against the wave numbers at several stations along the open channel.

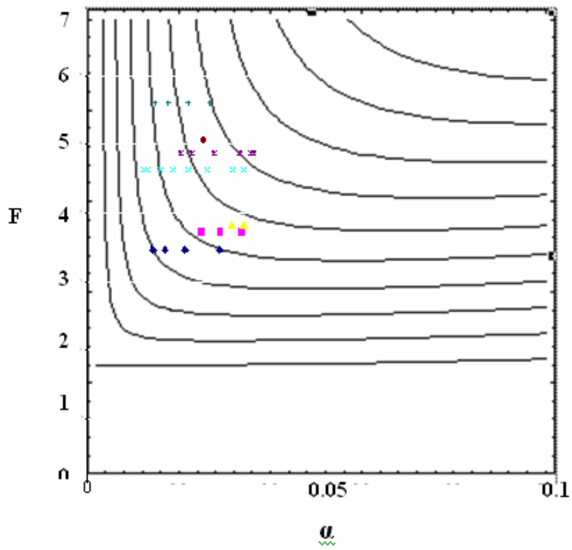

Figure 2. The contours of Im[Ω]. C^{-1}=20 and experimental data

5. RESULTS AND DISCUSSION

It has been seen from the Figure 2 that the experimental values are farley ongoing with the dominant wave numbers that have been obtained from the theoretical analysis. Also with the time the experimental values are deviated from the theoretical contour map. This may be due to the non linearity of the situation.

6. REFERENCES

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(3)Colombini, M. (2004). Revisiting the linear theory of sand dune formation. J. Fluid Mech. 502, 1-16