Constrained Kalman Filtering for Intersection Origin-Destination Matrices Estimation

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1. Introduction

The study of origin-destination flow (O-D flow) through networks and intersections provides basic information for planning and design as well as for traffic management and control. There have been many efforts attempted to estimate these flows using data from inexpensive technique such as vehicle counts from detectors. In control and management applications, the short-term estimation is more important compared to the long-term average values. In this regard, Kalman filtering technique has been used to automatically estimate dynamic O-D flow by a number of researchers [1-3]. However, the natural equality constraints for conservation of vehicles and the non-negativity constraints of O-D flows (or splits) were often neglected or at most were dealt with using heuristic method (truncation and normalization).

The objective of this paper is to present a new theory of constrained Kalman filter developed by Simon and Chia [4] and Simon and Simon [5] and using this theory to formulate a new framework for estimation of O-D flows considering equality and non-negativity constraints. In section 2, a brief review of standard Kalman filtering technique is provided. In section 3, the method is then modified in order to deal with constraints explicitly. In section 4, the problem of intersection O-D matrices estimation is formulated using this new technique and finally the concluding remarks are drawn.

2. Kalman Filtering Technique

Consider the discrete linear system given by:

$$\mathbf{x}_{k+1} = \mathbf{A}_k \mathbf{x}_k + \mathbf{w}_k \tag{1}$$

$$\mathbf{y}_k = \mathbf{C}_k \mathbf{x}_k + \mathbf{v}_k \tag{2}$$

where $\mathbf{x}$ is the ($n*1$) state vector, $\mathbf{y}$ is the ($m*1$) measurement, k is the time index, n and m are respectively the number of state and observation variables, $\mathbf{A}$ and $\mathbf{C}$ are coefficients matrices of state and measurement equations, $\mathbf{w}$ and $\mathbf{v}$ are noises of state and measurement equations and are assumed to be Gaussian with covariance matrices as $\mathbf{W}$ and $\mathbf{V}$, respectively. Further, it is assumed that $\mathbf{w}$ and $\mathbf{v}$ are uncorrelated.

The equations of Kalman filter for this type of system are given as below:

Time update:

$$\hat{\mathbf{x}}_k^- = \mathbf{A}_k \hat{\mathbf{x}}_{k-1} \tag{3}$$

$$\mathbf{P}_k^- = \mathbf{A}_k \mathbf{P}_{k-1} \mathbf{A}_k^T + \mathbf{W}_k \tag{4}$$

$$\mathbf{K}_k = \mathbf{P}_k^- \mathbf{C}_k^T \left(\mathbf{C}_k \mathbf{P}_k^- \mathbf{C}_k^T + \mathbf{V}_k \right)^{-1} \tag{5}$$

Measurement update:

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K}_k (\mathbf{y}_k - \mathbf{C}_k \hat{\mathbf{x}}_k^-) \tag{6}$$

$$\mathbf{P}_k = \mathbf{P}_k^- - \mathbf{K}_k \mathbf{C}_k \mathbf{P}_k^- \tag{7}.$$

Here, $\mathbf{K}$ is the Kalman gain, $\mathbf{P}$ is the covariance matrix of estimation error.

3. Constrained Kalman Filter

In the previous section, state variables are estimated without considering any constraint. In reality, the estimation is sometimes not allowed to fall into the infeasible regions. The constraints for the general O-D flow estimation problem are equality and non-negativity constraints. They will be explained separately in the following subsection.

3.1 Equality Constraints

For the dynamic system in (1) and (2), consider additional constraints

$$\mathbf{D}\mathbf{x}_k = \mathbf{d}_k \tag{8}$$

where $\mathbf{D}$ is a known ($s*n$) matrix, $\mathbf{d}$ is a known ($s*1$) vector, and s is the number of constraints. Hereafter, to simplify the equations, subscript k will not be shown except required. Simon and Chia [4] proposed the method to incorporate equality constraints into the standard Kalman filter as follows:

$$\tilde{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{U}^{-1}\mathbf{D}^T \left(\mathbf{D}\mathbf{U}^{-1}\mathbf{D}^T \right)^{-1} \left(\mathbf{D}\hat{\mathbf{x}} - \mathbf{d} \right) \tag{9}$$

$$\tilde{\mathbf{P}} = \mathbf{P} - \mathbf{U}^{-1}\mathbf{D}^T \left(\mathbf{D}\mathbf{U}^{-1}\mathbf{D}^T \right)^{-1} \mathbf{D}\mathbf{P} \tag{10}$$

where $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{P}}$ are the new state vector and error covariance matrices after imposing constraints, $\mathbf{U}$ is any symmetric positive definite weighting matrix. In Simon and Chia [4], if $\mathbf{U} = \mathbf{P}^{-1}$, the model is equivalent to the maximum probability method, and if $\mathbf{U} = \mathbf{I}$, it is equivalent to the mean square minimization method. The method considered in this paper follows the maximum probability method. Both equations (9) and (10) have two terms; the first term is the estimate obtained from unconstrained Kalman filter (equation (3) to (7)), the second term can be considered as the correction term for equality constraints.

3.2 Non-negativity Constraints

Consider now the following non-negativity constraints with system equation (1) and (2)

$$\mathbf{x}_k \geq \mathbf{0} \tag{11}$$

Simon and Simon [5] have proposed the method to solve inequality problem using the preliminary estimated result from unconstrained Kalman filter. The problem is redefined as

$$\min_{\tilde{\mathbf{x}}}\left(\tilde{\mathbf{x}}^T\mathbf{U}\tilde{\mathbf{x}} - 2\hat{\mathbf{x}}^T\mathbf{U}\tilde{\mathbf{x}}\right) \text{ such that } \tilde{\mathbf{x}} \geq \mathbf{0} \tag{12}$$

where $\mathbf{U}$ is the same as in previous. This problem is known as a quadratic programming problem and can be solved using an active set method [5]. This method uses the fact that it is only those constraints that are active at the solution of problem that are significant in the optimality conditions. Thus the non-negativity problem is reduced to the equality constraints problem by defining a new set of equality constraints which correspond to the active set of non-negativity problem. Using the new set of equality constraints, equations (9) and (10) can be used and we can then obtain the estimate of non-negativity constraints problem.

4. Intersection O-D Matrices Estimation

This section explains the framework of using constrained Kalman filter to estimate intersection O-D matrices. Denote $q_i(k)$ as volume entering at entrance i during time interval k, $y_j(k)$ as volume which leaves exit j during time interval k, $\mathbf{q}(k)$ and $\mathbf{y}(k)$ as vectors containing $q_i(k)$ and $y_j(k)$ respectively, $b_{ij}(k)$ as fraction of O-D flow (O-D split) entering at entrance i and leaves at exit j during time interval k, $\mathbf{b}_j(k)$ as vector containing all split which are related to the jth exit flow, $\mathbf{b}(k)$ as the vector containing all split parameters. The following relationships should be hold:

$$y_j(k) = \sum_{i=1}^{n_o} q_i b_{ij} \tag{13}$$

$$0 \leq b_{ij}(k) \leq 1 \qquad \text{for all } i, j, k \tag{13}$$

$$\sum_{j=1}^{n_d} b_{ij}(k) = 1 \qquad \text{for all } i, k \tag{14}$$

There are two alternative approaches to formulate the above equations as state-space model. The first method, regarded as the simultaneous method, is to estimate all split parameters at the same time. The second approach, regarded as the exit-based method, is to separate the estimation according to each exit jth.

The state-space formulation of the first approach is defined as

$$\mathbf{b}(k) = \mathbf{b}(k-1) + \mathbf{w}(k) \tag{15}$$

$$\mathbf{y}(k) = \mathbf{Q}(k)\mathbf{b}(k) + \mathbf{v}(k) \tag{16}$$

where $\mathbf{Q}(k)$ is an entry flow matrix which has a one-to-one mapping with $\mathbf{q}(k)$ while $\mathbf{w}(k)$ and $\mathbf{v}(k)$ are as before. Equations (3) – (7) are used to obtain the unconstrained estimate and equations (9) – (10) are then used to impose equality constriants of equation (14). Next is to check for the non-negativity constraint of the estimate. If there exist an active set, include this set into the equality constraints and perform again equations (9) – (10) until the non-negativity constraint is satisfied. The state-space model of the second approach is

$$\mathbf{b}_j(k) = \mathbf{b}_j(k-1) + \mathbf{w}(k) \tag{17}$$

$$\mathbf{y}_j(k) = \mathbf{q}(k)\mathbf{b}_j(k) + \mathbf{v}(k) \tag{18}$$

In this approach, similar to the work of Cremer and Keller [1], Kalman gain, error covariance matrix, and split parameters are calculated separately for each exit flow during the unconstrained estimate. The result of split and error covariance is then combined into a single vector and matrix. We then use the same procedure as the first approach to obtain the correction for equality and non-negativity constraints. After the correction, error covariance matrix is then separated in a similar way as when it is combined.

5. Concluding Remarks

In this paper, a new framework for intersection O-D matrices estimation based on constrained Kalman filter is presented. The framework will be tested with simulated and real traffic data as well as the comparison with some other estimation methods in the near future.

6. References

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