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Free Vibrations of Cylindrical Shells Containing Liquid on Elastic Foundations

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1. Introduction

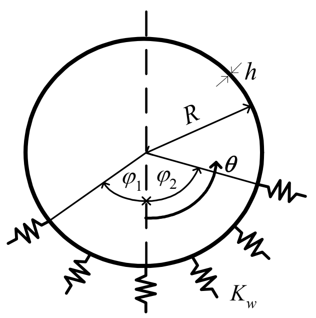
Cylindrical shells as structural components are used in many engineering fields in the form of storage tanks, water ducts, storage, vessels, pipelines, and many others. They are used because of their strength and effectiveness. This paper examines the interactive behavior between shell, foundation, and internal liquid.

Lakis *et al*.[1-2] have developed the hybrid finite element formulation for circular cylindrical shells based on the analytical shape functions which are derived from governing equation of shell. The analysis of whole buried pipeline subjected to sinusoidal seismic wave, differential settlement, and dislocation of ground has been investigated by Yang *et al*.[3] using the shell finite element. Paliwal *et al*.[4-5] have studied the free vibrations of the whole buried cylindrical shells in Winkler and Pasternak foundations by direct solution to the governing equations of motion. In the paper, the elastic foundations are distributed uniformly both in the circumferential and in the longitudinal directions. However, cylindrical shells are generally laid on elastic foundation, so that the foundation only covers certain parts of the shell in the circumferential direction. This leads to more complex problem. Amabili *et al*.[6] have investigated the free vibrations of cylindrical shell simply supported at both ends with a non-uniform elastic foundation in the circumferential direction based on the Rayleigh-Ritz method. The elastic foundation has to be assumed distributed uniformly over the whole cylinder length in the longitudinal direction. Gunawan *et al*.[7-8] have studied the static and free vibration of cylindrical shells partially buried in the elastic foundation based on the semi-analytical finite element method where the simple polynomials were used as shape function in the longitudinal direction.

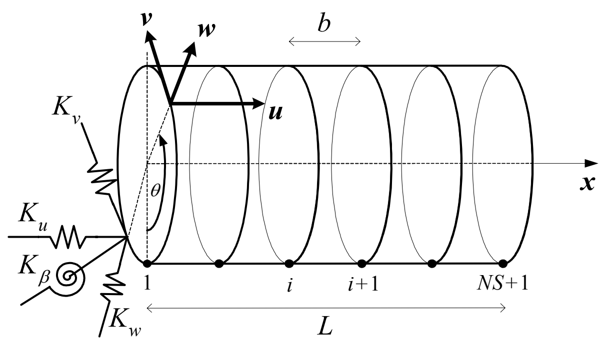
This paper presents the dynamic characteristics (i.e. natural frequencies and mode shapes) of fluid-filled cylindrical shells on elastic foundation by means on the semi-analytical finite element method where shape functions based on the governing equations of shell are used in the analysis. The shell is discretized into cylindrical finite elements. Soil as a foundation is represented by four parameter elastic springs and may be distributed by a Fourier series and an element mesh strategy in the circumferential and in the longitudinal directions, respectively. The internal fluid is described by the potential flow. Hydrodynamic pressure acting on the shell is derived from the condition for dynamic coupling of the fluid-structure. The influence of internal fluid, shell geometries, and foundation parameters (i.e. spring stiffness and enclosed angle) on the natural frequency of the vibrating system is presented systematically for both symmetrical and asymmetrical vibrations.

2. Model and formulation

The structure is an isotropic thin elastic cylindrical shell with Young's modulus E, Poisson's ratio υ, radius of the middle surface R, thickness h, and length L. The foundation is represented by continuous elastic (axial, circumferential, radial, and rotational) springs and distributed on a limited arc. The axial, circumferential, radial, and rotational spring coefficients are denoted by K_u, K_v, K_w, and K_β, respectively. In the analysis, all the spring coefficients are assumed to be constant along the enclosed arc. The angles that define the enclosed arc are denoted by φ_1 and φ_2. The geometry and generalized model of the structure are shown in Fig.1.



(a) Geometry



(b) Generalized model

Fig.1. Cylindrical shell on elastic foundation.

The displacement of a point on the middle surface in the axial, circumferential, and radial directions is indicated by u, v, and w, respectively. The displacement functions which include the symmetrical (superscript S) and asymmetrical (superscript U) deformations with respect to the the θ = 0 axis are given by

$$
\begin{aligned}
u(x,\theta) &= \sum_{m=0}^{M}\left[U_m^{\mathrm{S}}(x)\cos(m\theta)+U_m^{\mathrm{U}}(x)\sin(m\theta)\right] \\
v(x,\theta) &= \sum_{m=0}^{M}\left[V_m^{\mathrm{S}}(x)\sin(m\theta)+V_m^{\mathrm{U}}(x)\cos(m\theta)\right] \\
w(x,\theta) &= \sum_{m=0}^{M}\left[W_m^{\mathrm{S}}(x)\cos(m\theta)+W_m^{\mathrm{U}}(x)\sin(m\theta)\right]
\end{aligned}
\tag{1}
$$

where m is a typical circumferential wave number. For the sake of brevity, formulation is explained only for a symmetrical system since formulation for the asymmetrical system is completely analogous to that of the symmetrical one. The shape functions in the longitudinal direction are assumed [10] to be in the form of:

$$
\begin{aligned}
U_m^{\mathrm{S}}(x) &= A_m^{\mathrm{S}}\, e^{\mu_m x/R} \\
V_m^{\mathrm{S}}(x) &= B_m^{\mathrm{S}}\, e^{\mu_m x/R} \\
W_m^{\mathrm{S}}(x) &= C_m^{\mathrm{S}}\, e^{\mu_m x/R}
\end{aligned}
\tag{2}
$$

where A_m^{S}, B_m^{S}, and C_m^{S} are constants for a typical circumferential wave, m. μ_m is the characteristic value which can be found by substituting Eq.(2) into the following Sanders equations of thin cylindrical shells:

$$
L_i\left(u,v,w\right)=0 \tag{3}
$$

where L_i (i = 1, 2, 3) are the differential operators of the shell equation (without the foundation) with respect to x and θ. Details of the operators may be found in Ref.[2]. On the substitution of Eq.(2) into Eq.(3), three linear simultaneous equations in A_m^{S}, B_m^{S}, and C_m^{S} can be obtained. For a non-trivial solution, determinant of the coefficient matrix has to be zero. After simplifications, for $m \neq 0$, the bi-fourth polynomial can be obtained and given below:

$$
\mu_m^8 + a_{m1}\mu_m^6 + a_{m2}\mu_m^4 + a_{m3}\mu_m^2 + a_{m4} = 0 \tag{4}
$$

where a_{mi} (i = 1, 2, 3, 4) are the coefficients of the polynomial. Solution of Eq.(4) leads to eight complex characteristic roots μ_{mj} (j = 1, 2, 3, … 8). As the constants A_m^{S}, B_m^{S}, and C_m^{S} are not independent, the complete set of the shape functions in the longitudinal direction can be rewritten as follows:

$$
\begin{aligned}
U_m^{\mathrm{S}}(x) &= \sum_{j=1}^{8}\alpha_{mj}^{\mathrm{S}}\, C_{mj}^{\mathrm{S}}\, e^{\mu_m x/R} \\
V_m^{\mathrm{S}}(x) &= \sum_{j=1}^{8}\gamma_{mj}^{\mathrm{S}}\, C_{mj}^{\mathrm{S}}\, e^{\mu_m x/R} \\
W_m^{\mathrm{S}}(x) &= \sum_{j=1}^{8} C_{mj}^{\mathrm{S}}\, e^{\mu_m x/R}
\end{aligned}
\tag{5}
$$

where α_{mj}^{S} and γ_{mj}^{S} are constants which can be determined by back substitution into Eq.(3). It is worthwhile to mention that the asymmetrical system leads to the identical characteristic polynomial as for the symmetrical system (Eq.(4)) so that $\mu_{mj}^{\mathrm{S}} = \mu_{mj}^{\mathrm{U}} = \mu_{mj}$, $\alpha_{mj}^{\mathrm{S}} = \alpha_{mj}^{\mathrm{U}}$, but $\gamma_{mj}^{\mathrm{S}} = -\gamma_{mj}^{\mathrm{U}}$. For m = 0, the system is separated into torsional and non-torsional systems [2].

Discretization is done in the usual way as in the finite element method. The following nodal displacement parameters at the element boundaries are used:

$$
\boldsymbol{\delta}\mathbf{e}_m^{\mathrm{S}} = \left\{u_i^{\mathrm{S}} \quad v_i^{\mathrm{S}} \quad w_i^{\mathrm{S}} \quad \beta_i^{\mathrm{S}} \quad u_{i+1}^{\mathrm{S}} \quad v_{i+1}^{\mathrm{S}} \quad w_{i+1}^{\mathrm{S}} \quad \beta_{i+1}^{\mathrm{S}}\right\}^{\mathrm{T}} \tag{6}
$$

where rotation angle β is defined as the first derivative of w with respect to x.

The stiffness and mass matrices of a shell element are given by [7]

$$
\mathbf{K}_{\mathbf{S}} = \iint_A \mathbf{B}^{\mathrm{T}}\,\mathbf{P}\,\mathbf{B}\,\mathrm{d}A \tag{7}
$$

and,

$$
\mathbf{M}_{\mathbf{S}} = \rho_{\mathrm{S}} h \iint_A \mathbf{N}^{\mathrm{T}}\,\mathbf{N}\,\mathrm{d}A \tag{8}
$$

where ρ_{S} is mass per unit volume of the shell. The integration in thickness direction has already been carried out in Eq.(7).

The stiffness matrix of foundation for an element is given by

$$
\mathbf{K}_{\mathbf{F}} = \iint_A \mathbf{N}^{\mathrm{T}}\,\boldsymbol{\Psi}\,\mathbf{N}\,\mathrm{d}A \tag{9}
$$

where

$$
\boldsymbol{\Psi} = diag\left[\kappa_u(\theta) \quad \kappa_v(\theta) \quad \kappa_w(\theta) \quad \kappa_\beta(\theta)\right] \tag{10}
$$

Note that, m and n systems in Eq.(12) are coupled due to partial distribution of the foundation in the circumferential direction. Full explanation on the details of derivation for stiffness and mass matrices can be found in the paper by Gunawan [7].

The fluid is assumed to be incompressible, invicid, and the fluid motion is irrotational so that the flow can be described by a velocity potential, Φ, which satisfies the following Laplace equation:

$$
\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial x^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} = 0 \tag{11}
$$

The hydrodynamic pressure, p, acting on the wall of the shell can be determined from the linearized Bernoulli equation and is given by

$$
p = -\rho_L \frac{\partial\Phi}{\partial t}\Big|_{r=R_i} \tag{12}
$$

where $R_i = R - h/2$ is inner radius of the shell and ρ_{L} is density of the fluid. The motion of the shell and fluid is fully coupled by the radial velocities on the interface between shell and fluid so that

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=R_i} = \frac{\partial w}{\partial t} \tag{13}$$

The velocity potential function is can be written as follows:

$$\Phi(x,\theta,r,t) = \sum_{m=0}^{\infty}\sum_{j=1}^{8} \Phi_{mj}^{\mathrm{S}} + \Phi_{mj}^{\mathrm{U}} \tag{14}$$

where $\Phi_{mj}^{\mathrm{S}} = R_{mj}^{\mathrm{S}}(r)\, S_{mj}^{\mathrm{S}}(x,\theta,t)$ and $\Phi_{mj}^{\mathrm{U}} = R_{mj}^{\mathrm{U}}(r)\, S_{mj}^{\mathrm{U}}(x,\theta,t)$.

By using Eq.(13) and Eq.(14), the following equation can be obtained:

$$\Phi_{mj}^{\mathrm{S}} = \frac{R_{mj}^{\mathrm{S}}(r)}{\dfrac{\partial R_{mj}^{\mathrm{S}}(R_i)}{\partial r}} \frac{\partial w_{mj}^{\mathrm{S}}}{\partial t} \tag{15}$$

where similar expression still holds for Φ_{mj}^{U}. Henceforth, derivation of the fluid equation will be explained only for a symmetrical system since the expressions for the asymmetrical system can be obtained analogously. By the substitution of Eq.(15) into Eq.(11) and considering the shell and fluid are in motion, the following equation can be obtained:

$$\frac{\mathrm{d}^2 R_{mj}^{\mathrm{S}}(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R_{mj}^{\mathrm{S}}(r)}{\mathrm{d}r} + \left(\lambda_{mj}^{\mathrm{S}} - \frac{m^2}{r^2}\right) R_{mj}^{\mathrm{S}}(r) = 0 \tag{16}$$

where $\lambda_{mj}^{\mathrm{S}} = \mu_{mj}^{\mathrm{S}} / R$. By consideration to the flow condition, the solution of Eq.(16) can be expressed as

$$R_{mj}^{\mathrm{S}}(r) = D_{mj}^{\mathrm{S}}\, J_m(\lambda_{mj}^{\mathrm{S}} r) \tag{17}$$

where $J_m(\lambda_{mj}^{\mathrm{S}} r)$ and D_{mj}^{S} are the m-th modified Bessel function of the first kind and constant, respectively. The velocity potential function can be obtained by substituting Eq.(17) into Eq.(15) and is rewritten as

$$\Phi_{mj}^{\mathrm{S}} = \frac{J_m(\lambda_{mj}^{\mathrm{S}} r)}{\dfrac{\partial}{\partial r}\left(J_m(\lambda_{mj}^{\mathrm{S}} R_i)\right)} \frac{\partial w_{mj}^{\mathrm{S}}}{\partial t} \tag{18}$$

The hydrodynamic pressure expression can be obtained by substituting Eq.(18) into Eq.(12) and is given by

$$p = -\rho_L \sum_{m=0}^{\infty}\sum_{j=1}^{8} F_m(\lambda_{mj}, R_i)\left[\frac{\partial^2 w_{mj}^{\mathrm{S}}}{\partial t^2} + \frac{\partial^2 w_{mj}^{\mathrm{U}}}{\partial t^2}\right] \tag{19}$$

where

$$F_m(\lambda_{mj}, R_i) = \frac{J_m(\lambda_{mj} R_i)}{\dfrac{\partial}{\partial r}\left(J_m(\lambda_{mj} R_i)\right)} \tag{20}$$

note that $\lambda_{mj} = \lambda_{mj}^{\mathrm{S}} = \lambda_{mj}^{\mathrm{U}}$. By taking Eq.(19) as an external load, the energy expression can be established, and by using the finite element method, the mass matrix of fluid can be obtained and written as

$$M_L = \rho_{\mathrm{L}} \iint_{A_i} \mathbf{N}_w^{\mathrm{T}}\, \mathbf{N}_w^{*}\, \mathrm{d}A_i \tag{21}$$

where $\mathbf{N}_w$ and A_i are total shape function of the radial displacement and internal surface of the shell, respectively. Eq.(20) has been incorporated in $\mathbf{N}_w^{*}$ for each circumferential wave number.

In the analysis, an approximate solution is obtained by truncating the series of wave number m to a finite number of waves, M. From Eqs.(7), (8), (9), and (21), the global equation of the problem can be written as

$$[\,\mathbb{K}_{\mathrm{S}} + \mathbb{K}_{\mathrm{F}}\,]\ \mathbb{d} \;=\; \omega^2\,[\,\mathbb{M}_{\mathrm{S}} + \mathbb{M}_{\mathrm{L}}\,]\ \mathbb{d} \tag{22}$$

where $\mathbb{K}_{\mathrm{S}}$, $\mathbb{M}_{\mathrm{S}}$, $\mathbb{K}_{\mathrm{F}}$, $\mathbb{M}_{\mathrm{L}}$, $\mathbb{d}$, and ω are the global stiffness and mass matrices of the shell, the global stiffness matrix of the foundation, the global mass matrix of the fluid, the total nodal displacement vector, and the natural frequency of the vibrating system, respectively. For convenience, the non-dimensional frequency parameter $\Omega = \omega L\sqrt{\rho_{\mathrm{S}}(1-\upsilon^2)/E}$ is used through out this paper.

3. Numerical results

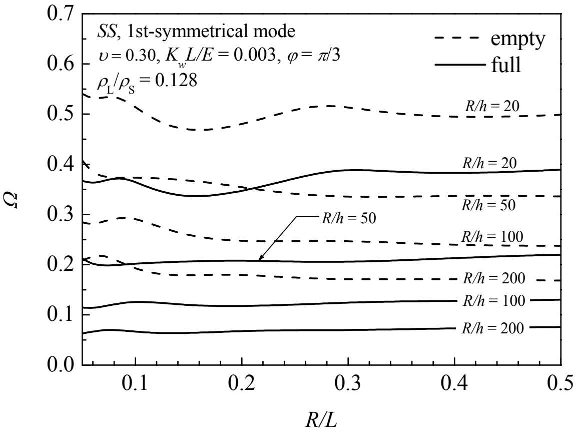
From convergence studies which are not shown here, total number of elements (NS) is equal to 20 and M is equal to 20 are used to obtained sufficient accuracy of the results.

For an example, a shell simply supported at both ends with the following parameters: $\upsilon = 0.30$, $K_u = K_v = K_\beta = 0$, $K_w = 0.003$, $\varphi_1 = \varphi_2 = \varphi = \pi/3$, and $\rho_{\mathrm{L}}/\rho_{\mathrm{S}} = 0.128$ are analyzed. Variations in Ω with R/L for different values of R/h are shown in Fig.2. The results for empty shells are also plotted in the figure. As R/L increases, Ω fluctuates. These fluctuations are caused by the changes in most dominant waves. However, the fluctuations diminish as R/h increases. The existence of internal fluid lowered down the natural frequencies, however the curves for certain values of R/h are similar.

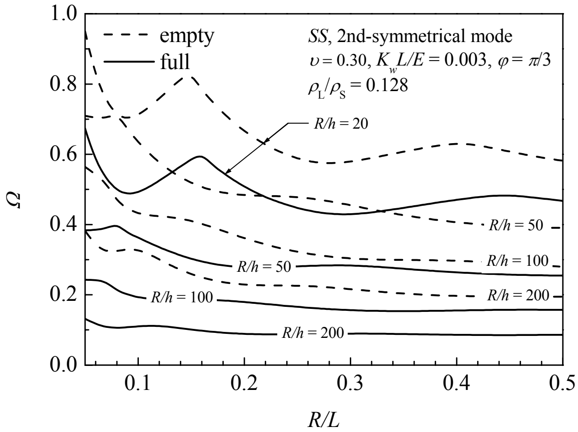
More results will be explained in detail during the presentation.

4. Conclusions

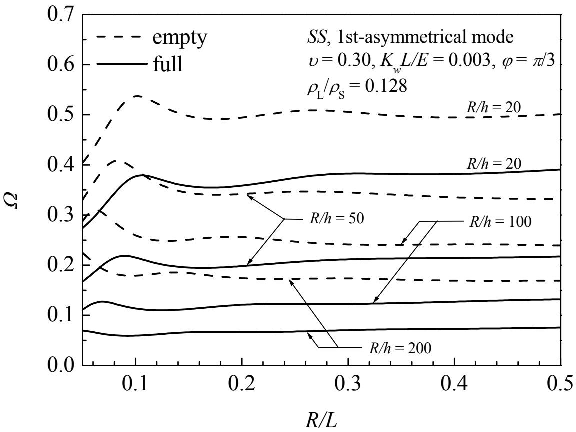
Free vibration analysis of cylindrical shells filled with fluid and partially buried on elastic foundations by using the semi-analytical finite element method are presented. The analytical shape functions used in the longitudinal direction are derived from the governing equations of the empty shell without the foundation terms. The present formulation is still applicable and gives significant improvements in the convergence behavior when compared with the usual simple polynomials based formulation. Distribution of the foundation in the circumferential and in the longitudinal directions may be handled by the Fourier series and an element mesh strategy, respectively. The fluid in the sectional plane is treated analytically without discretization into finite elements. The present method is suitable for the problem considered due to its generality, simplicity, and further development possibilities.



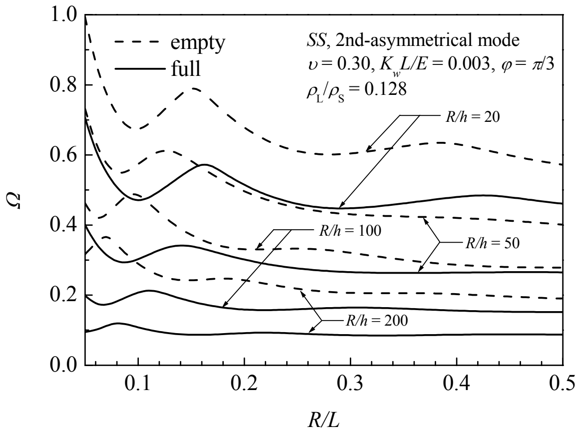
(a) First symmetrical mode



(b) Second symmetrical mode



(c) First asymmetrical mode



(d) Second asymmetrical mode

Fig.2. Variations in Ω with R/L for different values of R/h
(SS, $\upsilon = 0.30$, $K_w L/E = 0.003$, $\varphi = \pi/3$, and $\rho_L/\rho_S = 0.128$).

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