The Equivalent Frequency Transfer Function of Runoff Models Considering River Network

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ABSTRACT

Luai and Fujita had derived the equivalent frequency transfer function (EFTF) of several runoff models such as the Saint Venant equation, diffusion wave equation and kinematic wave equation on a mountainous slope. Fujita proposed the EFTF of kinematic wave model considering river network. This paper aims to lead the method to calculate the EFTF of the Saint Venant equation and diffusion wave equation.

INTRODUCTION

Equivalent Frequency Transfer Function (EFTF) has originally been used among control engineers to analyze non-linear elements such as threshold and saturation elements. Luai H. and Fujita M. had derived the EFTF between rainfall and discharge from slope using kinematic wave equation. This study has developed an application of EFTF based on diffusion wave model with zero depth gradient lower boundary condition. Furthermore, we have developed the application of EFTF to diffusion wave model from slope to river network.

SLOPE CONDITION

Continuity and momentum equations of river slope model are shown as eq. (1) and (2).

$$\frac{\partial h}{\partial t}+\frac{\partial q_s}{\partial x}=r(t) \qquad 0 \le x \le l_s \tag{1}$$

$$q_s=\frac{\sqrt{i_s}}{n_s}h^{5/3}=\alpha h^{5/3} \tag{2}$$

where h : water depth　q_s : discharge per unit width

r : rainfall　t : time

x : distance along slope

i_s: slope gradient　n_s : roughness coefficient

Initial and boundary conditions are

$$q_s(t,0)=0 \ , \quad q_s(0,x)=0 \tag{3}$$

NETWORK BASIN ANALYSIS

In this study, we focus on river network, which consists of 3 unit basins. Definition of unit basin is a basin consisting of 2 singles slopes and 1 channel. The examples of unit basin and network basin are shown in Figure 1 (A) and (B).

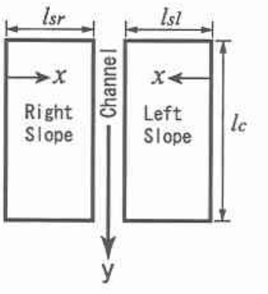


Fig 1(A). Unit Basin

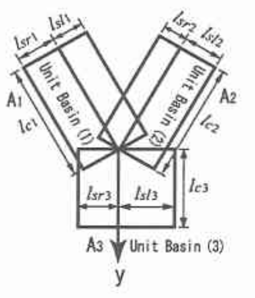


Fig 1(B). Network Basin

Continuity and momentum equations of river network are

$$\frac{\partial a_i}{\partial t}+\frac{\partial q_i}{\partial y}=2q_s(t,l_s) \qquad 0 \le y \le l_c \tag{4}$$

$$q_i=\frac{1}{n_c W^{2/3}}\sqrt{i_c-\frac{1}{W}\frac{\partial a_i}{\partial y}}\,a_i^{5/3} \qquad i=1,2,3 \tag{5}$$

Initial condition is

$$q_i(t,0)=0 \qquad i=1,2 \tag{6}$$

Where W and a are width and cross-sectional area of channel. Subscript s, c and i indicate slope, channel and number of basin respectively. To reduce the complicate in derivation, we assume the condition at the junction of each basin as eq. (7) while eq.(8) is the initial condition of basin no. 3.

$$a_1(t,l_c)=a_2(t,l_c)=a_3(t,0) \tag{7}$$

$$q_3(t,0)=q_1(t,l_c)+q_2(t,l_c) \tag{8}$$

Downstream condition is denoted in eq. (9).

$$\left[\frac{\partial a_3}{\partial y}\right]_{y=l_c} = 0 \tag{9}$$

We assume that

$$r(t) = \bar{r} + Be^{j\omega t}$$
$$q_s(t,x) = \bar{q}_s(x) + C_s(x)e^{j\omega t}$$
$$h(t,x) = \bar{h}(x) + D_s(x)e^{j\omega t} \tag{10}$$

Consider steady-state condition, we obtain

$$\bar{q}_s(x) = \bar{r}x \tag{11}$$

Refer to previous study, EFTF , $Z_s(j\omega t_s)$,of slope is

$$Z_s(j\omega t_s) = \frac{C_s(l_s)}{Bl_s} = e^{-j\omega t_s}\,{}_1F_1[p_s, p_s+1, j\omega t_s] \tag{12}$$

t_s in eq.(12) is expressed as

$$t_s = \left(\frac{\bar{r}^{1-p_s} l_s}{\alpha}\right)^{1/p_s} \tag{13}$$

α and p_s are constants. ${}_1F_1(a,b,c)$ denotes Kummer's confluent hypergeometric function. We simplify the calculation by assume that each basin contain the same topographical characteristics for example length and gradient.

Therefore from eq.(4), (10) and (12) lead to eq. (14).

$$\frac{\partial a_i}{\partial t} + \frac{\partial q_i}{\partial y} = 2\bar{q}_s(l_s) + 2C_s(l_s)e^{j\omega t_s}$$
$$= 2\bar{r}l_s + 2Bl_s Z_s(j\omega t_s)e^{j\omega t_s} \tag{14}$$

Non-Dimensional derivation is introduced in order to reduce related parameters. Non-Dimensional Form of eq.(4) and (7) are

$$a_* A_i = a_i \qquad t_* T = t \tag{15}$$

$$q_* Q_i = q_i \qquad y_* Y = y \qquad q_{s*} Q_s = q_s \tag{16}$$

$$y_* = l_c \qquad q_{s*} = 2\bar{r}l_s \qquad q_* = 2\bar{r}l_s l_c \tag{17}$$

From eq. (5) and (15)-(17) give,

$$a_* = \left\{\frac{2n_c W^{2/3}\bar{r}l_s l_c}{\sqrt{i_c}}\right\}^{0.6} \tag{18}$$

$$t_* = \frac{1}{\bar{r}l_s}\left\{\frac{2n_c W^{2/3}\bar{r}l_s l_c}{\sqrt{i_c}}\right\}^{0.6} \tag{19}$$

The continuity and momentum equations can be rewritten in Non-Dimensional form as denoted in eq. (20) and (21).

$$\frac{\partial A_i}{\partial T} + \frac{\partial Q_i}{\partial Y} = 2Q_s \qquad 0 \le Y \le 1 \tag{20}$$

$$Q_i = \sqrt{1 - \frac{1}{F}\frac{\partial A_i}{\partial Y}}\, A_i^{5/3} \tag{21}$$

Where $$F = W i_c l_c \left\{\frac{\sqrt{i_c}}{n_c W^{2/3}\bar{r} l_s l_c}\right\}^{0.6} \tag{22}$$

The EFTF derived from Non-Dimensional equation is

$$Z_s(j\Omega) = \frac{C_s(1)}{Bl_s} = e^{-j\Omega T_s}\,{}_1F_1[5/3, 8/3, j\Omega T_s] \tag{23}$$

Similar to dimensional analysis, we assume that

$$Q_i(T,Y) = \bar{Q}_i(Y) + C_{c,i}(Y)e^{j\Omega T} \tag{24}$$

$$A_i(T,Y) = \bar{A}_i(Y) + D_{c,i}(Y)e^{j\Omega T} \tag{25}$$

From eq. (10), (11), (16) and (17)

$$Q_s(T,1) = 1 + C_s(1)e^{j\Omega T} = 0.5 + \frac{B}{\bar{r}}Z_s(j\Omega) \tag{26}$$

Initial and boundary condition for non-dimensional are

$$\frac{d\bar{Q}_i}{dY} = 1 \;,\quad \bar{Q}_i(0) = 0 \qquad i = 1,2 \tag{27}$$

$$\bar{Q}_3(0) = \bar{Q}_1(1) + \bar{Q}_2(1) \tag{28}$$

Eq.(29) are obtained from steady-state condition

$$\bar{Q}_i = Y \quad i = 1,2 \;,\quad \bar{Q}_3 = Y + 2 \tag{29}$$

From eq.(14) and (26)-(29)

$$\frac{dC_{c,i}}{dY} + j\Omega D_{c,i} = \frac{B}{\bar{r}}Z_s(j\Omega) \tag{30}$$

$$\bar{Q}_i = \sqrt{1 - \frac{1}{F}\frac{\partial \bar{A}_i}{\partial Y}}\, \bar{A}_i^{5/3} \tag{31}$$

Eq. (31) can be rewritten as

$$\frac{d\bar{A}_i}{dY} = F\left(1 - \frac{\bar{Q}_i^2}{\bar{A}_i^{10/3}}\right) \tag{32}$$

From eq. (9), (29) and (32) yield non-dimensional downstream condition as expressed in eq. (33) and (34).

$$\bar{A}_3(1) = 3^{3/5} \tag{33}$$

$$\bar{A}_1(1) = \bar{A}_2(1) = \bar{A}_3(0) \tag{34}$$

Eq. (35) is derived from eq. (24),(25) and (31).

$$C_{c,i} = \frac{5\bar{Q}_i D_{c,i}}{3\bar{A}_i} - \frac{\bar{A}_i^{10/3}}{2F\bar{Q}_i}\frac{dD_{c,i}}{dY} \tag{35}$$

Substitution of eq. (35) into eq.(30), we can obtain

$$\frac{d^2C_{c,i}}{dY^2} - f_{i,1}\frac{dC_{c,i}}{dY} - j\Omega f_{i,2}C_{c,i} = -f_{i,1}\frac{B}{\bar{r}}Z_s(j\Omega) \tag{36}$$

Where $$f_{i,1} = \frac{10F\bar{Q}_i^2}{3\bar{A}_i^{13/3}} \tag{37}$$

$$f_{l,2} = \frac{2F\overline{Q}_i}{\overline{A}^{10/3}} \tag{38}$$

To solve eq. (36), we need initial condition as following;

$$C_{c,1}(0) = C_{c,2}(0) = 0 \tag{39}$$

$$C_{c,3}(0) = C_{c,1}(1) + C_{c,2}(1) \tag{40}$$

$$D_{c,1}(1) = D_{c,2}(1) = D_{c,3}(0) \tag{41}$$

At the junction of each basin, we consider

$$\left[\frac{dC_{c,1}}{dY}\right]_{Y=1} = \left[\frac{dC_{c,2}}{dY}\right]_{Y=1} = \left[\frac{dC_{c,3}}{dY}\right]_{Y=0} \tag{42}$$

$$\left[\frac{d^2C_{c,3}}{dY^2}\right]_{Y=1} = 0 \tag{43}$$

Relationship between dimensional C_d and non-dimensional C_{nd} is

$$C_{nd} = \frac{C_d}{2\overline{r}l_s l_c} \tag{44}$$

The definition of $Z_{i,1}(j\omega)$ is

$$Z_{c,i}(j\omega) = \frac{C_d(l_c)}{2Bl_s l_c} \qquad i = 1,2 \tag{45}$$

We can rewrite eq.(36) for basin no.3 as

$$\frac{d^2C_{c,3}}{dY^2} - f_{l,1}\frac{dC_{c,3}}{dY} - j\Omega f_{l,2}C_{c,3} = \frac{-f_{l,1}Z_i(j\Omega)}{3} \tag{46}$$

$$Z_2(j\omega) = C_{c,3}(1) \tag{47}$$

The gain and time lag characteristics are

$$G(\omega) = |Z_2(j\omega)|, \quad T_L(\omega) = -\frac{\angle Z_2(j\omega)}{\omega} \tag{48}$$

We need numerical calculation in order to check the accuracy of theoretical method by assuming the following rainfall.

$$r(t) = \overline{r} + A\sin(\omega t) \tag{49}$$

Where A means the constant amplitude. Figure 1 shows the schematic relationship between a sinusoidal input and its output. The gain function is calculated numerically as

$$G(w) = \frac{B}{A} \tag{50}$$

The time lag function, $T_L(\omega)$ is calculated by the time interval between both sinusoidal peaks as shown in Fig 2. Solid line in Figure 3 show the vector locus, gain and time lag function obtained from theoretical method (eq.48) while the circled dhow the results from numerical calculation. It is observed that both results agree together. Numerical data is expressed in table 1.

	slope	channel
Length	1000(m)	5000(m)
n	0.5	0.05
i	0.106	0.022
p_s	5/3	5/3
width		2(m)

Table 1. Applied Data for numerical calculation

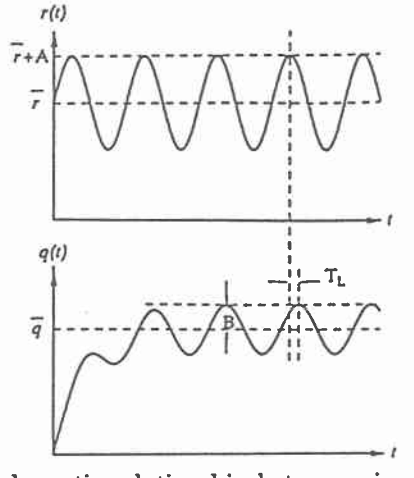
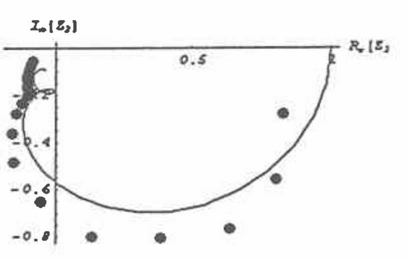
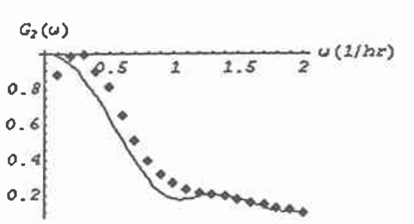


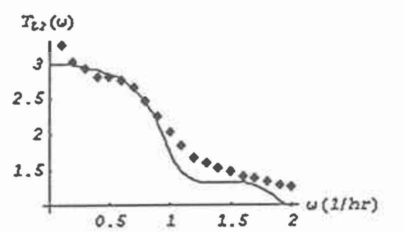
Fig.2 Schematic relationship between sinusoidal input and its output.



(A) Vector locus



(B) Gain



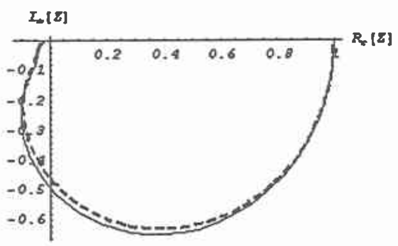
(C) Time lag function

Fig.3 (A), (B) and (C) Vector Locus, gain and Time lag

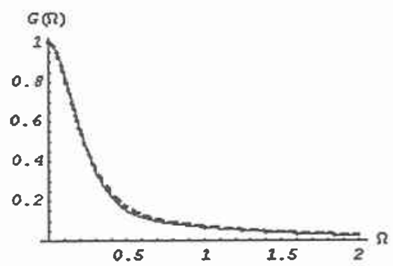
To simplify and reduce calculation time of the method above, we neglect second order derivative at the junction of basin No. 1 and 2 as expressed in eq. (51).

$$\left[\frac{d^2C_{c,1}}{dy^2}\right]_{Y=1} = \left[\frac{d^2C_{c,2}}{dy^2}\right]_{Y=1} = 0 \tag{51}$$

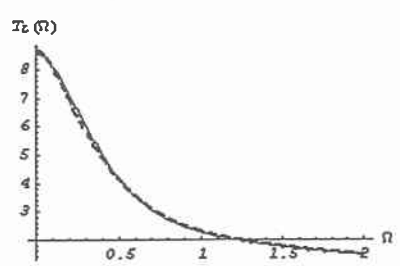
The obtained vector locus, gain and time lag function, without considering of eq.(51),have been compared to the considered one. Figure 4 illustrates the comparison of results.



(A) Vector locus



(B) Gain



(C) Time lag function

Fig.4 Comparison of results with (solid line) and without (dashed line) considering of eq. (51)

From the figure, it is noticed that the computational results agree together. Therefore it would be useful when we simplify the calculation by including eq. (51). The achieved EFTF of general basin can be leaded to simpler estimation of discharge. Its vector locus shows that we can apply second order differential equation to describe runoff system as shown in eq.(52)

$$G_3\frac{d^2q}{dt} + G_2\frac{dq}{dt} + q = r \tag{52}$$

$$q(0) = 0, \left.\frac{dq}{dt}\right|_{t=0} = 0 \tag{53}$$

EFTF of eq.(52) is

$$Z\ (j\omega) = \frac{1}{1 - G_3\omega^2 + jG_2\omega} \tag{54}$$

$$G_3 = \frac{1}{\omega^2}\left\{1 - \frac{R_e[Z]}{R_e^2[Z] + I_m^2[Z]}\right\}, G_2 = \frac{-I_m[Z]}{\omega(R_e^2[Z] + I_m^2[Z])} \tag{55}$$

Therefore we suggest the possibility to use parameters G_3 and G_2 to estimate amount of discharge in further study.

CONCLUSION

The diffusion wave, one of distributed parameter runoff models, was analyzed by using the equivalent frequency transfer function (EFTF). The word "EFTF" means relationship between rainfall input and discharge. In this study, we had derived EFTF for the simplest model or slope. The theoretical results agree with the one calculated from numerical method. Therefore the model was extended. Consequently EFTF for general basin was derived.

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