Lumping Process of Kinematic Wave Model Based on Equivalent Frequency Transfer Function

等価周波数伝達関数を用いた Kinematic Wave モデルの集中化

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1.ABSTRACT

There are two ways to described deterministic rainfall runoff model : first is a lumped parameter model based on ordinary differential equations and second is a distributed parameter model obtained from solving partial differential equations. This study proposes lumping process of kinematic wave model through "Equivalent Frequency Transfer Function" (EFTF). This method, which is originally used among control engineering, is applied to obtain the lumped parameter model for kinematic wave equation.

2.INTRODUCTION

EFTF is used to obtain the dynamic characteristics of non-linear elements such as saturation and hysteresis elements in the field of control engineering. In this paper, we adopt non-linear runoff model such as kinematic wave model instead of above non-linear elements and obtain EFTF after expansion of original EFTF's notion. The so-called equivalent frequency transfer function will be employed in order to specify the relationship between rainfall and discharge.

3.SINGLE SLOPE ANALYSIS

First of all,the continuity and momentum equations of kinematic wave runoff model are

$$\frac{\partial h}{\partial t}+\frac{\partial q_s}{\partial x}=r(t) \qquad 0\le x\le l \tag{1}$$

$$q_s=\alpha h^{p_s} \tag{2}$$

where h : water depth q : discharge per unit width

l : slope length t : time

α , p_s : constants x : distance along slope

To determine the solution requires the initial and boundary condition;

$$q_s(0,x)=0 \tag{3}$$

$$q_s(t,0)=0 \tag{4}$$

In order to derive EFTF, we assume that

$$r(t)=\bar{r}+Ae^{j\omega t} \quad ; \quad \bar{r}>A \tag{5}$$

$$h(t,x)=\bar{h}(x)+B(x)e^{j\omega t} \tag{6}$$

$$q_s(t,x)=\bar{q}_s(x)+C(x)e^{j\omega t} \tag{7}$$

in which j : imaginary unit ω : frequency

$\bar{r}$, $\bar{h}(x)$ and $\bar{q}_s(x)$ show rainfall , water depth and discharge at steady state and satisfy equations below.

$$\bar{h}(0)=0 \quad , \quad \bar{q}_s(0)=0 \tag{8}$$

$B(x)$ and $C(x)$ are unknown complex functions.

Eq. (9) is derived from eqs.(6)-(8).

$$B(0)=0 \quad , \quad C(0)=0 \tag{9}$$

The substitution of eqs. (5), (6) and (7) into eqs. (1) and (2) yields eq. (10).

$$\frac{dC}{dx}+\frac{j\omega}{\alpha p_s}\left(\frac{\bar{r}x}{\alpha}\right)^{(1-p_s)/p_s}C=A \tag{10}$$

The solution of equation (10) is;

$$C(x)=e^{-\int_0^x \frac{j\omega}{\alpha p_s}\left(\frac{\bar{r}x_1}{\alpha}\right)^{(1-p_s)/p_s}dx_1}\int_0^x Ae^{\int_0^{x_2}\frac{j\omega}{\alpha p_s}\left(\frac{\bar{r}}{\alpha}\right)^{(1-p_s)/p_s}x_3^{1/p_s}dx_3}dx_2$$

$$=Ae^{-\frac{j\omega}{\alpha}\left(\frac{\bar{r}}{\alpha}\right)^{(1-p_s/p_s)}x^{1/p_s}}\int_0^x e^{\frac{j\omega}{\alpha}\left(\frac{\bar{r}}{\alpha}\right)^{(1-p_s)/p_s}x_1^{1/p_s}}dx_1 \tag{11}$$

The EFTF, $Z_s(j\omega)$ between $r(t)$ and $q_s(t,l)$ is defined by;

$$Z_s(j\omega)=\frac{C(l)}{Al} \tag{12}$$

From eqs.(11)and(12),we obtain

$$Z_s(j\omega)=e^{-j\omega t_{c1}}\int_0^1 e^{j\omega t_{c1}y_1^{1/p_s}}dy_1 \tag{13}$$

$$= e^{-j\omega t_{c1}} {}_1F_1(p, p_s + 1, j\omega t_{c1}) \qquad (14)$$

$$t_{c1} = \left(\frac{\bar{r}^{-1-p_s} l}{\alpha} \right)^{1/p_s} \qquad (15)$$

where ${}_1F_1(a,b,c)$ denotes Kummer's confluent hypergeometric function defined by

$${}_1F_1(a,b,c) = 1 + \frac{az}{b} + \frac{a(a+1)c^2}{2!b(b+1)} + \frac{a(a+1)(a+2)c^3}{3!b(b+1)(b+2)} + \dots \qquad (16)$$

t_{c1} in eq.(15) means concentration time over $\bar{r}$ on the slope.

4.UNIT BASIN ANALYSIS

Definition of unit basin is a basin consisting of 2 single slopes and 1 channel shown in Fig.1.

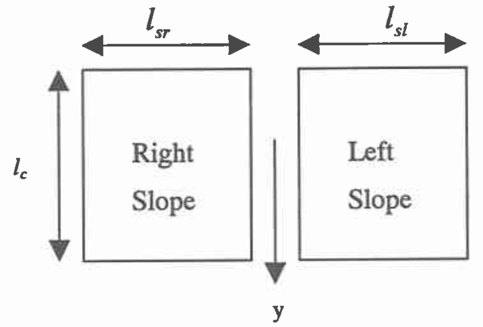


Fig.1.Top view of Unit Basin

We also adopt kinematic wave equation for channel flow. Continuity equation and momentum equation can be written as;

$$\frac{\partial a}{\partial t} + \frac{\partial q_c}{\partial y} = q_{sr}(t, l_{sr}) + q_{sl}(t, l_{sl}) \qquad (17)$$

$$q_c = \beta a^{p_c} \qquad (18)$$

Subscript sr,sl and c specify right slope,left slope and channel respectively. Initial and boundary condition are;

$$q_c(t,0) = 0, \quad q_c(0,y) = 0 \qquad (19)$$

we assume that

$$a(t,y) = \bar{a}(y) + B(y)e^{j\omega t} \qquad (20)$$

$$q_c(t,y) = \bar{q}_c(y) + C_c(y)e^{j\omega t} \qquad (21)$$

Refer to eq.(7),right side of eq.(17) is described by;

$$q_{sr}(t, l_{sr}) = \bar{q}_{sr}(l_{sr}) + C_{sr}(l_{sr})e^{j\omega t} \qquad (22)$$

$$q_{sl}(t, l_{sl}) = \bar{q}_{sl}(l_{sl}) + C_{sl}(l_{sl})e^{j\omega t} \qquad (23)$$

Substitution eq.(20) - eq.(23) into eq.(17)-(18) gives

$$\frac{dC_c}{dy} + j\omega \frac{1}{\beta p_c} \left\{ \frac{1}{\beta} \bar{r}(l_{sr} + l_{sl}) y \right\}^{\frac{1-p_c}{p_c}} C_c = C_{sr}(l_{sr}) + C_{sl}(l_{sl}) \qquad (24)$$

$$C_c(0) = 0 \qquad (25)$$

The EFTF, $Z_u(j\omega)$,between $\bar{r}$ and $q_c(t, l_c)$ is defined by,

$$Z_u(j\omega) = \frac{C_c(l_c)}{A l_c (l_{sr} + l_{sl})} \qquad (26)$$

Similar to previous calculation, $Z_u(j\omega)$ is obtained.

$$Z_u(j\omega) = \frac{Z_c(j\omega)}{l_{sr} + l_{sl}} \{ l_{sr} Z_{sr}(j\omega) + l_{sl} Z_{sl}(j\omega) \} \qquad (27)$$

where

$$Z_{sr}(j\omega) = e^{-j\omega t_{csr}} {}_1F_1[p_s, p_s + 1, j\omega t_{csr}] \qquad (28)$$

$$Z_{sl}(j\omega) = e^{-j\omega t_{csl}} {}_1F_1[p_s, p_s + 1, j\omega t_{csl}] \qquad (29)$$

$$Z_c(j\omega) = e^{-j\omega t_{cc}} {}_1F_1[p_c, p_c + 1, j\omega t_{cc}] \qquad (30)$$

and

$$t_{csr} = \left(\frac{\bar{r}^{-1-p_s} l_{sr}}{\alpha_{sr}} \right)^{1/p_s} \qquad (31)$$

$$t_{csl} = \left(\frac{\bar{r}^{-1-p_s} l_{sl}}{\alpha_{sl}} \right)^{1/p_s} \qquad (32)$$

$$t_{cc} = \left(\frac{\{ \bar{r}(l_{sr} + l_{sl}) \}^{1-p_c} l_c}{\beta} \right)^{1/p_c} \qquad (33)$$

5.NETWORK BASIN ANALYSIS

We focus on river network shown in Fig.2.It is possible to extend the results from such river network to further complex one.

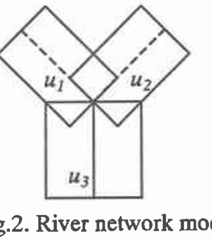


Fig.2. River network model

Continuity equation and momentum equation are

$$\frac{\partial a_3}{\partial t} + \frac{\partial q_{c3}}{\partial y} = q_{sr3}(t, l_{sr3}) + q_{sl3}(t, l_{sl3}) \qquad (34)$$

$$q_{c3} = \beta_3 a_3^{p_c} \qquad (35)$$

Subscript sr,sl,c and number specify right slope, left slope, channel and number of basin respectively.

Initial condition is

$$q_{c3}(t,0) = q_{c1}(t, l_{c1}) + q_{c2}(t, l_{c2}) \qquad (36)$$

we assumed that,

$$q_{sr3}(t, l_{sr3}) = \bar{q}_{sr3}(l_{sr3}) + C_{sr3}(l_{sr3})e^{j\omega t} \qquad (37)$$

$$q_{c1}(t, l_{c1}) = \bar{q}_{c1}(l_{c1}) + C_{c1}(l_{c1})e^{j\omega t} \qquad (38)$$

$$q_{c2}(t, l_{c2}) = \bar{q}_{c2}(l_{c2}) + C_{c2}(l_{c2})e^{j\omega t} \qquad (39)$$

From single slope and unit basin analysis,

$$\bar{q}_{sr3}(l_{sr3}) = \bar{r}l_{sr3}, \bar{q}_{sl3}(l_{sl3}) = \bar{r}l_{sl3} \tag{40}$$

$$\bar{q}_{c1}(l_{c1}) = \bar{r}l_{c1}(l_{sr1} + l_{sl1}) = \bar{r}A_1 \tag{41}$$

$$\bar{q}_{c2}(l_{c2}) = \bar{r}l_{c2}(l_{c2} + l_{c2}) = \bar{r}A_2 \tag{42}$$

Additionally, we assume that

$$a_3 = \bar{a}_3(y) + B(y)e^{j\omega t} \tag{43}$$

$$q_{c3} = \bar{q}_{c3}(y) + C(y)e^{j\omega t} \tag{44}$$

After substitution eqs.(36) to (44) into eqs.(34) and (35),we obtain

$$\frac{dC}{dy} + j\omega\frac{1}{\beta_3 p_c}\left\{\frac{1}{\beta_3}\bar{r}(l_{sr3}+l_{sl3})y + \bar{r}(A_1+A_2)\right\}^{\frac{1-p_c}{p_c}} C$$

$$= C_{sr3}(l_{sr3}) + C_{sl3}(l_{sl3}) \tag{45}$$

where $C(0) = C_{c1}(l_{c1}) + C_{c2}(l_{c2})$ (46)

and $A_i = l_{ci}(l_{sri} + l_{sli}), i = 1,2,3$ (47)

Definition of the EFTF, $Z_w(j\omega)$,between $\bar{r}$ and $q_{c3}(t,l_{c3})$ is denoted by

$$Z_w(j\omega) = \frac{C(l_{c3})}{A.A_r} \tag{48}$$

in which $A_r = A_1 + A_2 + A_3$ (49)

Finally,

$$Z_w(j\omega) = \frac{1}{A_r}[\{A_1Z_{u1}(j\omega) + A_2Z_{u2}(j\omega)\}e^{-j\omega(t_{c3}-t_{c2})}$$

$$+ \frac{e^{-j\omega t_{c3}}}{l_{sr3}+l_{sl3}}\{l_{sr3}Z_{sr3} + l_{sl3}Z_{sl3}\}\{A_r\,{}_1F_1[p_c, p_c+1, j\omega t_{c3}]$$

$$-(A_1+A_2)\,{}_1F_1[p_c, p_c+1, j\omega t_{c2}]\}] \tag{50}$$

where

$$t_{c2} = \frac{1}{\bar{r}(l_{sr3}+l_{sl3})}\left(\frac{\bar{r}(A_1+A_2)}{\beta_3}\right)^{1/p_c} \tag{51}$$

$$t_{c3} = \frac{1}{\bar{r}(l_{sr3}+l_{sl3})}\left(\frac{\bar{r}A_r}{\beta_3}\right)^{1/p_c} \tag{52}$$

6.CASE STUDY

In this section,we show gain and time lag function using topographical data.Eq.(53) denotes an example of topographical data obtained from Rumoi river basin.

$L_1 = (1486,1007,10256)$

$L_2 = (976,1188,7547)$

$L_3 = (567,167,1373)$

$\theta_1 = (0.067,0.113,0.007)$

$\theta_2 = (0.143,0.087,0.020)$

$\theta_3 = (0.072,0.202,0.004)$ (53)

L_i (right slope,left slope,channel,m)means slope and channel length at i^{th} unit basin shown in Fig.2.

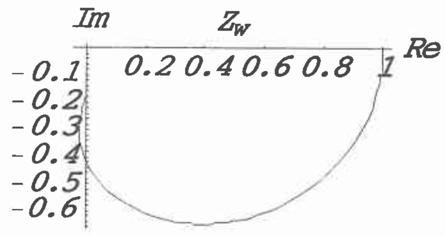
θ_i (right slope,left slope,channel,rad.)means slope and channel gradient at i^{th} unit basin shown in Fig.2.

Definitions of gain and time lag function of EFTF, $Z(j\omega)$ are

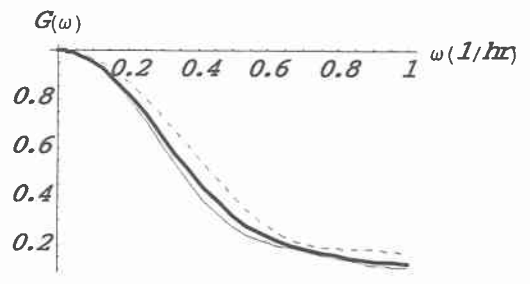
$$G(\omega) = |Z(j\omega)| \tag{54}$$

$$T_G(\omega) = \frac{-\angle\{Z(j\omega)\}}{\omega} \tag{55}$$

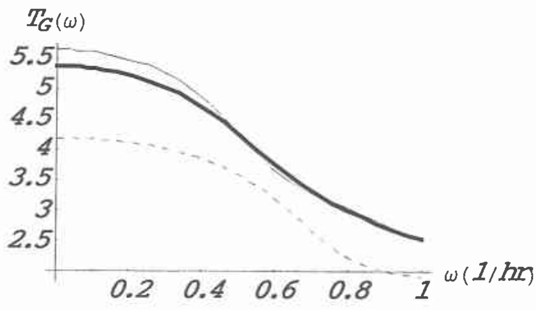
Fig.4 (A) illustrates vector locus(relationship between Re(Z) and Im(Z)) of basin3. Fig.4 (B) and (C) show gain and time lag function of basin 1,2 and 3.



(A)



(B)



(C)

Fig 4.Vector locus,gain and time lag function

Slim solid line represent gain and time lag function of basin 1, dashed line represent gain and time lag function of basin 2 and thick solid line is used to signify gain and time lag of basin 3.

The vector locus depicted from eq.(48) is comparable to the one obtained from second order differential equation.Therefore,the assumption of second order differential equation will be adopted to describe the kinematic wave runoff system as described by eq.(56).Initial equations are shown in eq.(57).

$$f_1\frac{d^2q}{dt^2}+f_2\frac{dq}{dt}+q=r \tag{56}$$

$$q(0)=0,\left[\frac{dq}{dt}\right]_{t=0}=0 \tag{57}$$

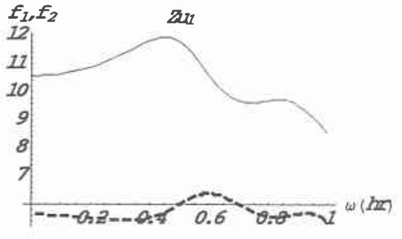
EFTF of eq.(56) is

$$Z(j\omega)=\frac{1}{1-f_1(\omega^2)+jf_2(\omega)}=\mathrm{Re}(Z(j\omega))+\mathrm{Im}(Z(j\omega)) \tag{58}$$

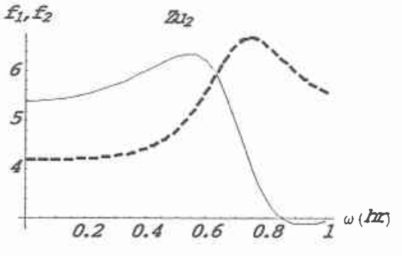
Eq.(59) is derived from eqs.(56) to (58).

$$f_1=\frac{-\mathrm{Re}(Z)+\mathrm{Re}^2(Z)+\mathrm{Im}^2(Z)}{\omega^2(\mathrm{Re}^2(Z)+\mathrm{Im}^2(Z))},f_2=\frac{\mathrm{Im}(Z)}{\omega(\mathrm{Re}^2(Z)+\mathrm{Im}^2(Z))} \tag{59}$$

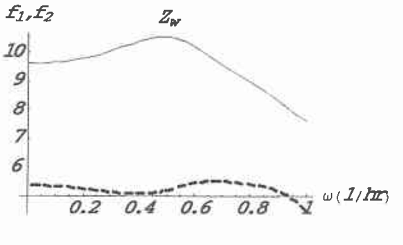
Fig.5 (A),(B) and (C) show relationship between f_1, f_2 and ω.



(A)



(B)



(C)

Fig.5 Relationship between f_1, f_2 and ω

In figures above, solid lines stand for f_1 while dashed lines stand for f_2.

It has been proved that if eq.(58) is practical, f_1 and f_2 will be constant along ω.According to fig.5,it is observed that only low frequency domain can explain this phenomena. However, from fig.4 (B),gain tend to decrease as frequency increase. This means main effect is based on low frequency.Then it is possible to adopt eq.(56) to evaluate discharge of runoff model as shown in example below.

Consider at frequency equal to 0.1 of basin 3, Approximated f_1 and f_2 from fig.5(C) are 9.6 and 5.4 respectively. Rainfall is expressed by rectangular rainfall shown in fig.6. Comparison of results from eqs.(34),(35) and from eq.(56) are illustrated in fig.(7).

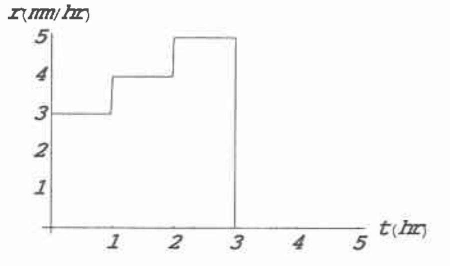


Fig.6. Adopted rainfall

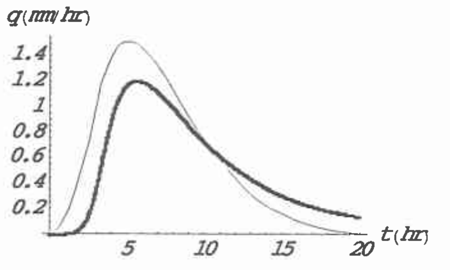


Fig.7. Comparison of discharge from eqs.(34),(35) (thick solid line) and from eq.(56)(slim solid line)

7. CONCLUSION

Previously, several papers focus on lumping process of the mountainous slope of basin. However, the proposed method here provides the lumped model which its basin consists of many slopes and channels. This paper shows obtained lumped model can be expressed by the second order delay system.

References

R.Hamouda and M.Fujita:Application of the equivalent frequency response method to nonlinear runoff system-St.Venant Equation and its related models-under contribution.