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Relationship Between Over-land Flow Equations  
And Their Topographic Conditions

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**Abstract**

In previous works St.Venant Equations and their approximations were studied under time dominant for simple rectangular rainfall.

Real rainfall input which include different types of frequency was studied in this paper by employing the frequency response method, for this aim an experimental numerical approach was conducted and the results were confirmed by a theoretical analysis. Also the results of St.Venant model were compared with the kinematic wave model which is considered as a special case of the former one.

**Key words:** Shallow Waves Equations, Frequency Response, CIP Method, Gain

**Basic Equations And Their Approximations**

The full St.Venant equations describing the propagation of shallow waves are expressed by the continuity and the momentum equations, their dimensionless form in the x-direction is given by eqs. (1) and (2) (woolhiser and Liggett, 1967)

$$\partial H / \partial T + U(\partial H / \partial X) + H(\partial U / \partial X) = R \quad (1)$$

$$\partial U / \partial T + U(\partial U / \partial X) + F_0^{-2}(\partial H / \partial X) = K(1 - U^2 / H) - U / H \quad (2)$$

where  $H = h / H_0$ ,  $U = u / U_0$ ,  $X = x / L_0$ ,  $T = tU_0 / L_0$ ,  $S_0 = U_0^2 / C^2 H_0$  (3)

Capital letters denote dimensionless amounts, small letters denote dimensional ones, and the letters with the sub-script 0 denote normal flow amounts at the end of the slope  $L_0$ .

where  $T, H, U, R$  are the time, the depth, the average velocity and lateral inflow per unit area per unit time. The two independent parameters are the Froude number,  $F_0^2 = U_0 / \sqrt{gH_0}$ , and the

Kinematic Wave number  $K = S_0 L_0 / (H_0 F_0^2)$ .

1) Kinematic Wave Approximation:

For  $K > 50$ , the terms in the momentum eq.(2) are negligible in comparison to the term  $K(1 - U^2 / H)$ . Therefore eq.(2) reduces to  $K(1 - U^2 / H) = 0 \Rightarrow U^2 = H$  (4)

Substituting eq.(4) into eq.(1), the kinematic wave equation is obtained

$$\partial H / \partial T + \partial(H^{3/2}) / \partial X = 1 \quad (5)$$

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2) Diffusion Wave Approximation:

When  $F_0 \ll 1.0$ , and  $K$  is large, but  $F_0^2 K$  is not negligible, then eq.(2) reduces to:

$$\frac{\partial H}{\partial X} \approx F_0^2 K (1 - U^2 / H) \quad (6)$$

Substituting eq.(6) into eq.(1), the diffusion wave equation is obtained:

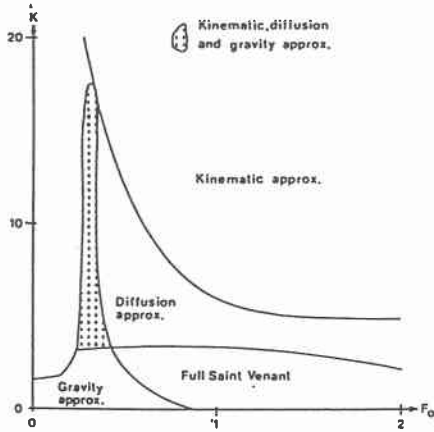
$$\frac{\partial H}{\partial T} + \frac{\partial}{\partial X} \left[ H^{\frac{3}{2}} \left\{ 1 - (F_0^2 K)^{-1} \frac{\partial H}{\partial X} \right\}^{\frac{1}{2}} \right] / \partial X = 1 \quad (7)$$

3) Gravity Wave Approximation:

When  $K$  is very small, eq.(2) reduces to the gravity wave equation:

$$\frac{\partial U}{\partial T} + U \left( \frac{\partial U}{\partial X} \right) + F_0^{-2} \left( \frac{\partial H}{\partial X} \right) = - \frac{U}{H} \quad (8)$$

for various constant values of  $F_0$ , and  $K$ , the region of applying every approximation was obtained by J.H.Daluz Vieira 1981, for critical flow boundary condition as shown in Fig-1.



### Fundamental Equations

Let's assume that  $R(T)$  in eq.(1) is

$$R = \bar{r} + A \sin(\omega T) \quad (9)$$

where  $\bar{r} \gg A$ ;  $\bar{r}, A$  are constants. Under this

assumption the solution  $H(X, T), U(X, T)$

include a periodic component as given by:

$$H = \bar{h}(X, T) + B(X) \sin(\omega T + \Phi_1) \quad (10)$$

$$U = \bar{u}(X, T) + C(X) \cos(\omega T + \Phi_2) \quad (11)$$

discharge is expressed by eq.(12)

$$Q(X, T) = U * H = \bar{u} * \bar{h} + \bar{h} * C \cos(\omega T + \Phi_2) + \bar{u} * B \sin(\omega T + \Phi_1) \quad (12)$$

terms with period of  $2\omega$  were neglected in eq.(12) because it has no influence on  $Q$ .

periodic function in eq.(12) can be written as:

$$\begin{aligned} & \sqrt{(\bar{h}C)^2 + (\bar{u}B)^2 + 2(\bar{h}C)(\bar{u}B)\sin(\Phi_1 - \Phi_2)\sin(\omega T + \theta)} \\ \cos\theta = & \frac{B\cos\Phi_1 - C\sin\Phi_2}{\sqrt{(\bar{h}C)^2 + (\bar{u}B)^2 + 2(\bar{h}C)(\bar{u}B)\sin(\Phi_1 - \Phi_2)}} \quad (13) \end{aligned}$$

Gain the periodic function of the output divided on the periodic function of the input is given by

$$G = \frac{\sqrt{(\bar{h}C)^2 + (\bar{u}B)^2 + 2(\bar{h}C)(\bar{u}B)\sin(\Phi_1 - \Phi_2)}}{A} \quad (14)$$

Substituting eqs.(11) and (13) into continuity equation yields the following two equations:

$$\bar{u}\bar{h} = \bar{r}X \quad (15)$$

$$\begin{aligned} A\sin\omega T = & B\omega\cos(\omega T + \Phi_1) + \frac{\partial(\bar{h}C)}{\partial X} \cos(\omega T + \Phi_2) \\ & + \frac{\partial(\bar{u}B)}{\partial X} \sin(\omega T + \Phi_1) \quad (16) \end{aligned}$$

by adding the first and the third term to the right in eq.(16) we obtained:

$$\begin{aligned} & \sqrt{\left( \frac{\partial(\bar{u}B)}{\partial X} \right)^2 + (B\omega)^2} \sin(\omega T + \Phi_1 + \Psi_1) = \\ & D\sin(\omega T + \Phi_1 + \Psi_1); \cos\Psi_1 = \frac{\left( \frac{\partial(\bar{u}B)}{\partial X} \right)}{D} \quad (17) \end{aligned}$$

adding Eq.(17) to the second term in eq.(16)

$$\begin{aligned} & \sqrt{D^2 + \left( \frac{\partial\bar{h}C}{\partial X} \right)^2 + 2D \frac{\partial\bar{h}C}{\partial X} \sin(\Phi_1 + \Psi_1 - \Phi_2)} * \\ & \sin(\omega T + \Psi_2) \quad (18) \end{aligned}$$

$$E\sin(\omega T + \Psi_2) = A\sin(\omega T)$$

to achieve the presumed equality in eq.(18)

$$\cos\Psi_2 = 1 \quad \sin\Psi_2 = 0 \quad (19)$$

$$\sin\Psi_2 = \frac{D\sin(\Phi_1 + \Psi_1) + \left( \frac{\partial\bar{h}C}{\partial X} \right) \cos\Phi_2}{E}$$

$$D\sin(\Phi_1 + \Psi_1) + \left( \frac{\partial\bar{h}C}{\partial X} \right) \cos\Phi_2 = 0 \quad (20)$$

Substituting eq.(20) into the condition  $E=A$  from eq.(18), gives eq.(21)

$$D^2 = \frac{\Lambda^2 \text{Cos}\Phi_2^2}{\text{Cos}\Phi_2^2 + \text{Sin}(\Phi_1 + \Psi_1)\text{Sin}(2\Phi_2 - \Phi_1 - \Psi_1)}$$

Another equation can be obtained by substituting eqs.(10) and (11) into eq.(2) as

$$\frac{\partial \bar{u}}{\partial X} (\bar{u} + C \text{Cos}(\omega T + \Phi_2)) + \frac{\partial C}{\partial X} (\bar{u} \text{Cos}(\omega T + \Phi_2)) + \frac{1}{F_0^2} \frac{\partial \bar{h}}{\partial X} + \frac{1}{F_0^2} \frac{\partial B}{\partial X} \text{Sin}(\omega T + \Phi_1) = \quad (22)$$

$$K(1 - \bar{u}^2 \bar{h}) - \bar{u} \bar{h} (\bar{r} + A \text{Sin}\omega T) + C \omega \text{Sin}(\omega T + \Phi_2) + \bar{u} C (\bar{u} + \bar{r}) \text{Cos}(\omega T + \Phi_1) - \bar{h} C (2\bar{u} - \bar{r}) \text{Cos}(\omega T + \Phi_2)$$

Eqs.(21) and (22) are two simultaneous differential equations which are obtained from both continuity and momentum equations. Solving Eq.(21) and (22) gives  $C(X), B(X)$ . But we still have to find the values of both  $\bar{h}, \bar{u}$ . For this aim lets assume that  $\bar{h}$  is defined as

$$\bar{h} = \alpha X^\beta \quad (23)$$

$$\bar{u} = \bar{r} X / \bar{h} = (\bar{r} / \alpha) X^{1-\beta} \quad (24)$$

momentum eq. in the steady state is given by

$$\bar{u} \frac{d\bar{u}}{dX} + \frac{\bar{h}}{F_0^2} \frac{d\bar{h}}{dX} = K(\bar{h} - \bar{u}^2) - \bar{r}\bar{u} \quad (25)$$

Multiplying eq.(25) by  $\bar{r}$ , then substituting both eq.(23) and (24), and taking the terms in the right side to the left one gives the error value as

$$\epsilon(X) = \frac{\bar{r}^2}{\alpha} (1-\beta) X^{1-\beta} + \frac{\alpha^2 \beta}{F_0^2} X^{2\beta-1} + \frac{\bar{r}^2}{\alpha} X^{1-\beta} - K \left( \alpha X^\beta - \left( \frac{\bar{r}}{\alpha} \right)^2 X^{2-2\beta} \right) \quad (26)$$

assuming the error = 0 at  $X=1$ , yields eq.(27)

$$\epsilon(1) = \frac{\bar{r}^2}{\alpha} (1-\beta) + \frac{\alpha^2 \beta}{F_0^2} - K \left( \alpha - \left( \frac{\bar{r}}{\alpha} \right)^2 \right) + \frac{\bar{r}^2}{\alpha} = 0$$

from eq.(27),  $\alpha, \beta$  relation is obtained

$$\beta = \frac{K \left( \alpha - \left( \frac{\bar{r}}{\alpha} \right)^2 \right) - \frac{2\bar{r}^2}{\alpha}}{\frac{\alpha^2}{F_0^2} - \frac{\bar{r}^2}{\alpha}} \quad (28)$$

still we need another equation, let's use the following approximation

$$(x - \Delta)^m = x^m + mx^{m-1}\Delta + \text{neglect} \quad (29)$$

in the same way

$$X^\beta = (1 - X)^\beta = 1 - \beta X$$

$$X^{1-\beta} = (1 - X)^{1-\beta} = 1 - (1-\beta)X \quad (30)$$

$$X^{2\beta-1} = (1 - X)^{2\beta-1} = 1 - (2\beta-1)X$$

Substituting eq.(30) into eq.(26) gives

$$\frac{\bar{r}^2}{\alpha} \left( 1 - (1-\beta)^2 X \right) + \frac{\alpha^2 \beta}{F_0^2} (1 - (2\beta-1)X) \quad (31)$$

$$- K \left\{ \alpha(1-\beta X) - \left( \frac{\bar{r}}{\alpha} \right)^2 (1 - (2-2\beta)X) \right\} = 0$$

Solving eqs. (28) and (31) gives  $\alpha, \beta$ .

### Cubic-Polynomial Interpolation (CIP)

CIP method was used to calculate water depth, velocity and hence discharge. In this method, St.Venant Eq. is written as

$$\frac{\hat{u} - u^n}{\Delta t} = -\frac{1}{F_0^2} \frac{\partial h^n}{\partial x} + K \left( 1 - \left( \frac{u^n}{h} \right)^n \right) - r \left( \frac{u}{h} \right)^n \quad (32)$$

$$\frac{\tilde{u} - \hat{u}}{dt} = -\frac{1}{F_0^2} \frac{\partial \Phi}{\partial X} \quad (33) \quad h^{n+1} = h^n + \Phi \quad (34)$$

$$\frac{\Phi}{\Delta t} + \frac{\partial \tilde{u} h^{n+1}}{\partial X} = 0 \quad (35) \quad \frac{h^{n+1} - h^n}{\Delta t} = r \quad (36)$$

$$\frac{u^{n+1} - \tilde{u}}{\Delta t} + \frac{\tilde{u} \partial \tilde{u}}{\partial x} = 0 \quad (37)$$

Where  $\hat{u}, \tilde{u}$  are predictor and corrector respectively,  $n$  denotes the present time.

The advection velocity vector is defined as

$$u_{ave} = 0.5 * (u^{n+1} + u^{n-1}) \quad ; \quad \Delta x = u_{ave} \Delta t \quad (38)$$

Also the following boundary conditions were employed, initial condition:  $h(x,0)=u(x,0)=0$ , which means a dry slope or empty channel. Upper boundary condition:  $u(0,t)=h(0,t)=0$ , and no lower boundary condition, different from previous works.

### Numerical Approach

The authors calculated water depth by using CIP method for different rainfall inputs 0.1, 1.0, 5.0, 10.0, 20.0, different kinematic wave numbers  $K= 5, 10, 20, 50$ , and  $0.5 \leq F_0^2 \leq 2$ , as shown in Fig-2.  $\alpha, \beta$  are functions of  $X, K$ , and  $F_0^2$

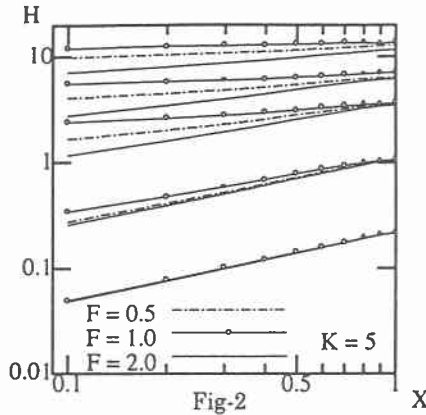


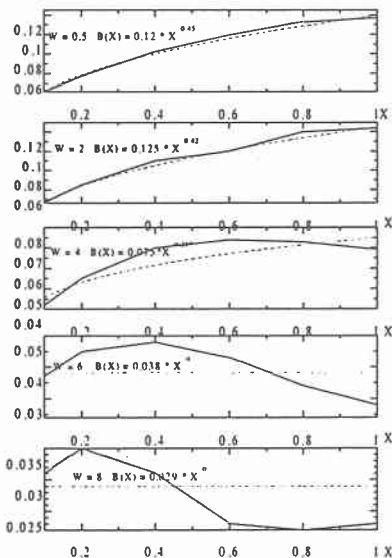
Fig-2

$\alpha, \beta$  can be expressed by the following numerical equations

$$\alpha = 1.1 * K^{-0.03} * R^{0.94K^{-0.081}} * F_0^{2(-0.022k^{-0.88}R)} \quad (39)$$

$$\beta = 0.52 * K^{0.07} R^{-4K^{-1.3}} * F_0^{2(K^{-1.3}R^{0.8})} \quad (40)$$

Also the authors found  $B(X), C(X)$  for different periods  $0.5 \leq \omega \leq 10, K=5, 50, F_0^2 = 1$  for rainfall input  $R = 1 + 0.2 \sin \omega T$ , see Fig-3 for  $K=5$ .



For the same previous condition Gain was plotted and compared with the gain characteristic of kinematic wave model (The authors 1997) which is expressed by the following dimensionless equations, Fig-4.

$$G_K^{n0}(\Omega) = \frac{2}{X R^{p-1} \Omega} * \left\{ \sum_{i=1}^n \frac{\sin^2(e_i^{n0} \Omega)}{C_i^{n0^2}} + \right. \quad (41)$$

$$\left. 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{\sin(e_i^{n0} \Omega) \sin(e_j^{n0} \Omega) \cos((f_i^{n0} - f_j^{n0}) \Omega)}{C_i^{n0} C_j^{n0}} \right\}^{\frac{1}{2}}$$

$$D_i^{n0} = (C_{i-1}^{n0} - C_i^{n0})(i-1)(X/n) + D_{i-1}^{n0}, D_1^{n0} = 0$$

$$C_i^{n0} = \left\{ i^p - (i-1)^p \right\} \left( \frac{X}{n} \right)^{p-1}, e_i^{n0} = \frac{C_i^{n0} X}{2n} R^{p-1}$$

$$f_i^{n0} = \left( \frac{C_i^{n0} i X}{2n} + D_i^{n0} - X^p \right) R^{p-1} - e_i^{n0}$$

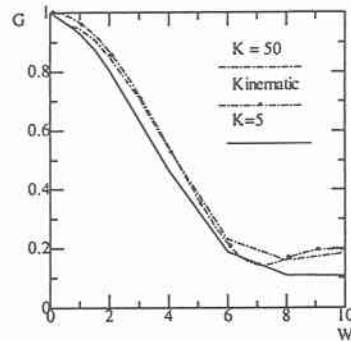


Fig-4

### Conclusion

Water depth and velocity are functions of  $K, F_0, X$ , and time. The shorter the wave is the bigger the values of  $C(X)$  and  $B(X)$ . For  $K=50$  St.venant equations' solution agrees with kinematic wave model solution not only for simple rectangular rainfall input, but also for a rain with a periodic component. Numerical solution agrees with the theoretical approach.

### References:

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