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Stochastic Response of Kinematic Wave Model
- Regionalised variable includes random component-

by
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Abstract

Slope length is believed to be one of the regionalised variables. Such regionalised variables are best described by probabilistic theory. In this paper, we employ the well-known kinematic Wave equation to evaluate the effects of random slope length. And we found that the obtained average discharge by using rectangular rainfall is not explained by kinematic Wave equation.

Basic Theory:

The kinematic wave model for a uniform slope is expressed by

$$\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=r \qquad 0\le x\le l \tag{1}$$

$$q=\alpha h^{m} \tag{2}$$

$$h(0,x)=0,q(0,x)=0,h(t,0)=0,q(t,0)=0$$

h: water depth, q: discharge per unit width, r: rainfall intensity, l: slope length, α, and m kinematic wave parameters.

Fujita had derived the following storage function equation from Eq.(1) and (2).

$$\frac{ds_h}{dt}+q_h=r \tag{3}$$

$$s_h=\frac{m}{m+1}\left(\frac{l}{\alpha}\right)^{1/m} q_h^{1/m} \tag{4}$$

where r, q_h, and s_h are rainfall intensity, discharge and storage respectively.

If the slope length in Eq.(1) is supposed to be a random variable, h and q are also random variables. In this paper, we assume the rectangular rainfall input as shown in Fig-1 .The water profile on the slope is shown schematically in Fig.-2.

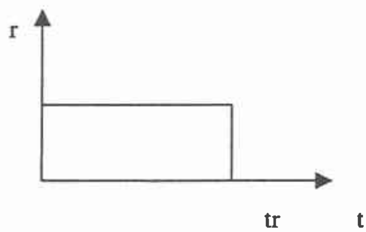

Fig -1

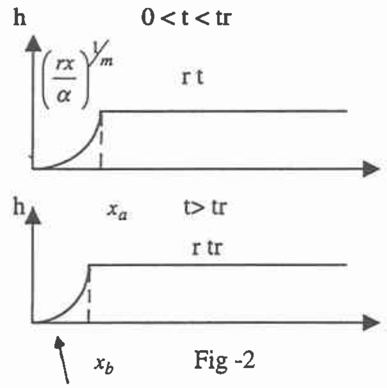

Fig -2

See equation (7)

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Theoretical approach:

During the rainfall time $0 \le t \le t_r$ water height is defined as

$$h(x,t)=\begin{cases}\left(\dfrac{rx}{\alpha}\right)^{1/m} & 0 \le x \le x_a \\ rt & x \ge x_a\end{cases} \quad (5)$$

where $x_a = \alpha r^{m-1} t^m$ (6)

After the rain stops $t > t_r$

$0 \le x \le x_b$

$$x = \alpha m h(x,t)^{m-1}(t-t_r)+\frac{\alpha}{r}h(x,t)^m \quad (7)$$

$$x \ge x_b \qquad h(x,t)=rt_r \quad (8)$$

$$x_b = \alpha r^{m-1}\left\{m t_r^{m-1}(t-t_r)+t_r^m\right\} \quad (9)$$

An average discharge and storage is defined as

$$E\{q(t)\}=\int_0^\infty q(x,t)f(x)dx \quad (10)$$

$$E\{s(t)\}=\int_0^\infty\int_0^x h(y,t)dy\, f(x)dx \quad (11)$$

The function $f(x)$ is probability density function of slope length. It is well known that $f(x)$ is approximated by Gamma distribution or Log-normal distribution. In this paper, we employ thelog-normal distribution described by

$$f(x)=\frac{1}{\sigma x\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}(\log x-\mu)^2} \quad (12)$$

and its average and variance are described by

$$\bar{x}=e^{\mu+\frac{\sigma^2}{2}} \quad (13)$$

$$\sigma_x^2=e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right) \quad (14)$$

In order to simplify the development of this theory, we use the non-dimensional form from Eq.(1) and Eq.(3)

$$\frac{\partial H}{\partial T}+\frac{\partial Q}{\partial x}=R \qquad 0 \le x \le L \quad (15)$$

$$Q=H^m \quad (16)$$

$$h_* H = h\ ,\ t_* T = t\ ,\ q_* Q = q\ ,\quad x_* X = x\ ,\ r_* R = r \quad (17)$$

The capital letters such as H, T, Q, X, R denote non-dimensional quantities corresponding to h, t, q, x, r. The definition of the new parameters is

$$x_* = \bar{l}\ ,\ r_* = \bar{r}\ ,\ q_* = \bar{r}\bar{l}\ ,$$

$$h_* = \left(\frac{\bar{r}\bar{l}}{\alpha}\right)^{1/m},\ t_* = \left(\frac{\bar{r}^{1-m}\bar{l}}{\alpha}\right)^{1/m} \quad (18)$$

where $\bar{r}$: average rainfall intensity.

Using Eq.(15) and (16), we rewrite Eq.(10),(11)

$$E\{Q(T)\}=\int_0^\infty Q(X,T)F(X)dX \quad (19)$$

$$E\{S(T)\}=\int_0^\infty\int_0^X H(Y,T)dY\,F(X)dX \quad (20)$$

$F(X)$: is the non-dimensional probability density function of slope length, described as

$$F(x)=\frac{1}{\sigma X\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}\left(\log X+\frac{1}{2}\sigma^2\right)^2} \quad (21)$$

Again we can rewrite Eq.(19) , (20) as follows

During rainfall time $0 \le T \le T_R$

$$E\{Q(T)\}=R\int_{-\infty}^{Y_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{Y^2}{2}}dY+(RT)^m\int_{Y_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{Y^2}{2}}dY \quad (22)$$

$$Y_1=\frac{1}{\sigma}\left\{\log\left(R^{m-1}T^m\right)-\frac{1}{2}\sigma^2\right\}$$
$$Y_2=\frac{1}{\sigma}\left\{\log\left(R^{m-1}T^m\right)+\frac{1}{2}\sigma^2\right\} \quad (23)$$

$$E\{S(T)\}=\frac{m}{m+1}R^{1/m}e^{\frac{\sigma^2(m+1)}{2m^2}}\int_{-\infty}^{Y_3}\frac{1}{\sqrt{2\pi}}e^{-\frac{Y^2}{2}}dy+$$
$$RT\int_{Y_1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{Y^2}{2}}dY-\frac{R^m}{m+1}T^{m+1}\int_{Y_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{Y^2}{2}}dY \quad (24)$$

$$Y_3=\frac{1}{\sigma}\left\{\log\left(R^{m-1}T^m\right)-\sigma^2\frac{m+2}{2m}\right\} \quad (25)$$

$T > T_R$

$$E\{Q(T)\}=\int_0^{x_2} Q(X,T)F(X)dX+(RT_R)^m\int_{Y_4}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{Y^2}{2}}dY$$

(26)

$$Y_4=\frac{H^m}{R}+mH^{m-1}(T-T_R) \qquad (27)$$

$$E\{S(T)\}=\int_0^{X_b}\int_0^{X}H(Y,T)dYF(X)dX+$$

$$\int_{X_b}^{\infty}\begin{bmatrix}RT_R(X-X_b)+\\(RT_R)^m\left\{(T-T_R)(m-1)+\frac{m}{m+1}T_R\right\}\end{bmatrix}F(x)dX \qquad (28)$$

we can rewrite Eq.(28) as

$$E\{S(T)\}=\int_0^{X_b}\int_0^{X}H(Y,T)dYF(X)dX+$$

$$RT_R\int_{Y_4}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{Y^2}{2}}dY+$$

$$+(RT_r)^m\begin{Bmatrix}(T-T_R)(m-1)+\frac{m}{m+1}T_R\\-RT_RX_b\end{Bmatrix}\int_{Y_4}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{Y^2}{2}}dY$$

(29)

Fig-3, shows, water discharge E{Q(Tr)}, for rectangular rainfall, with intensity 3,and duration period 2,when slope length is a random variable. (Theoretical solution)

From Fig-3, it is noticeable that Q(Tr) decreases when σ_L^2 increases, if we can assume new kinematic wave equation with γ,β parameters and slope length l_n, the solution of Eq.(30) yields E{Q(T)}.

$$\frac{\partial H}{\partial T}+\frac{\partial Q}{\partial X}=R \qquad 0\le X\le l_n$$

$$Q=\gamma H^{\beta} \qquad (30)$$

The solution of this equation is:

E{Q(t)} = R l_n when tr>tc

E{Q(t)} = R tr when tc<tr

We depicted E{q(t)} graph in Fig-4, for rectangular rain fall with intensity 2, and duration tr= 0.5 < tc, tr= 4 >tc and l_n=1.

So, it is noticeable in case tr < tc, E{Q(T)},

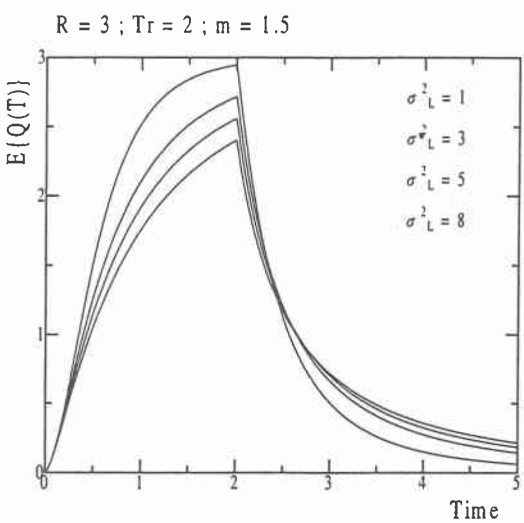

Fig - 3

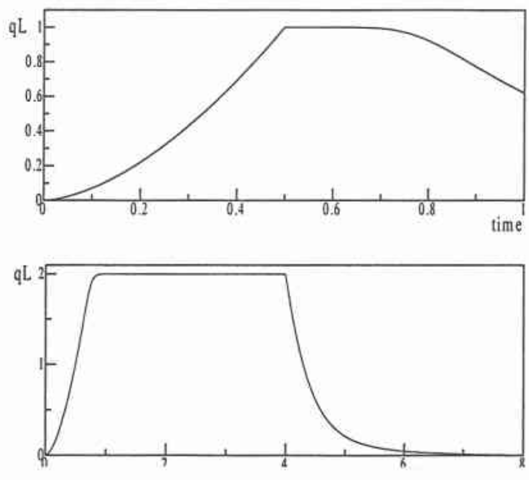

Fig - 4

is independent of slope length, and kinematic wave model is unable to explain E{Q(T)} as a function of slope length.

As shown in Fig-5, we ploted E{Q(T)}, E{S(T)} relation according to the previous theoretical solution. For rectangular rainfall = 1,3, and 5. E{Q(T)}, E{S(T)} relation can be approximated by the dotted line, whose equation

$$E\{S(T)\}=\frac{m}{m+1}E\{Q(T)\}^{1/m} \qquad (31)$$

from Eq.(31), we can introduce Storage

Function model.

The non-dimensional form of Storage function model is expressed by

$$\frac{dS}{dT} = R - Q \tag{32}$$

$$S = K Q^p \tag{33}$$

where $K = S(T_r)/Q^{1/m}(T_r)$; p = 1/m (34)

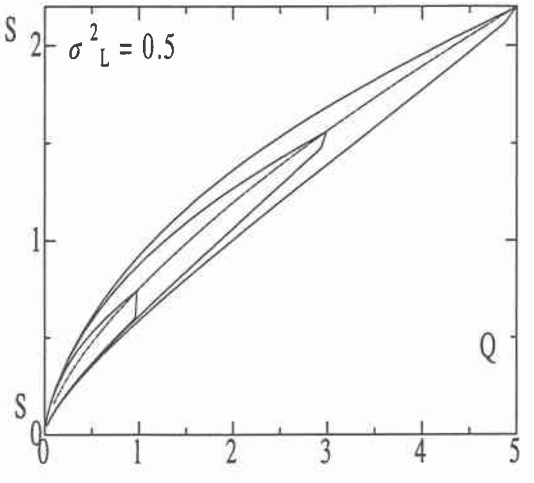

Fig - 5

Now, let's define K for the case $T_r = \infty$, substituting $T_r = \infty$, into Eq.(23) and (25) yields $Y_1 = Y_2 = Y_3 = \infty$, then E{Q(T)}, E{S(T)}, in Eq.(22) and (24), will be expressed by

$$E\{Q(T)\} = R \tag{35}$$

$$E\{S(T)\} = \frac{m}{m+1} R^{1/m} e^{\frac{\sigma^2(m+1)}{2m^2}} \tag{36}$$

from Eq.(35) and (36), we can define K as

$$K = \frac{m}{m+1} e^{\frac{\sigma^2(m+1)}{2m^2}} \tag{37}$$

Using Eq.(34)- applying previous theoretical solution-,and Eq.(37), we can plot S, Q relation against σ_L^2 for different rainfall duration ,as shown in Fig-6 .(Using Log. Scale).

First check Method:

Fig-4 shows that S,Q relation has two ranges, the first one when $\sigma_L^2 < 1$, and K is not affected by rainfall duration, and the other one, where K takes different values according to rainfall duration.

Unfortunately, it is impossible to find theoretical way to calculate E{Q(Tr)}, when rain fall has a triangular shape.

So, we introduced the following numerical two methods, to define E{Q(T)}, the first one by using storage function model's equations, defined by Eq.(32) and (33) where K is defined by Eq.(37). And the second method, by solving Kinematic Wave model's equations, defined by Eq.(30) , simultaneously, with applying Eq.(19). In this calculation Storage function model shows E{Q(T)} for the case when $T_r = \infty$, and Kinematic wave model shows E{Q(T)}, for special tr.

In Fig-7, we show the results obtained by this numerical methods, into two regions, where K is affected by rainfall duration, and the case when K is an independent.

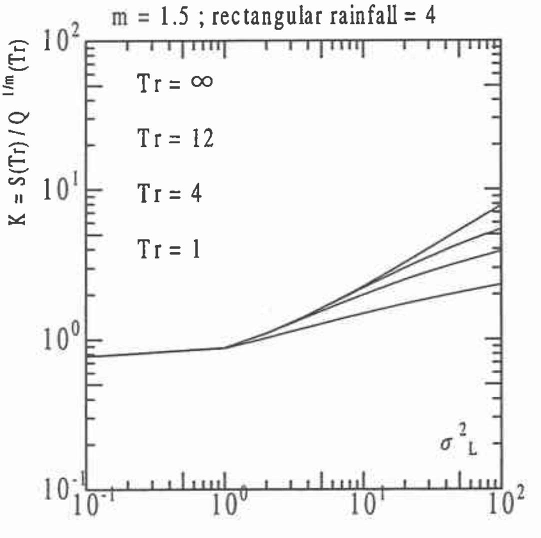

Fig - 6

Simulation Method:

Our previous results, were cross-checked again by a simulation method.

In this method we generated slope length according to log normal distribution function for 1000 times- see Eq.(21), then we introduced this random variable into Storage Function Model – see Eq.(32) and (33) – through K, whose definition:

$$K_i = \frac{m}{m+1} L_i \qquad (38)$$

where K_i is Storage function random coefficient which corresponds to random L_i .

Then we calculated $E\{Q_i(T)\}$ for each L_i , then E{Q(T)} was calculated according to Eq.(39)

$$E\{Q(T)\} = \frac{1}{n} \sum_{1}^{n} E\{Q_i(T)\} \qquad (39)$$

where n = 1000, the number of simulated slope lengths.

In Fig-7, we show the results of our simulated data into two regions, where K is affected by rainfall duration, and the case when K is an independent.

Conclusion:

Fig –7 , shows that the results of simulation and the results from Kinematic Wave Model and Storage Function Model does not agree with each other.

Discussing the results:

1) Storage function model was calculated for the case $Tr=\infty$, and as predicted from Fig – 6, its results agree with Kinematic wave model when σ_L^2=1,where K is independent of rainfall duration, but not when σ_L^2= 5.

2) now, we try to explain the difference between simulated E{Q(T)} and E{Q(T)} obtained by storage and kinematic wave models.

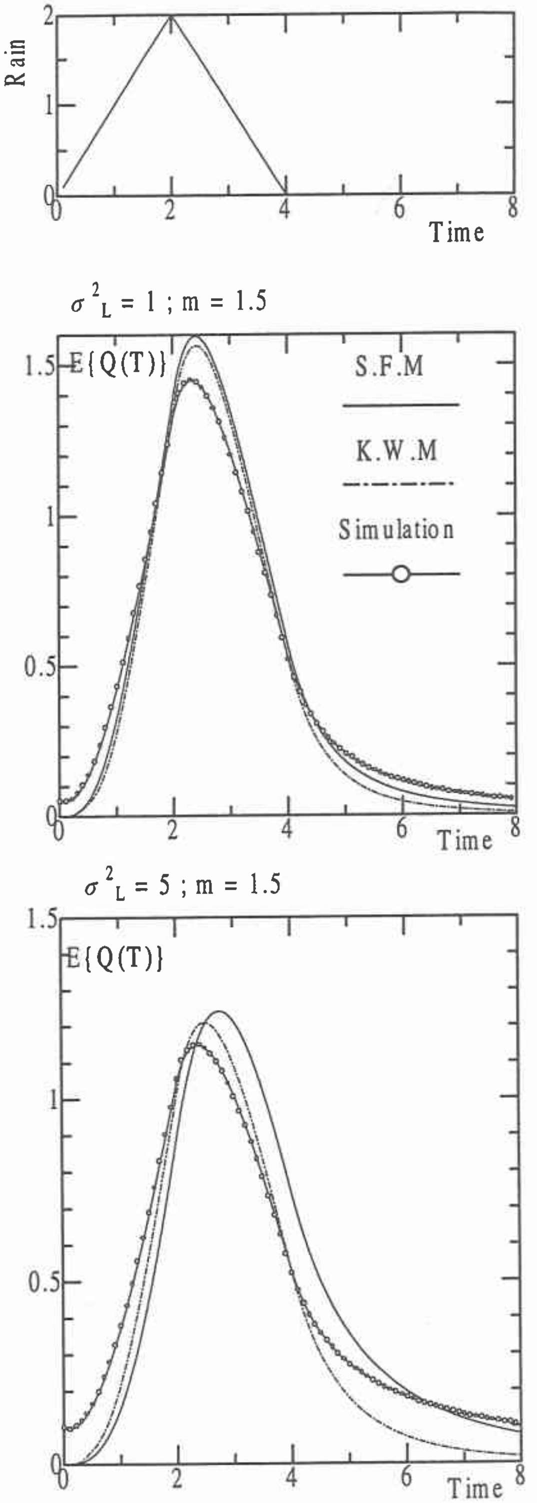

Fig - 5

If we consider L the slope length we can obtain Tc,- the time during which the height of water increases – as follows

$$T_c = \left(L R^{1-m}\right)^{1/m} \qquad (40)$$

if Tr > Tc ,we can define Q(L,T), S(L,T) as

$$Q(L,T)=\begin{cases}(RT)^m & 0 \le T \le T_c \\ RL & T_c < T \le T_R\end{cases} \qquad (41)$$

$$S(L,T)=\begin{cases} L\,H(L,T)-\dfrac{1}{R(m+1)}H(L,T)^{m+1} & 0 \le T \le T_c \\ \dfrac{m}{m+1}L\,H(L,T) & Tc < T \le T_r \end{cases}$$

(42)

if T> Tr

$$L = Q(L,T)/R + m(T-T_r)Q(L,t)^{\frac{m-1}{m}} \qquad (43)$$

$$S(L,T)=\frac{m-1}{m}L\,H(L,T)+\frac{1}{m(m+1)R}H(L,T)^{m+1} \quad (44)$$

if Tr < Tc, then Q(L,T), S(L,T) are defined as

$$Q(L,T)=\begin{cases}(RT)^m & 0 \le T \le T_R \\ (RT_r)^m & T_R < T \le T_g\end{cases}$$

$$L = Q(L,T)/R + m(T-T_R)Q(L,T)^{\frac{m-1}{m}} \qquad (44)$$

$$T_g = T_R + \left(L - R^{m-1}T_R^m\right)/\left\{m(RT_R)^{m-1}\right\}$$

$$S(L,T)=\begin{cases} L\,H(L,T)-\dfrac{1}{R(m+1)}H(L,T)^{m+1} & 0 \le T \le T_R \\ LH(L,T)-\dfrac{1}{R(m+1)}H(L,T)^{m+1} \\ \qquad -(T-T_R)H(L,T)^m & T_r < T \le T_g \\ \dfrac{m-1}{m}LH(L,T)+ \\ \dfrac{1}{Rm(m+1)}H(L,T)^{m+1} & T > T_g \end{cases} \qquad (45)$$

using the previous equations (40) through (45), if the rainfall has a rectangular shape, with different intensities,1,4,and 6, and rainfall time Tr=2, let's plot E{S(T)}, E{Q(T)} relation- solid line- for different slope length L =1,3,9,24, and 70, as shown in Fig-8, and the dotted line shows the approximation of E{S(T)}, E{Q(T)} relation, expressed by $m/(m+1) * L * Q^{1/m}$

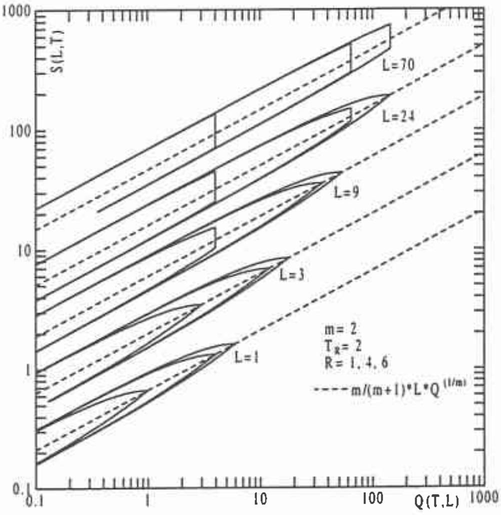

Fig - 8

Again, in Fig-9, we plot the approximation of E{S(T)}, E{Q(T)} relation for the same conditions mentioned in Fig-8, but for R=4.

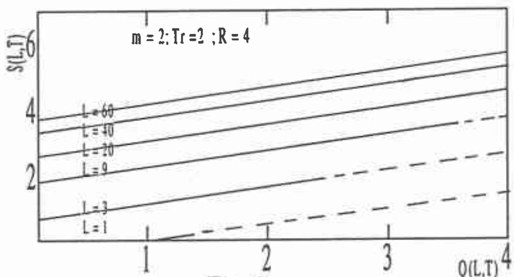

Fig -9

So, when we carry out simulation, we use complete length slope (solid and broken line, in Fig-9), but when we apply kinematic wave equation we limited our slope length to the solid line (in Fig-9), for that reason the average E{Q(T)} in simulation and kinematic wave model differ from each other.

References:

M.Fujita. the impact of variations of slope length on storage-discharge relationships. JSCE, Vol .314,P 75-86,1980.