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STOCHASTIC RESPONSE OF KINEMATIC WAVE MODEL -ESTIMATION OF HIGHER-ORDER MOMENTS-

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1. INTRODUCTION

It is well-known that runoff models are divided into two groups. One is a distributed parameter model and the other is a lumped parameter model. The distributed runoff model has more than two independent variables, so its fundamental equations are described by partial differential equations. On the other hand, lumped parameter runoff models has only one independent variable and the basic equations of this model are described by ordinary differential equations. These two types of runoff model have been developed independently. Since 1980s, the lumping process of distributed parameter runoff models has been developed. Fujita(1981) and Hoshi(1982) proposed storage function model based on kinematic wave equation. Mastubayashi(1994) and Budaghpur,S.(1995) reported storage function model derived from unsaturated flow equation. The relationship between the derived storage function models and original distributed parameter runoff models are evaluated cross by each proposer. However, adopted cross-checking is carried out through comparison between the both calculated discharges from two types of runoff model using rectangular rainfall input. A rectangular rainfall consists of only low frequency component. That is, a traditional cross-checking method focuses on rainfall pattern with low frequency component and bases on deterministic analysis. In this paper, authors introduce two new approaches to cross-check the obtained storage function model from kinematic wave equation. One is stochastic response and the other is frequency response.

2. FUNDAMENTAL THEORY

The kinematic wave model for a uniform slope consists of continuity and momentum equations as follows:

$$\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=r \qquad 0\le x\le l \tag{1}$$

$$h=\varepsilon q^{p} \tag{2}$$

h: water depth, q: discharge per unit width, r: rainfall intensity, l: slope length, ε , p: constants.

It is easy to evaluate the non-linearity of a runoff system through the value of p. Several types of storage function runoff model have been derived from eq.(1) and (2).

$$\frac{dS}{dt}+q=r \tag{3}$$

$$S=K_1 q^{p_1} \tag{4}$$

$$S=K_1 q^{p_1}+K_2\frac{dq}{dt} \quad (5)$$

$$S=K_1 q^{p_1}+K_2\frac{dq^{p_2}}{dt} \quad (6)$$

Fujta (1981) derived coefficients of eq.(4) based on kinematic wave equation theoretically. Eq.(5) is a storage funtion of Prasad's type. The coefficints included in this equation were obtained by Takara(1985). Hoshi(1982) proposed eq.(6). In this paper, authors focus on eq.(4) and analyze the relationship between kinematic wave equation and storage function model from the standpoin of frequency and stochastic response.

2.1 Stochastic Response

For the convenience of theoretical analysis, we divide random variables r, and q into a mean and a deviation from the mean:

$$r=\bar{r}+\tilde{r} \qquad E(\tilde{r})=0 \quad (7)$$

$$q=\bar{q}+\tilde{q} \qquad E(\tilde{q})=0 \quad (8)$$

Rainfall is assumed to be a stochastically independent random variable. The output q also belongs to a random process. The first four general moments of $r(t)$ are given by eq.(9) through (12):

$$E\{r(t)\}=\bar{r} \quad (9)$$

$$E\{\tilde{r}(\tau_1)\tilde{r}(\tau_2)\}=\sigma_r^2\delta(\tau_1-\tau_2) \quad (10)$$

$$E\{\tilde{r}(\tau_1)\tilde{r}(\tau_2)\tilde{r}(\tau_3)\}=\mu_{r3}\delta(\tau_1-\tau_2)\delta(\tau_2-\tau_3) \quad (11)$$

$$E\{\tilde{r}(\tau_1)\tilde{r}(\tau_2)\tilde{r}(\tau_3)\tilde{r}(\tau_4)\}=$$
$$(\mu_{r4}-3\sigma_r^4)\delta(\tau_1-\tau_2)\delta(\tau_2-\tau_3)\delta(\tau_3-\tau_4)$$
$$+\sigma_r^4\{\delta(\tau_1-\tau_2)\delta(\tau_3-\tau_4)+\delta(\tau_1-\tau_3)\delta(\tau_2-\tau_4)$$
$$+\delta(\tau_1-\tau_4)\delta(\tau_2-\tau_3)\} \quad (12)$$

where $\delta(\tau)$, σ_r^2 , μ_{r3} , μ_{r4} are delta function and the second through the fourth moment of $r(t)$ respectively.

If $p\neq 1$, it is difficult to solve eq.(1) and (2) theoretically because of the nonlinear term in eq.(2). Bras and Georgakakos(1980) proposed the following equations in regard to the random variable with an exponent

$$q^p=\alpha_1\bar{q}+\beta_1\tilde{q} \quad (13)$$

$$\alpha_1=\bar{q}^{p-1}\left\{1+\frac{1}{2}p(p-1)\frac{E(\tilde{q}^2)}{\bar{q}^2}+\frac{1}{6}p(p-1)(p-2)\frac{E(\tilde{q}^3)}{\bar{q}^3}+\right\} \quad (14)$$

$$\beta_1=\frac{\bar{q}^{p+1}}{E(\tilde{q}^2)}\left\{p\frac{E(\tilde{q}^2)}{\bar{q}^2}+\frac{1}{2}p(p-1)\frac{E(\tilde{q}^3)}{\bar{q}^3}+\right\} \quad (15)$$

Eq.(16) is derived by eq.(1),(2) , 1(7),(18) and (13).

$$\frac{\partial\{\varepsilon(\alpha_1\bar{q}+\beta_1\tilde{q})\}}{\partial t}+\frac{\partial(\bar{q}+\tilde{q})}{dx}=\bar{r}+\tilde{r} \quad (16)$$

Eq.(17) and (18) are derived from eq.(16).

$$\varepsilon\frac{\partial(\alpha_1\bar{q})}{\partial t}+\frac{\partial\bar{q}}{\partial x}=\bar{r} \quad (17)$$

$$\varepsilon\frac{\partial(\beta_1\tilde{q})}{\partial t}+\frac{\partial\tilde{q}}{\partial x}=\tilde{r} \quad (18)$$

It is possible to transform eq.(18) into a pair of simultaneous ordinary differential equation.

$$\frac{dx}{dt}=\frac{1}{\varepsilon\beta_1} \quad (19)$$

$$\frac{d\tilde{q}}{dt}+\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial t}\tilde{q}=\frac{\tilde{r}}{\varepsilon\beta_1} \quad (20)$$

The solution of eq.(20) is

$$\tilde{q}=e^{-\int D(\tau_1)d\tau_1}\int_0^t\frac{\tilde{r}(\tau_2)}{\varepsilon\beta_1(\tau_2)}e^{\int D(\tau_3)d\tau_3}d\tau_2 \quad (21)$$

$$D(t)=\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial t} \quad (22)$$

Taking the eq.(21) to the second power through the fourth power , then taking the expectation for the resulting equations, we obtain:

$$\frac{d\sigma_q^2}{dt}+\frac{2}{\beta_1}\frac{\partial\beta_1}{\partial t}\sigma_q^2=\frac{1}{(\varepsilon\beta_1)^2}C\sigma_r^2 \tag{23}$$

$$\frac{d\mu_{q3}}{dt}+\frac{3}{\beta_1}\frac{\partial\beta_1}{\partial t}\mu_{q3}=\frac{1}{(\varepsilon\beta_1)^3}C^2\mu_{r3} \tag{24}$$

$$\frac{d\mu_{q4}}{dt}+\frac{4}{\beta_1}\frac{\partial\beta_1}{\partial t}\mu_{q4}=\frac{1}{(\varepsilon\beta_1)^4}C^3(\mu_{r4}-3\sigma_r^4)$$

$$+\frac{6}{(\varepsilon\beta_1)^2}\sigma_q^2C\sigma_r^2 \tag{25}$$

The coefficient C in eq.(23) to eq.(25) is a constant with the dimension of time. This constant stems from the integration of delta function and ensures that both sides of these equations have the same dimension. If we focus on a pair of eq.(19) and (23), we derive eq.(26).

$$\frac{\partial\beta_1^2\sigma_q^2}{\partial t}+\frac{\beta_1}{\varepsilon}\frac{\partial\sigma_q^2}{dx}=\frac{1}{\varepsilon^2}C\sigma_r^2 \tag{26}$$

By the same way, we are able to obtain eq.(27) and (28).

$$\frac{\partial\beta_1^3\mu_{q3}}{\partial t}+\frac{\beta_1^2}{\varepsilon}\frac{\partial\mu_{q3}}{\partial x}=\frac{1}{\varepsilon^3}C^2\mu_{r3} \tag{27}$$

$$\frac{\partial\beta_1^4\mu_{q4}}{\partial t}+\frac{\beta_1^3}{\varepsilon}\frac{\partial\mu_{q4}}{\partial x}=\frac{1}{\varepsilon^4}C^3(\mu_{r4}-3\sigma_r^4)+\frac{6\beta_1^2}{\varepsilon^2}C\sigma_r^2\sigma_q^2 \tag{28}$$

On the other hand, the stochastic responses of eq.(3) and (4) are given by Fujita(1995). we shows results only.

$$\frac{d\bar{S}}{dt}+\left(\frac{1}{K}\right)^m\alpha_2\bar{S}=\bar{r} \qquad m=\frac{1}{p} \tag{29}$$

$$\frac{d\sigma_S^2}{dt}+2\left(\frac{1}{K}\right)^m\beta_2\sigma_S^2=C\sigma_r^2 \tag{30}$$

$$\frac{d\mu_{S3}}{dt}+3\left(\frac{1}{K}\right)^m\beta_2\mu_{S3}=C^2\mu_{r3} \tag{31}$$

$$\frac{d\mu_{S4}}{dt}+4\left(\frac{1}{K}\right)^m\beta_2\mu_{S4}=C^3(\mu_{r4}-3\sigma_r^4)+6C\sigma_r^2\sigma_S^2 \tag{32}$$

$$\bar{q}=\left(\frac{1}{K}\right)^m\alpha_2\bar{S} \qquad \sigma_q^2=\left(\frac{1}{K}\right)^{2m}\beta_2^2\sigma_S^2 \tag{33}$$

$$\mu_{q3}=\left(\frac{1}{K}\right)^{3m}\beta_2^3\mu_{S3} \qquad \mu_{q4}=\left(\frac{1}{K}\right)^{4m}\beta_2^4\mu_S^4 \tag{34}$$

$$S^m=\alpha_2\bar{S}+\beta_2\tilde{S} \tag{35}$$

$$\alpha_2=\bar{S}^{m-1}\left\{1+\frac{1}{2}m(m-1)\frac{E(\tilde{S}^2)}{\bar{S}^2}+\frac{1}{6}m(m-1)(m-2)\frac{E(\tilde{S}^3)}{\bar{S}^3}+\right\} \tag{36}$$

$$\beta_2=\frac{\bar{S}^{m+1}}{E(\tilde{S}^2)}\left\{m\frac{E(\tilde{S}^2)}{\bar{S}^2}+\frac{1}{2}m(m-1)\frac{E(\tilde{S}^3)}{\bar{S}^3}+\right\} \tag{37}$$

2.2 Examination Based on Simulation Method

The method considered here is carried out by directly solving eq.(1) and (2) and the first four moments are calculated at each time t by the simulation method.

Eq.(1) and (2) contain three parameters, that is, ε, p, l. We use the following non-dimensional equation to reduce the number of parameters.

$$\frac{\partial H}{\partial T}+\frac{\partial Q}{\partial x}=R \qquad 0\le X\le 1 \tag{38}$$

$$H=Q^p \tag{39}$$

Eq.(38) and (39) have only one parameter, p. These two equations are derived by introducing new parameters.

$$h=h_*H,\ q=q_*Q,\ t=t_*T,\ x=x_*X,\ r=r_*R \tag{40}$$

The capital letters such as H, Q, T, X and R denote non-dimensional quantites corresponding to h, q, t, x and r. The definitions of new parameters are

$$x_*=l,\ r_*=\bar{r},\ q_*=\bar{r}l,\ h_*=\varepsilon(\bar{r}l)^p,\ t_*=\varepsilon(\bar{r}^{1-1/p}l)^p \tag{41}$$

The storage function runoff model corresponding to eq.(38) and (39) are derived by Fujita.

$$\frac{dS}{dt}+Q=R \tag{42}$$

$$S=\frac{1}{1+p}Q^p \tag{43}$$

The non-dimensional discharge Q in eq.(42) shows discharge at X=1 in eq.(38).

The observed rainfall is discrete time series. Discrete rainfall input arises from accumulating a variable over a period of time.

$$r_d(t)=\frac{1}{\Delta t}\int_{t-\Delta t}^{t} r(\tau)d\tau \qquad (44)$$

In the case that continuous rainfall, $r(t)$ belongs to independent random variable, The relationships between the first four moments of $r(t)$ and $r_d(t)$ are given by Fujita.

$$E\{r_d(t)\}=\bar{r} \qquad \sigma_{r_d}^2=\frac{C\sigma_r^2}{\Delta t} \qquad (45)$$

$$\mu_{r3_d}=\frac{C^2\mu_{r3}}{(\Delta t)^2} \qquad \mu_{r4_d}=\frac{C^3(\mu_{r4}-3\sigma_r^4)}{(\Delta t)^3}+\frac{3C^2\sigma_r^4}{(\Delta t)^2} \qquad (46)$$

$\sigma_{r_d}^2$, μ_{r3_d} and μ_{r4_d} indicate the second , third and forth moment of $r_d(t)$. The non-dimensional first four moments of $R(T)$ are

$$\bar{R}=1 \quad \sigma_R^2=\frac{\sigma_r^2}{\bar{r}^2} \quad \mu_{R3}=\frac{\mu_{r3}}{\bar{r}^3} \quad \mu_{R4}=\frac{\mu_{r4}}{\bar{r}^4} \qquad (47)$$

We use the following exponential distribution for the non-dimensional discrete rainfall $R_d(T)$

$$f(\tilde{R}_d)=\begin{cases}\lambda e^{-\lambda\left(\tilde{R}_d+\frac{1}{\lambda}\right)} & -\frac{1}{\lambda}\le\tilde{R}_d \\ 0 & elsewhere\end{cases} \qquad (48)$$

$\tilde{R}_d$ is the deviation from the mean, $\bar{R}_d$. The first four moments of $\tilde{R}_d$ are

$$E(\tilde{R}_d)=0 \qquad \sigma_{\tilde{R}d}^2=\frac{1}{\lambda^2} \qquad (49)$$

$$\mu_{\tilde{R}3_d}=\frac{2}{\lambda^3} \qquad \mu_{\tilde{R}4_d}=\frac{9}{\lambda^4} \qquad (50)$$

The following parameters are set to evaluate moments for discharge.

$$\Delta T=0.2 \quad \lambda=\sqrt{5} \quad \bar{R}=1 \qquad (51)$$

The first four moments of the non-dimensional discrete rainfall are able to calculated by eq.(49)~(51).

$$\sigma_{R_d}=0.2 \qquad \mu_{R3_d}=0.179 \qquad \mu_{R4_d}=0.360 \qquad (52)$$

The first four moments of the non-dimensional continuous rainfall are obtained by eq.(45),(46) and (51).

$$\sigma_R^2=0.04 \qquad \mu_{R3}=7.16\times10^{-3} \qquad \mu_{R4}=6.72\times10^{-3} \qquad (53)$$

If we set $\varepsilon=1$ in eq.(2), this equation is just the same with eq.(39), the theoretical first four moments of non-dimensional discharge $Q(T)$ are obtained by substituting $\varepsilon=1$ into eq.(17) and (26)~(28). In the calculation of eq.(26)~(28), we use eq.(53) instead of σ_r^2 , μ_{r3} , μ_{r4}. Fig.-1 shows the comparison between the theoretical moments(dotted line) of discharge and simulated one(solid line) using kinematic wave equation. Fig.-2 indicates the comparison of the theoretical moments between kinematic wave equation(dotted line) and storage function model derived from kinematic wave equation(solid line).

2.3 Frequency Response

The theory of frequency response is applied to a linear system in principle. If $p=1$, then eq.(38) and (39) satisfy eq.(54).

$$Q(X,T)=\int_0^T R(\tau)J(X,T-\tau)d\tau \qquad (54)$$

$J(X,T)$ denotes a impulse response function. Fig.-3 shows $J(X,T)$. It is easily calculate frequency transform function, $V(j\omega)$.

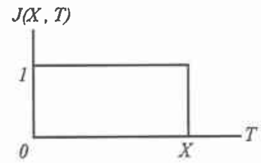

Fig.-3 Impulse Response Function

$$V(j\omega)=\frac{1}{j\omega}(1-e^{-jX\omega}) \quad j\text{: imaginary unit} \tag{55}$$

The gain characteristics, $G(\omega)$ is

$$G=\frac{1}{\omega}\sqrt{2(1-\cos(X\omega))} \tag{56}$$

If $p\neq1$, it is impossible to derive G theoretically. We adopt the following method to calculate, G. It is assumed $R(T)$ in eq.(38) is expressed by

$$R(T)=\bar{R}+B\sin(\omega T) \tag{57}$$

By substituting eq.(57) into eq.(38), we calculate $Q(X,T)$ numerically. It is found that $Q(X,T)$ at steady state is approximated by

$$Q(X,T)=\bar{R}+B'\sin(\omega T+\phi) \tag{58}$$

The gain, G is defined by

$$G=\frac{B'}{B} \tag{59}$$

Equation (60) is obtained by using eq.(59).

$$G=\frac{1}{f(p)(\bar{R}X)^{p-1}\omega}\sqrt{2(1-\cos\{f(p)\bar{R}^{p-1}X^{p}\omega\}} \tag{60}$$

$$f(p)=1-e^{-\frac{3.664p}{1-p}} \tag{61}$$

On the other hand, $G-\omega$ relation for storage function model is given by Fujita.

$$G=\frac{1}{\sqrt{1+(Kp\bar{R}^{p-1}\omega)^2}} \tag{62}$$

Fig.-4 shows $G-\omega$ relation. The parameters to calculate eq.(60) and (62) are

$$p=0.6 \quad \bar{R}=2 \quad X=1 \quad K=\frac{1}{1+p} \tag{63}$$

The symbol ● of fig.-4 indicates a direct calculation using eq.(59). There is

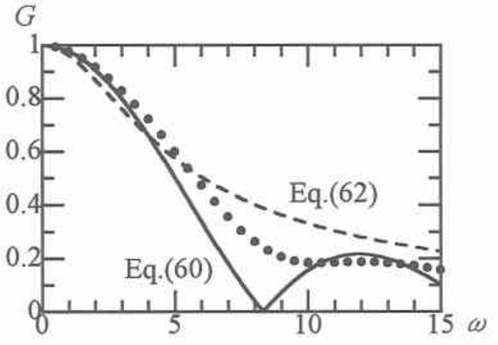

Fig.-4 G-ω Relation

process is independently distributed. The theoretical moments obtained from the proposed equations shows good agreement with simulated ones. One of distinctive features is that the second, third and fourth moments from kinematic wave model have a peak value at $T=1$. The non-dimensional time $T=1$ corresponds to the time of concentration in dimensional form. On the other hand, the stochastic response from storage function model do not show such phenomena. The reason why the stochastic responses from kinematic wave model show this phenomenon and the stochasic responses from storage function model do not show them is not clear at present. Authors consider an approach from the standpoint of frequency response plays an important role to solve this phenomenon.

Conclusion

Partial differential equations whose solutions give the first four moments discharge from kinematic wave model are proposed under the condition that rainfall input from a random

References

Fujita, M., Kudo, M. and Tanaka, G.(1995):Stochastic Response of Storage Function Model, J. JSCE, No.515,1-11

Fujita, M.;Study on Storage Function Model Considering the Variation of Slope Length(1981):J. JSCE, No.314,75-86

Bras,R.L. and Georgakakos, K.P.(1980);Real Time Nonlinear Filtering Techniques in Streamflow Forecasting-A Statistical Linearization Approach-:Third Int. Symp. on

Stochastic Hydraulics,95-105

Hoshi, K. and Yamaoka, I.:Relationship Between Kinematic Wave and Storage Function Model(1982),273-278

Takasao,T.,Takara,K. and Kusuhashi,Y.(1985):A Stochastic Method of Evaluating Short-Term Rainfall Models,Disaster Prevention Research Institute Annuals Vol.28 B-2,221-235

Matsaubayashi,U.,Takagi,F. and Huruta,S.(1994):On the Lumping of Runoff Model Based on the Unsaturated Flow Theory, J. of JSCE,N0.497,11-20

Budaghpur,S.,Fujita,M. and Shimizu,Y.: Lumping Process Based on Unsaturated Infiltration Theory(1995),J. of Hydraulic Eng.,JSCE,Vol.39,209-214

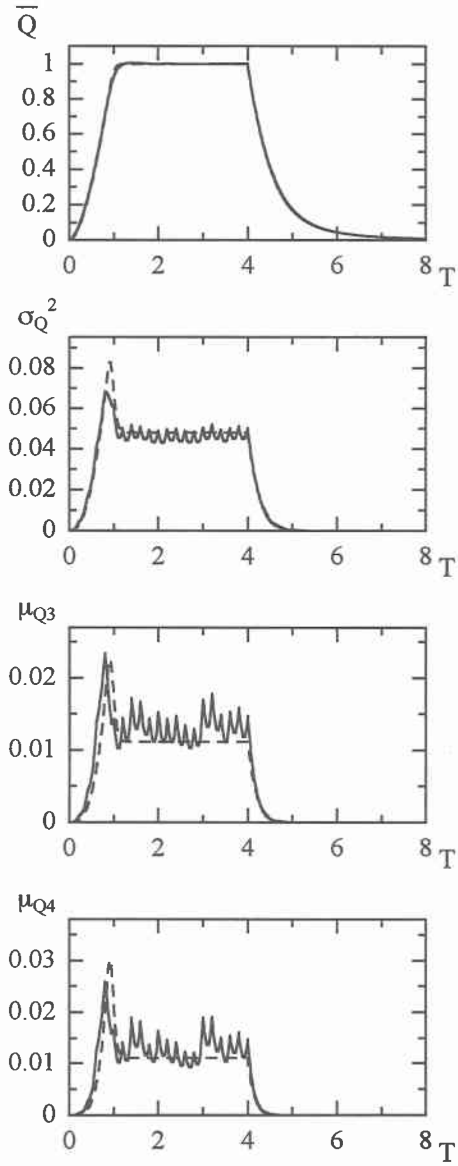

Fig.-1 Comparison between Simulated and Theoretical Moments in Kinematic Wave Model

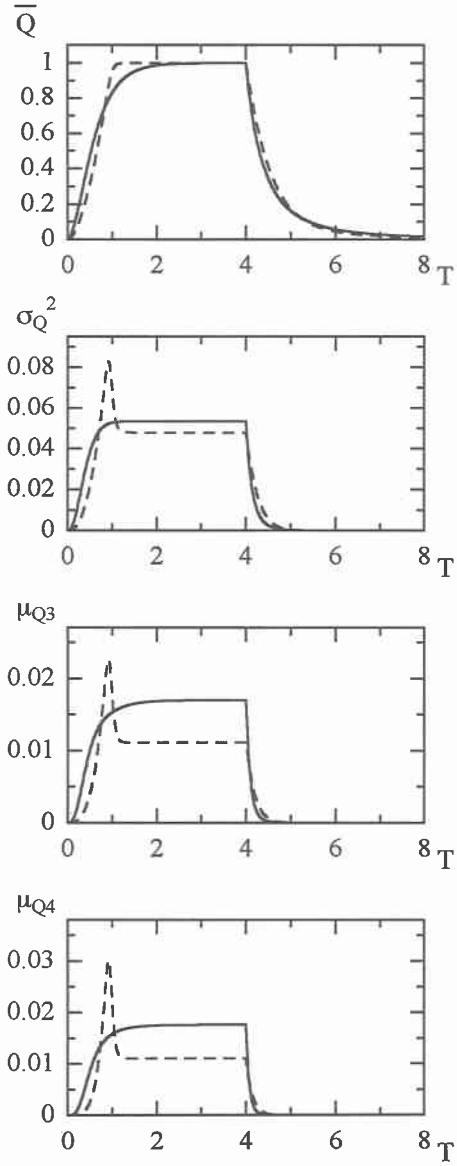

Fig.-2 Comparison between Theoretical Moments from Kinematic Wave Model and Storage Function One