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NUMERICAL ANALYSIS OF UNSATURATED FLOW

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INTRODUCTION:

In general, saturated and unsaturated infiltration of rain water is effected by confined air in the soil. And to evaluate it's effect is the major part of the present study. For this purpose, uniform rainfall (45mm/hr) with three hours duration is applied to the surface of unsaturated soil column. The soil column is situated in circular shape with a circular hole in it's centeral part as it is illustrated in Figure(1). The hole inside the soil column is assumed as a path through which the air can be easily released.The phenomena of infilteration of rain water is considered by calculation of suction inside the soil column in two cases (with and without centeral hole) to realize the hole's effect. The calculation of suction is through the solution of Richard's equation which is adopted as a fundamental equation. However, the mechanism of rain water infiltration and air escape is not considered in the present calculation because it is very difficult to creat a mathematical model to represent this phenomena.Therefore, the calculation is done through a solution of ordinary equations. The result of the calculation is compared experimentaly.

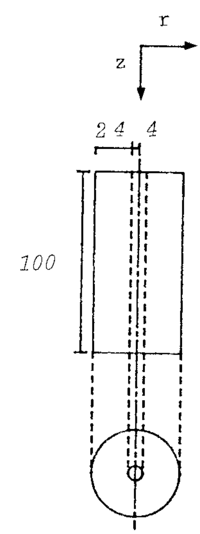

Figure.1:Profile of The Soil Column
(Unit=Centimeter)

STATEMENT OF THE PROBLEM:

Infiltration of rain water into the soil column without gap is assumed as a case of one dimensional infiltration where two dimensional infiltration is considered for the case of the gap inside the soil column. For this purpose movement of rainwater into unsaturated homogeneous soil column is described by the Richard's equation as following:

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$$\frac{\partial \theta}{\partial t} = \frac{\partial V_z}{\partial z} \qquad (1)$$

$$\frac{\partial \theta}{\partial t} = -\left(\frac{\partial V_z}{\partial z} + \frac{V_r}{r} + \frac{\partial V_r}{\partial r} \right) \qquad (2)$$

θ is moisture content
Z is depth of column
Vz is velocity in Z direction

t is time
r is radius of soil column
Vr is velocity in r direction

Equation (1) expresses the condition of one dimensional infiltration, which is penetration of rain water through the depth of soil column (vertical direction). In one dimensional calculation the boundary condition at top and bottom of the soil column is considered. Equation(2) expresses the condition of two dimensional infiltration which is penetration of rain water through length and radius of the soil column. However, according to the darcy's equation the velocities in vertical and circular directions can be written as:

$$V_z = -K \cdot \frac{\partial \phi}{\partial z}$$

$$V_r = -K \cdot \frac{\partial \phi}{\partial r}$$

Where K represents the hydrulic conductivity of the soil and ϕ is the water potential inside the soil column. ϕ can be expressed in such a way that $\phi = \psi - z$ where ψ is the suction. The followings relationships have been used to elaborate Darcy's equation.

$$\theta = (\theta_s - \theta_r)(\psi/\psi_0 + 1)\,EXP(-\psi/\psi_0) + \theta_r \qquad (5)$$

$$K = K_s \left(\frac{\theta - \theta_r}{\theta_s - \theta_r} \right)^{\beta} \qquad (6)$$

θ_s is maximum soil moistur, θ_r is minimum soil moistur, ks is soil conductivity and β is depended to the soil property. With the respect to Equation (5), Equation(6) can be re-write into the following form:

$$K = K_s \left[(\psi/\psi_0 + 1)\,EXP(-\psi/\psi_0) \right]^{\beta}$$

To show the actual behavior of soil moisture inside the soil column, fundamental equations (equations 1 and 2) are converted into the ψ-mode equations.
Equation (8) is the ψ-mode equation form of Equation (1) and Equation (9) is the ψ-mode equation of Equation (2). The finite difference methode is used to solve Equations (8) and (9) as following:

$$C \cdot \frac{\partial \psi}{\partial t} = K \cdot \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial K}{\partial z} \cdot \frac{\partial \psi}{\partial z} - \frac{\partial K}{\partial z} \qquad (8)$$

$$C \cdot \frac{\partial \psi}{\partial t} = \frac{C_{i,n+1} + C_{i,n}}{2} \cdot \frac{\psi_{i,n+1} - \psi_{i,n}}{\Delta t}$$

$$K \cdot \frac{\partial^2 \psi}{\partial z^2} = \frac{K_{i,n+1} + K_{i,n}}{2} \cdot \frac{1}{2} \left[\frac{\psi_{i,n+1} - 2\psi_{i,n} + \psi_{i-1,n}}{\Delta z^2} + \frac{\psi_{i+1,n+1} - 2\psi_{i,n+1} + \psi_{i-1,n+1}}{\Delta z^2} \right]$$

$$\frac{\partial K}{\partial z}\cdot\frac{\partial \Psi}{\partial z} = \tfrac{1}{2}\left[\frac{K_{i+1,n}-K_{i-1,n}}{2\Delta z} + \frac{\Psi_{i+1,n+1}-\Psi_{i-1,n+1}}{2\Delta z}\right]$$

$$-\frac{\partial K}{\partial z} = -\tfrac{1}{2}\left[\frac{K_{i+1,n}-K_{i-1,n}}{2\Delta z} + \frac{K_{i+1,n+1}-K_{i-1,n+1}}{2\Delta z}\right]$$

$$C\cdot\frac{\partial \Psi}{\partial t} = K\cdot\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial K}{\partial z}\cdot\frac{\partial \Psi}{\partial z} - \frac{\partial K}{\partial z} + K\cdot\frac{\partial \Psi}{\partial r}\cdot\frac{1}{r} + K\cdot\frac{\partial^2 \Psi}{\partial r^2} + \frac{\partial K}{\partial r}\cdot\frac{\partial \Psi}{\partial r} - \frac{\partial K}{\partial r} \quad (9)$$

$$K\cdot\frac{\partial^2 \Psi}{\partial z^2} = \frac{K_{i,j,n+1}+K_{i,j,n}}{2}\cdot\tfrac{1}{2}\left(\frac{\Psi_{i+1,j,n}-2\Psi_{i,j,n}+\Psi_{i-1,j,n}}{\Delta z^2}\right.$$

$$\left.+\frac{\Psi_{i+1,j,n+1}-2\Psi_{i,j,n+1}+\Psi_{i-1,j,n+1}}{\Delta z^2}\right)$$

$$\frac{\partial K}{\partial z}\cdot\frac{\partial \Psi}{\partial z} = \tfrac{1}{2}\left(\frac{K_{i+1,j,n}-K_{i-1,j,n}}{2\Delta z} + \frac{K_{i+1,j,n+1}-K_{i-1,j,n+1}}{2\Delta z}\right)$$

$$* \tfrac{1}{2}\left(\frac{\Psi_{i+1,j,n}-\Psi_{i-1,j,n}}{2\Delta z} + \frac{\Psi_{i+1,j,n+1}-\Psi_{i-1,j,n+1}}{2\Delta z}\right)$$

$$-\frac{\partial K}{\partial z} = -\tfrac{1}{2}\left(\frac{K_{i+1,j,n}-K_{i-1,j,n}}{2\Delta z} + \frac{K_{i+1,j,n+1}-K_{i-1,j,n+1}}{2\Delta z}\right)$$

$$\frac{1}{r}*K\cdot\frac{\partial \Psi}{\partial r} = \frac{K_{i,j,n+1}+K_{i,j,n}}{2}\cdot\tfrac{1}{2}\left(\frac{\Psi_{i,j+1,n}-\Psi_{i,j-1,n}}{2\Delta r} + \frac{\Psi_{i,j+1,n+1}-\Psi_{i,j-1,n+1}}{2\Delta r}\right)$$

$$* \frac{1}{\Delta r * r_i}$$

$$K\cdot\frac{\partial^2 \Psi}{\partial r^2} = \frac{K_{i,j,n+1}+K_{i,j,n}}{2} * \tfrac{1}{2}\left(\frac{\Psi_{i,j+1,n}-2\Psi_{i,j,n}+\Psi_{i,j-1,n}}{\Delta r^2}\right.$$

$$\left.+\frac{\Psi_{i,j+1,n+1}-2\Psi_{i,j,n+1}+\Psi_{i,j-1,n+1}}{\Delta r^2}\right)$$

$$\frac{\partial K}{\partial r}\cdot\frac{\partial \psi}{\partial r} = \frac{1}{2}\left(\frac{K_{i,j+1,n}-K_{i,j-1,n}}{2\Delta r}+\frac{K_{i,j+1,n+1}-K_{i,j-1,n+1}}{2\Delta r}\right)$$

$$*\frac{1}{2}\left(\frac{\psi_{i,j+1,n}-\psi_{i,j-1,n}}{2\Delta r}+\frac{\psi_{i,j+1,n+1}-\psi_{i,j-1,n+1}}{2\Delta r}\right)$$

$$-\frac{\partial K}{\partial r} = -\frac{1}{2}\left(\frac{K_{i,j+1,n}-K_{i,j-1,n}}{2\Delta r}+\frac{K_{i,j+1,n+1}-K_{i,j-1,n+1}}{2\Delta r}\right)$$

Following equations have been used to express the boundary conditions.
At the surface of soil column: $V_z=R(t)$, $Z=0$ (R is rainfall)

At the outter side of the soil column: $V_r=0$ or $\frac{\partial \varphi}{\partial r}=0$

At the inner side of the soil column: $\frac{\partial V_r}{\partial r}=0$ or $\frac{\partial^2 \psi}{\partial r^2}=0$

At the bottom of the soil column: $\frac{\partial V_z}{\partial z}=0$ or $\frac{\partial^2 \psi}{\partial z^2}=0$

Initial condition $t=0$, $\psi=Z-100$ (10)
Following parameters have been used for calculations:
$\theta_s=.44$, $\theta_r=.04$, $\psi_0=-40$cm, $\beta=2.5$, Ks=.12cm/minute
Dt (time interval)=.5 minute Dz(depth interval)=1 cm
Dr(width interval)= 1 cm

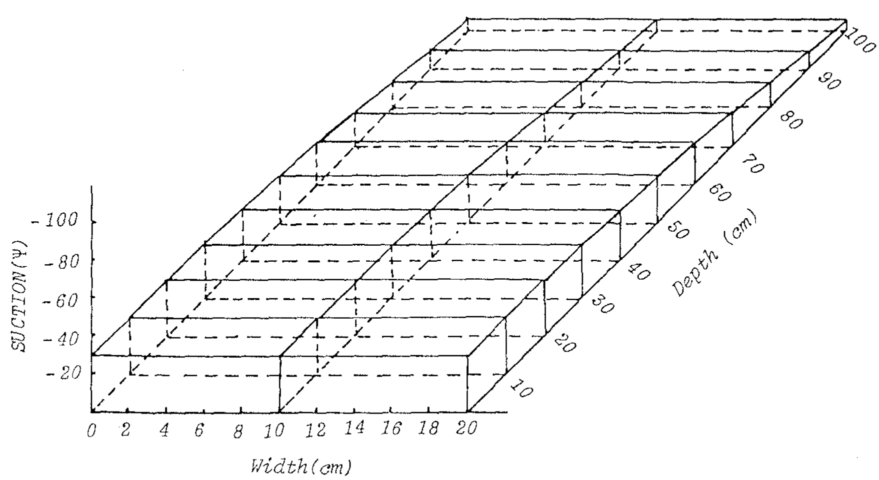

Figure.2:Profile of Suction Through the Width of Soil Column at 2:40 hrs

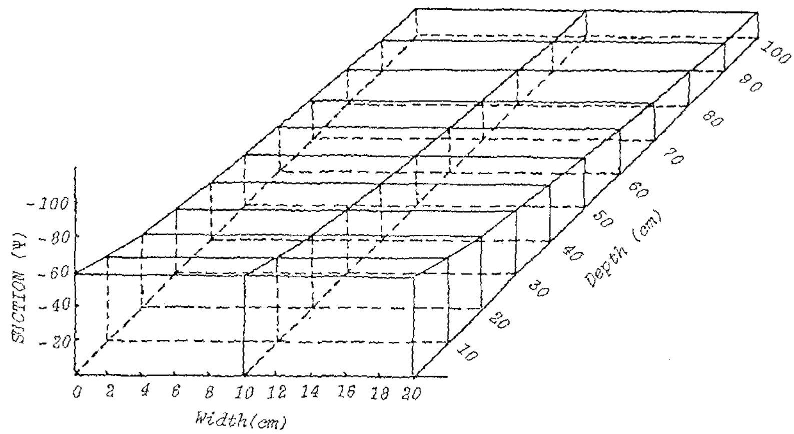

Figure.3:Profile of Suction Through the Width of Soil Column at 3:40 hrs

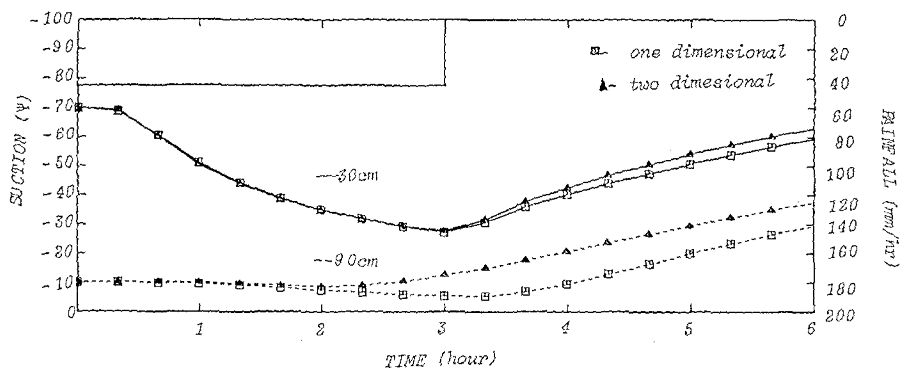

Figure.4:Profile of Suction at 30 and 90 cm Depth

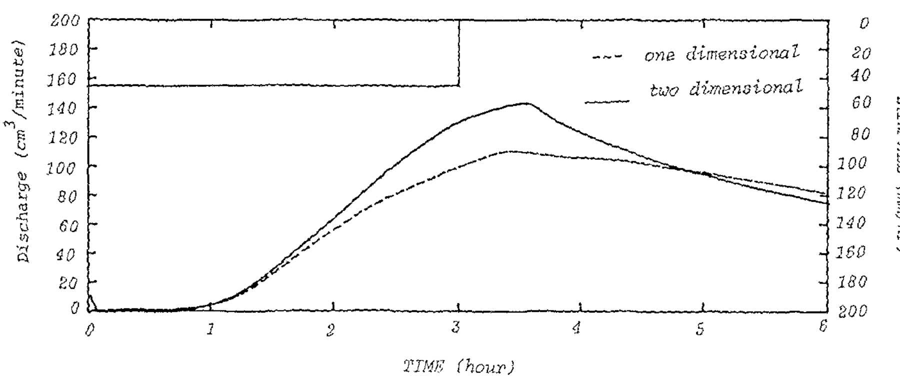

Figure.5 :Comparison of Discharge at The Bottom of Soil Column

CONCLUSION:

Uniform rainfall(45 mm/hr)is applied for three hours to the surface of the soil column and calculation of suction continued upto the six hours, following conlusions have been achived:

The presence of circular gap in the centeral part of the soil column is not effective over the discharge of rain water at the inner side of the column. This can be clearly shown in Figures (2) and (3). Suction is uniform through the width of the soil column at different depths and time intervals. Therefore there is not any discharge at inner side of the soil column.

Figure (4) shows the profile of suction at 30 and 90 centimeters depth of the soil column (one and two dimensional calculations). At the 30 centimeter Suctions are very close to each other but at 90 centimeter depth, suctions are deviated. It means the depth of the soil colum is effective in deviation of suctions and the presence of the whole is not effective.

Figure (5) shows the comparison of discharge at the bottom of soil column. In case of two dimensional calculations, discharge shows slightly higher value as compare to one dimensional. The reason might be due to the accumulated of error in numerical calculations. Suction is uniform through the width of soil column and disharge can not be occured (as it is discussed in Figures 2 and 3). However, as the result of study, the presence of gap in the centeral part of the soil is ineffective to change the suction through the width of the soil column (horizental direction) and therefore, discharge could not appeare at the inner side of the soil column. But the result of experiment shows a high discharge through the centeral hole of the soil column. This inconsistency of numerical and experimental results might be due to the effect of water inter and air escape mechanism of the soil which is not considered in numerical calculations.

REFERENCES:

(1) Numerical Analysis of two Dimensional Unsaturated Flow, by Hiroshi Saga, Pro. of Hokkaido Branch of JSCE, 1985

(2) Numerical Solution of Partial Differential Equation, by G.D Smith, Oxford Mathematical Hand Book, 1965