An Optimization Problem to Reinforce the Road Network Connectivity

Shoichiro NAKAYAMA[1], Shun-ichi KOBAYASHI[2] and Hiromichi YAMAGUCHI[3]

[1]Member of JSCE, School of Geosciences & Civil Engineering, Kanazawa University
(Kakuma-machi, Kanazawa 920-1192, Japan)
E-mail: nakayama@staff.kanazawa-u.ac.jp
[2]Member of JSCE, School of Geosciences & Civil Engineering, Kanazawa University
(Kakuma-machi, Kanazawa 920-1192, Japan)
E-mail: koba@se.kanazawa-u.ac.jp
[3]Member of JSCE, School of Geosciences & Civil Engineering, Kanazawa University
(Kakuma-machi, Kanazawa 920-1192, Japan)
E-mail: hyamaguchi@se.kanazawa-u.ac.jp

In this study, we propose an equal distribution problem for method to evaluate the network connectivity using the Laplacian matrix, and discuss why the second minimum eigenvalue of the Laplacian matrix of the network represents an indicator of network connectivity. Then, we develop an optimization problem to reinforce the road network connectivity using the second minimum eigenvalue of its Laplacian matrix, and prove that it is formulated as a convex optimization problem. Furthermore, we show that the optimization problem is also formulated as a semi-definite programming, which is an extension of linear programming.

Key Words : *reinforcement optimization problem, Laplacian matrix, semi-definite programming*

1. INTRODUCTION

Japan has been struck by several huge earthquakes, such as Hyogoken-Nambu Earthquake in 1995, the Great East Japan Earthquake in 2011 and most recently the 2016 Kumamoto Earthquake. In the event of a natural disaster, some roads are sometimes destroyed, causing many problems in evacuating people rapidly and transporting relief supplies. Additionally, troubles with the road network can have serious consequence for society and for the business community over the wide areas. Thus, we need to improve the connectivity of the road network so that passable roads connect principal (disaster) bases even in the events of natural disasters. It is substantially important to evaluate the network connectivity quantitatively, and, then, the evaluation of connectivity is one guide for preparations for disasters. For example, we can make a reinforcement plan of the road network to effectively improve the connectivity or take measures to prevent disaster damage to vulnerable sections of the network.

There are dozens of connectivity indicators. Connectivity has often been quantified as a passable probability between a pair of origin and destination in the road network (e.g. Bell & Iida, 1997). It is highly demanding to calculate the passable probabilities of all origin-destination pairs (ODs).

As other indicators, Kurauchi et al. (2009) evaluated the interconnectivity between cities based on the number of distinct paths that connect ODs and evaluated the vulnerability based on the number of exclusive paths that have been reduced because of disruption of each link. The evaluation is performed based on the idea of an n-connected network, which indicates that the connection between ODs is guaranteed even if $n - 1$ links are disrupted (Grötschel, 1995).

Clustering is an approach to find links or node sets that have strong and efficient connectivity (e.g. von Luxburg, 2007). Then, links between the clusters is critical because their disruption contribute to the division of the network. Demsar et al. (2008) focused on the betweenness centrality, clustering coefficient, and cut vertices of the line graph. They verified this using the Helsinki network and concluded that the betweenness centrality and cut vertices are useful measures for identifying critical locations. Akbarzadeh et al. (2019) verified that links between the clusters are most critical as compared with other indicators of link importance.

Several studies exist which evaluate vulnerable links using eigenvalue. Nouzard et al. (2016) identified the criticality of the links based on the largest eigenvalue of the capacity and traffic volume weighted adjacent matrix, and detected clusters of the road network by considering the criticality of the weights using the modularity optimization method. Bell et al. (2017) have proposed the identification of flow bottlenecks using capacity-weighted eigenvalue analysis. This method introduces the network partitions exhibiting the least capacity of cutting links by considering the relative sizes of the subnetworks on either side of the cut. The advantage of this approach that uses eigenvalue analysis is that it is possible to identify potential bottlenecks without setting various ODs or routes.

In this study, we focus on the quantitative identification of critical roads that affect network connectivity. In a huge disaster, such as the one that has been mentioned previously, it is important to evaluate the weakness of large-scale network connectivity with large number of links and nodes because of extensive damage. One of the tasks is to efficiently identify the weaknesses in such networks. In this study, we adopt a spectral analysis (eigenvalue analysis) to do so. Thus, the objective of this study is to examine a method for evaluating network connectivity, based on analyses of the eigenvalues of a Laplacian matrix, and to verify its usefulness. This method requires only the geometrical information about a road network and determines quantitatively which roads have the greatest consequences for network connectivity.

Quantifying the network connectivity only is not enough to improve the road network connectivity. It is also important to examine an improvement measure, e.g., to make a reinforcement plan. We propose an optimal reinforcement problem for the rod network connectivity using the eigenvalue of its Laplacian matrix.

First, we present the Laplacian matrix and show how to evaluate network connectivity using this method. Second, we discuss why the second minimum eigenvalue of the Laplacian matrix of the network represents an indicator of network connectivity. Finally, we develop an optimization problem to reinforce the road network connectivity using the second minimum eigenvalue of its Laplacian matrix, and prove that it is formulated as a convex optimization problem. Furthermore, we show that the optimization problem is also formulated as a semi-definite programming, which is an extension of linear programming.

2. EQUAL DSITRIBUTION PROBLEM

(1) Adjacency, degree, and Laplacian matrices

A road network is modeled as a plane graph with nodes and links. Let n denote the total number of nodes in a road network. The relationship between the links and nodes can be written as an adjacency matrix $\mathbf{A}$, which is a square matrix with n rows and columns. Let a_{ij} denote the strength to connect nodes i and j ($a_{ij} \geq 0\ \forall i, j$), and a_{ij} is component (i, j) of $\mathbf{A}$. It is assumed in this study that there is no loop link whose start and end nodes are the same, and $a_{ii} = 0\ (i = 1, 2, \cdots, n)$ and that each link is undirected, that is, $a_{ij} = a_{ji}\ (i, j = 1, 2, \cdots, n)$. Therefore,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & 0 \end{bmatrix}. \quad (1)$$

The degree matrix $\mathbf{D}$ is a diagonal matrix, each diagonal component of which is given as follows:

$$d_i = \sum_{j=1}^{n} a_{ij} \quad (2)$$

Therefore,

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} & 0 & \cdots & 0 \\ 0 & \sum_{j=1}^{n} a_{2j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{j=1}^{n} a_{nj} \end{bmatrix}. \quad (3)$$

The Laplacian matrix $\mathbf{L}$ is then defined as follows:

$$\mathbf{L} = \mathbf{D} - \mathbf{A} \quad (4)$$

In this study, we propose an equal distribution problem with the Laplacian matrix to characterize and quantify the connectivity of the network.

(2) Equal Distribution Problem

Imagine that (relief) supplies must be distributed evenly among the locations in the network in case of emergency. The followings are assumed:

A1. Relief supplies must be distributed to all nodes in the network

A2. The required amount of supplies at the node is proportional to the node's attribute, e.g. population at the node

A3. A part of supplies is transferred to adjacent nodes that are suffering worse shortage of supplies.

Let $x_i(t)$ denote the present amount of supplies at node i until time t. The required amount of supplies at node i is b_i.

In emergency, the relief supplies have to be distributed. Even though the amount of supplies does not reach the required level, they are transferred to an adjacent node if the fulfilment rate at the adjacent node is less than that at this node. This is myopic, but, in emergency, we only have local information rather than full information of the whole network area.

A4. The transfer "strength" is proportional to the difference between the fulfilment rates of connected two nodes.

The transfer strength of supplies is given by

$$\varphi^{i\to j}(t) = -a_{ij}\left[\frac{x_i(t)}{b_i} - \frac{x_j(t)}{b_j}\right] \quad (5)$$

where $\varphi^{i\to j}(t)$ is the transfer strength from node i to node j at time t. Clearly, $\varphi^{i\to i}(t) = 0$ and $\varphi^{j\to i}(t) = a_{ij}\left[x_i(t)/b_i - x_j(t)/b_j\right]$. Therefore, for any nodes,

$$\varphi^{i\to j}(t) = -\varphi^{j\to i}(t). \quad (6)$$

The time derivative of the amount of supplies on node i is given as follows:

$$\frac{d}{dt}x_i(t) = \sum_{j=1}^{n}\varphi^{i\to j}(t) = -\sum_{j=1}^{n}a_{ij}\left[\frac{x_i(t)}{b_i} - \frac{x_j(t)}{b_j}\right] \quad (7)$$

We can confirm the followings:

$$\sum_{i=1}^{n}\frac{d}{dt}x_i(t) = \sum_{i=1}^{n}\sum_{j=1}^{n}\varphi^{i\to j}(t) = 0, \quad (8)$$

because $\varphi^{i\to j}(t) = -\varphi^{j\to i}(t)$ in Eq. (6) and $\varphi^{i\to i}(t) = 0$. This indicates that the total of supplied is always constant. Let $\mathbf{x}(t)$ and $\frac{d}{dt}\mathbf{x}(t)$ define as follows:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \frac{d}{dt}\mathbf{x}(t) = \begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \\ \vdots \\ \frac{d}{dt}x_n(t) \end{bmatrix}. \quad (9)$$

Let $\mathbf{b}$ be the vector of the required amounts of supplies, whose component is b_i. As will be stated in Appendix, using $\mathbf{x}(t)$ and $\frac{d}{dt}\mathbf{x}(t)$, Eq. (7) gives the time derivative of $\mathbf{x}(t)$ as follows:

$$\frac{d}{dt}\mathbf{x}(t) = -\mathbf{L}\mathbf{B}^{-1}\mathbf{x}(t) \quad (10)$$

where $\mathbf{B}^{-1}$ is the inverse of $\mathbf{B}$, and $\mathbf{B} = \mathrm{diag}(\mathbf{b})$, that is,

$$\mathbf{B} = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{bmatrix}, \quad (11)$$

where $\mathrm{diag}(\mathbf{b})$ denotes the diagonal matrix whose diagonal components are the components of $\mathbf{b}$. Let

$$\mathbf{x}(t) = \mathbf{B}^{\frac{1}{2}}\mathbf{y}(t), \quad (12)$$

where

$$\mathbf{B}^{\frac{1}{2}} = \begin{bmatrix} \sqrt{b_1} & 0 & \cdots & 0 \\ 0 & \sqrt{b_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{b_n} \end{bmatrix}. \quad (13)$$

Substituting Eq. (11) into Eq. (9) yields

$$\mathbf{B}^{\frac{1}{2}}\frac{d}{dt}\mathbf{y}(t) = -\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{y}(t). \quad (14)$$

Multiplying the above equation with $\mathbf{B}^{-\frac{1}{2}}$ from the left gives

$$\frac{d}{dt}\mathbf{y}(t) = -\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{y}(t). \quad (15)$$

Let $\mathbf{L}_s$ define as follows:

$$\mathbf{L}_s \coloneqq \mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}} = \begin{bmatrix} \sum_{j=1}^{n}\frac{a_{1j}}{b_1} & -\frac{a_{12}}{\sqrt{b_1b_2}} & \cdots & -\frac{a_{1n}}{\sqrt{b_1b_n}} \\ -\frac{a_{12}}{\sqrt{b_1b_2}} & \sum_{j=1}^{n}\frac{a_{2j}}{b_2} & \cdots & -\frac{a_{2n}}{\sqrt{b_2b_n}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{1n}}{\sqrt{b_1b_n}} & -\frac{a_{2n}}{\sqrt{b_2b_n}} & \cdots & \sum_{j=1}^{n}\frac{a_{nj}}{b_n} \end{bmatrix}. \quad (16)$$

While $\mathbf{L}_b$ is asymmetric, where $\mathbf{L}_b = \mathbf{L}\mathbf{B}^{-1}$, $\mathbf{L}_s$ is a real symmetric matrix, because $\mathbf{L}_s = \mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}} = \left[\mathbf{B}^{-\frac{1}{2}}\right]^T\mathbf{L}\mathbf{B}^{-\frac{1}{2}}$, where $\mathbf{B}^T$ is the transpose of $\mathbf{B}$. Replacing $\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}$ in Eq. (15) with $\mathbf{L}_s$ gives

$$\frac{d}{dt}\mathbf{y}(t) = -\mathbf{L}_s\,\mathbf{y}(t). \quad (17)$$

A real symmetric matrix can be diagonalized (e.g. Strang, 1976). Therefore, the above linear ordinary differential equation is solved as

$$\mathbf{y}(t) = e^{-\mathbf{L}_s t}\mathbf{y}(0) \quad (18)$$

where $\mathbf{y}(0)$ is the initial of $\mathbf{y}(t)$, and, furthermore, the above can be decomposed as

$$\mathbf{y}(t) = e^{-\mathbf{L}_s t}\mathbf{y}(0)$$
$$= \tilde{c}_1 e^{-\lambda_1 t}\mathbf{v}_1 + \tilde{c}_2 e^{-\lambda_2 t}\mathbf{v}_2 + \cdots + \tilde{c}_n e^{-\lambda_n t}\mathbf{v}_n \quad (19)$$

where $\tilde{c}_i$ is the multiplier associated with $\mathbf{y}(0)$, and λ_i and $\mathbf{v}_i$ are the eigenvalue and eigenvector of $\mathbf{L}_s$, respectively (e.g. Strang, 1976). From Sylvester's law of inertia (e.g. Strang, 1976), the eigenvalues of $\mathbf{L}_s$, which is a congruent matrix of $\mathbf{L}$ because $\mathbf{L}_s = \left[\mathbf{B}^{-\frac{1}{2}}\right]^T \mathbf{L}\mathbf{B}^{-\frac{1}{2}}$, are satisfied with

$$0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n\,, \quad (20)$$

as with the Laplacian matrix $\mathbf{L}$. Thus, all eigenvalues of $\mathbf{L}_s$ are non-negative, and $\mathbf{L}_s$ is semi-definite (e.g. Strang, 1976). Therefore,

$$\mathbf{y}(t) = \tilde{c}_1\mathbf{v}_1 + \tilde{c}_2 e^{-\lambda_2 t}\mathbf{v}_2 + \cdots + \tilde{c}_n e^{-\lambda_n t}\mathbf{v}_n. \quad (21)$$

Multiplying the above equation with $\mathbf{B}^{\frac{1}{2}}$ from the left yields

$$\mathbf{x}(t) = \mathbf{B}^{\frac{1}{2}}\mathbf{y}(t) = \tilde{c}_1\mathbf{B}^{\frac{1}{2}}\mathbf{v}_1 + \tilde{c}_2 e^{-\lambda_2 t}\mathbf{B}^{\frac{1}{2}}\mathbf{v}_2 + \tilde{c}_3 e^{-\lambda_3 t}\mathbf{B}^{\frac{1}{2}}\mathbf{v}_3 + \cdots + \tilde{c}_n e^{-\lambda_n t}\mathbf{B}^{\frac{1}{2}}\mathbf{v}_n, \quad (22)$$

because $\mathbf{x}(t) = \mathbf{B}^{\frac{1}{2}}\mathbf{y}(t)$ of Eq. (12). Concludingly, the "speed" of convergence of the system is prescribed by the second minimum eigenvalue λ_2. Eq. (22) indicates that the eigenvalues of $\mathbf{L}\mathbf{B}^{-1}$ and $\mathbf{L}_s$ are all equal, and $\tilde{\mathbf{u}}_i = \mathbf{B}^{\frac{1}{2}}\mathbf{v}_i$, where $\tilde{\mathbf{u}}_i$ is a (non-normalized) eigenvector of $\mathbf{L}\mathbf{B}^{-1}$ for λ_i. These are also confirmed by $\mathbf{L}_s\mathbf{v}_i = \lambda_i\mathbf{v}_i$,. Multiplying $\mathbf{L}_s\mathbf{v}_i = \lambda_i\mathbf{v}_i$ with $\mathbf{B}^{\frac{1}{2}}$ from left yields $\mathbf{B}^{\frac{1}{2}}\mathbf{L}_s\mathbf{v}_i = \lambda_i\mathbf{B}^{\frac{1}{2}}\mathbf{v}_i$, and then, $\mathbf{B}^{\frac{1}{2}}\mathbf{L}_s\mathbf{v}_i = \lambda_i\tilde{\mathbf{u}}_i$ because $\tilde{\mathbf{u}}_i = \mathbf{B}^{\frac{1}{2}}\mathbf{v}_i$. On the other hand, $\mathbf{B}^{\frac{1}{2}}\mathbf{L}_s\mathbf{v}_i = \mathbf{B}^{\frac{1}{2}}\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{v}_i = \mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{v}_i = \mathbf{L}\mathbf{B}^{-1}\mathbf{B}^{\frac{1}{2}}\mathbf{v}_i = \mathbf{L}\mathbf{B}^{-1}\tilde{\mathbf{u}}_i$ due to $\mathbf{L}_s = \mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}$. Therefore, $\mathbf{L}\mathbf{B}^{-1}\tilde{\mathbf{u}}_i = \lambda_i\tilde{\mathbf{u}}_i$. Let $\mathbf{u}_i = \tilde{\mathbf{u}}_i/\|\mathbf{u}_i\|$, that is, $\mathbf{u}_i$ is the normalized eigenvector of $\mathbf{L}\mathbf{B}^{-1}$.

Concludingly, λ_i and $\mathbf{u}_i$ are the eigenvalue and eigenvector of $\mathbf{L}\mathbf{B}^{-1}$.

$$\mathbf{x}(t) = c_1\mathbf{u}_1 + c_2 e^{-\lambda_2 t}\mathbf{u}_2 + \cdots + c_n e^{-\lambda_n t}\mathbf{u}_n, \quad (23)$$

where $c_i = \tilde{c}_i\|\mathbf{u}_i\|$.

Eq. $(a5)$ in Appendix shows that a non-normalized eigenvector of $\mathbf{L}_s$ for the minimum eigenvalue, that is $\lambda_1 = 0$, is

$$\mathbf{b}^{\frac{1}{2}} = \sqrt{\mathbf{b}} = \begin{bmatrix}\sqrt{b_1}\\ \sqrt{b_2}\\ \vdots\\ \sqrt{b_n}\end{bmatrix}. \quad (24)$$

In this study, the eigenvector is normalized, and the (normalized) eigenvector with $\lambda_1 = 0$ is

$$\mathbf{v}_1 = \frac{\mathbf{b}^{\frac{1}{2}}}{\left\|\mathbf{b}^{\frac{1}{2}}\right\|} = \frac{1}{\sqrt{\sum_{i=1}^n b_i}}\begin{bmatrix}\sqrt{b_1}\\ \sqrt{b_2}\\ \vdots\\ \sqrt{b_n}\end{bmatrix}, \quad (25)$$

where $\|\cdot\|$ denotes Euclidean distance. Therefore, when $\lambda_2 > 0$,

$$\lim_{t\to\infty}\mathbf{x}(t) = \tilde{c}_1\mathbf{B}^{\frac{1}{2}}\boldsymbol{v}_1 = \frac{\tilde{c}_1}{\sqrt{\sum_{i=1}^n b_i}}\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix} = c_1\mathbf{b}, \quad (26)$$

where $c_1 = \tilde{c}_1\sqrt{\sum_{i=1}^n b_i}$. Because $\sum_{i=1}^n x_i(t) = Q$ have to be always satisfied, $c_1 = Q/\sum_{i=1}^n b_i$. It is known that the second minimum eigenvalue of the Laplacian matrix is positive for connected networks. Note that any node pair is connected by links in a connected network. From Sylvester's law of inertia (e.g. Strang, 1976), the second minimum eigenvalue of $\mathbf{L}_s$ for connected networks is positive, as with $\mathbf{L}$. Concludingly, for connected networks,

$$\lim_{t\to\infty}\mathbf{x}(t) = \frac{Q}{\sum_{i=1}^n b_i}\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix} = c_1\mathbf{b}. \quad (27)$$

As stated above, the convergence speed of the equal distribution problem is principally characterized by the second minimum eigenvalue, λ_2. Its eigenvector, $\mathbf{u}_2$, is also useful to investigate the network's connectivity. The transfer movement of supplies is decomposed as the equation (17), and the slowest movement is expressed as $c_2 e^{-\lambda_2 t}\mathbf{u}_2$, which principally prescribes the convergence speed. The second minimum eigenvector, $\mathbf{u}_2$, can partition the nodes in the network. As the value of component of $\mathbf{u}_2$ is closer, the dynamics are more similar in the sense of the slowest movement of supplies.

There are finite positive constants, c, γ, and λ. If

$$\|\mathbf{x}(t) - c\mathbf{b}\| \le \gamma e^{-\lambda t} \quad \forall t \ge 0, \forall\mathbf{x}(0) \in \Omega, \quad (28)$$

$\mathbf{x}(t)$ converges exponentially, where

$$\Omega = \begin{cases}\sum_{i=1}^n x_i(t) = Q > 0\\ x_i(t) \ge 0 \quad (i = 1,2,\cdots,n).\end{cases} \quad (29)$$

For any time ($\forall t \ge 0$), $\|\mathbf{x}(t) - c\mathbf{b}\| \ge 0$ is less than $\gamma e^{-\lambda t}$, that is, the supplies, $\mathbf{x}(t)$, converge exponentially with exponent λ. As exponent λ increases, the system converges faster. Thus, exponent λ stands for an exponential convergence speed. For connected neworks, $\lambda_2 > 0$, and the exponential convergence speed exponent is λ_2, according to Eq. (22). The exponential convergence speed does not depend on the initial state, and is widely available.

The second minimum eigenvector as well as the eigenvalue is also useful. It can be seen from Eq. (22) that the second minimum eigenvalue and eigenvector dominates over the third or subsequent eigenvalues and eigenvectors as an enough time passes. The dynamics of supplies at the nodes asymptotically approximate to

$$\mathbf{x}(t) \cong c_1\mathbf{b} + c_2 e^{-\lambda_2 t}\mathbf{u}_2. \quad (30)$$

The first term in the right hand of the above equation is constant, and the dynamics of supplies at nodes are prescribed by the second minimum eigenvalue and eigenvector.

$$\mathbf{r}(t) = \mathbf{B}^{-1}\mathbf{x}(t) \cong c_1\mathbf{1} + c_2 e^{-\lambda_2 t}\mathbf{B}^{-1}\mathbf{u}_2. \quad (31)$$

The dynamics of fulfillment rate of supplies at a node is closer to that of supplies at the node with a similar second minimum eigenvector component. The fulfillment rate of supplies at node i is expressed as $r_i(t) = x_i(t)/b_i \cong c_1 + c_2 \frac{u_{2i}}{b_i} e^{-\lambda_2 t}$, where u_{2i} is component i of $\mathbf{u}_2$, that is, i-th component of the second minimum eigenvector component. Hence, if $\frac{u_{2i}}{b_i}$ is close to $\frac{u_{2j}}{b_j}$, the dynamics of fulfillment rates at nodes i and j are similar. As the nodes are connected more strongly, their dynamics should be closer. Thus, connectivity between a pair of nodes can be given by

$$v_{ij} = \left\| \frac{u_{2i}}{b_i} - \frac{u_{2j}}{b_j} \right\|, \quad (32)$$

where v_{ij} is the connectivity from the standpoint of the dynamics of fulfillment rates at nodes. Thus, the component of the second minimum eigenvector provides the information on the connectivity between the nodes. For network connectivity analysis, the following is defined:

$$v_i = \frac{u_{2i}}{b_i}. \quad (33)$$

(3) Simple Example

Fig. 1 shows a 4-node U-shape network, where 3 links connect the 4 nodes. The link strengths are all 1, that is, $a_{12} = a_{23} = a_{34} = 1$. The Laplacian matrix of the 4-node U-shape network is

$$\mathbf{L} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \quad (34)$$

In case that the required amounts of supplies are all 1, that is, $b_1 = b_2 = b_3 = b_4 = 1$, the second minimum eigenvalue of the Laplacian matrix of the 4-node U-shape network, which is expressed in Eq. (34), is $2 - \sqrt{2}$. Fig. 2 presents a 4-node square network, in which 4 links connect 4 nodes, with $a_{12} = a_{23} = a_{34} = a_{14} = 1$. The connectivity of the 4-node square network shown in Fig. 2 is stronger than that of the 4-node U-shape network in Fig. 1. The second minimum eigenvalue of the Laplacian matrix of the 4-node square network is 2, while that of the U-shape network is $2 - \sqrt{2}$ as described above. These example networks exemplify that the second minimum eigenvalues of strongly connected network is larger than that of weakly connected network.

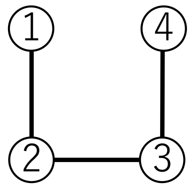

Fig. 1 4-node U-shape network

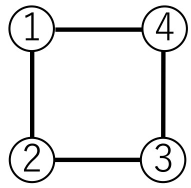

Fig. 2 4-node square network

Fig. 3 shows the dynamics of the amounts of supplies at the nodes in the 4-node U-shape network with $x_1(0) = 1.2, x_2(0) = 0.8, x_3(0) = 0.4$ and $x_4(0) = 0$. The figure illustrates that the equal distribution reaches as enough time passes.

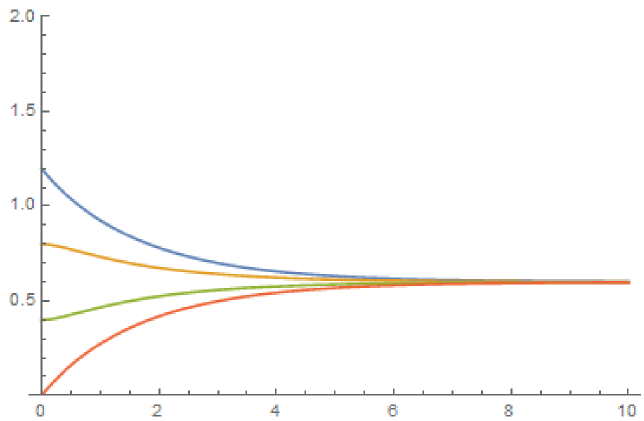

Fig. 3 The dynamics of supplies in the 4-node U-shape network

The dynamics of the amounts of supplies at the nodes in the 4-node square network with $x_1(0) = 1.2, x_2(0) = 0.8, x_3(0) = 0.4$ and $x_4(0) = 0$ are depicted in Fig. 4. The speed to converge to the equal distribution in the square network is much faster than the U-shape network. Thus, we can confirm from the dynamics as well as the second minimum eigenvalue that the supplies are equally distributed faster in the strongly connected square network than in the U-shape network.

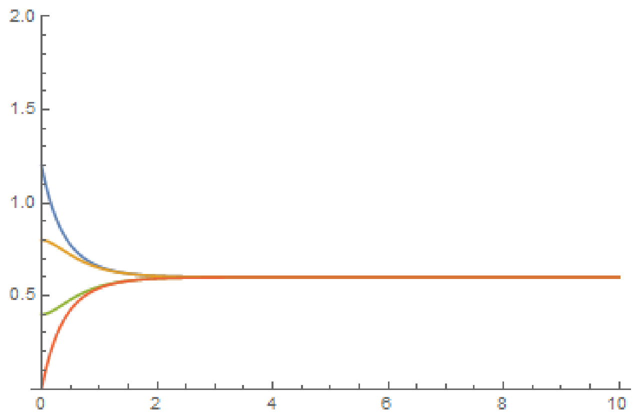

Fig. 4 The dynamics of supplies in the 4-node square network

Next, the case with different required amounts of supplies at the nodes are examined. In the 4-node U-shape network, the required amount of supplies for 2 nodes in the middle in the network is b while that for the both end 2 nodes is 1, that is,

$$\mathbf{b} = \begin{bmatrix} 1 \\ b \\ b \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{35}$$

The characteristic polynomial of $\mathbf{LB}^{-1}$ to yield the eigenvalues of the 4-node U-shape network is expressed as

$$|\lambda\mathbf{I} - \mathbf{LB}^{-1}| = \lambda\left[\lambda - \left(1 + \frac{1}{b}\right)\right]\left[\lambda^2 - \left(1 + \frac{3}{b}\right)\lambda + \frac{2}{b}\right] \tag{36}$$

Therefore, the eigenvalues of $\mathbf{LB}^{-1}$ for the 4-node U-shape network are

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = \dfrac{b + 3 - \sqrt{b^2 - 2b + 9}}{2b} \\ \lambda_3 = \dfrac{b + 1}{b} \\ \lambda_4 = \dfrac{b + 3 + \sqrt{b^2 - 2b + 9}}{2b} \end{cases} \tag{37}$$

Let $\lambda_2(b)$ denote $(b + 3 - \sqrt{b^2 - 2b + 9})/2b$. Fig. 5 indicates $\lambda_2(b)$, that is, the second minimum eigenvalue of Laplacian matrix of the 4-node U-shape network with varying b. The λ_2-intercept in Fig. 5 is given as follows:

$$\lim_{b \to 0} \lambda_2(b) = \frac{2}{3}. \tag{38}$$

The derivative of $\lambda_2(b)$ with respect to b is

$$\frac{d\lambda_2}{db} = \frac{-b + 9 - 3\sqrt{(b-1)^2 + 8}}{2b^2\sqrt{(b-1)^2 + 8}} < 0. \tag{39}$$

Clearly, the denominator of the second term is positive. The numerator is $-b + 9 - 3\sqrt{(b-1)^2 + 8} < 0 \ (\forall b > 0)$, because $(-b + 9)^2 - \left(3\sqrt{b^2 - 2b + 9}\right)^2 = -8b^2 \leq 0$. Therefore, the second minimum eigenvalue of Laplacian matrix of the 4-node U-shape network decreases as b increases.

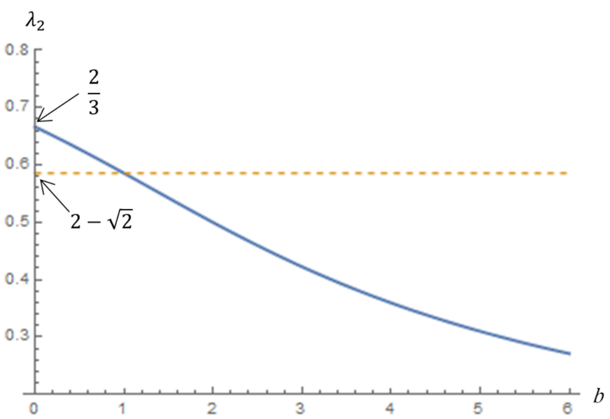

Fig. 5 The second minimum eigenvalue of the 4-node U-shape network with varying b

Fig. 6 shows the dynamics of supplies in the 4-node U-shape network with $b = \frac{1}{2}$. The amounts of supplies at nodes 2 and 3 converge to a half of those at nodes 1 and 4 because $b = \frac{1}{2}$. Fig. 7 presents the dynamics of supplies in the U-shape network with $b = \frac{1}{20}$.

It would seem that the 4-node U-shape network approaches the 2-node network which is shown in Fig. 8 as b goes to 0. The characteristic polynomial of the 2-node network with $a_{12} = 1/3$ in Fig. 8 is given by

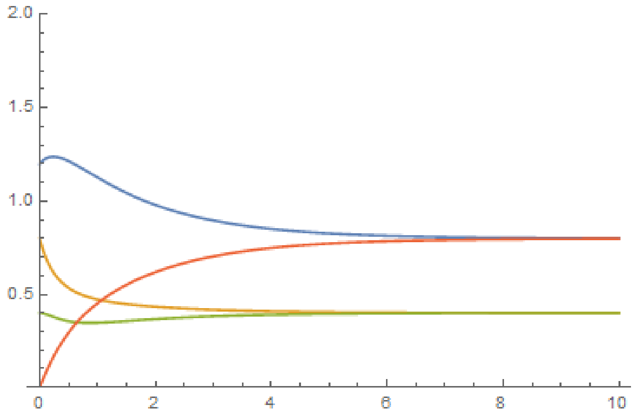

Fig. 6 The dynamics of supplies in the U-shape network with $b = \frac{1}{2}$

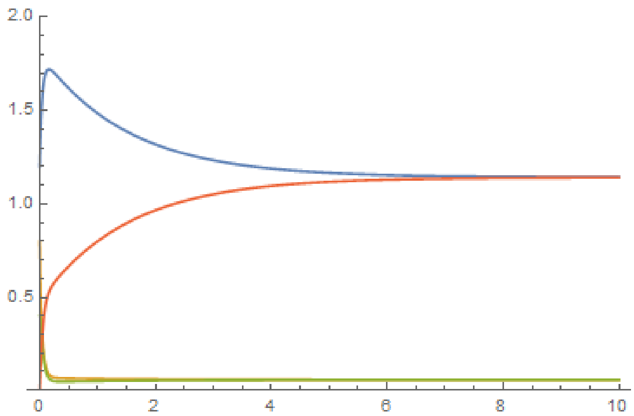

Fig. 7 $b = 1/20$ the dynamics of supplies in the U-shape network with $b = \frac{1}{20}$

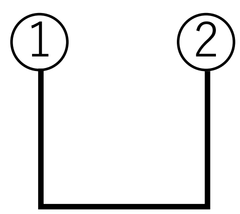

Fig. 8 2-node U-shape network

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \right| = \lambda \left(\lambda - \frac{2}{3} \right). \qquad (40)$$

Therefore, the eigenvalues are 0 and $2/3$. Thus, the second minimum eigenvalue of Laplacian matrix of the 2-node network with $a_{12} = 1/3$ is $2/3$. This is equal to $\lambda_2(0) = \lim_{b \to 0} \lambda_2(b) = 2/3$. As $b \to 0$, the 4-node U-shape network with $a_{12} = a_{23} = a_{34} = 1$ (the link strengths are all 1) corresponds to the 2-node U-shape network with $a_{12} = 1/3$. Fig. 9 shows the dynamics of supplies of the 2-node network with $a_{12} = 1/3$. Fundamental structure of the dynamics in Fig. 9 is similar to that in Fig. 7. Intuitively, we can understand that the 2-node network in Fig. 8 is a limit of the 4-node U-shape network, and their convergence speeds seems almost same.

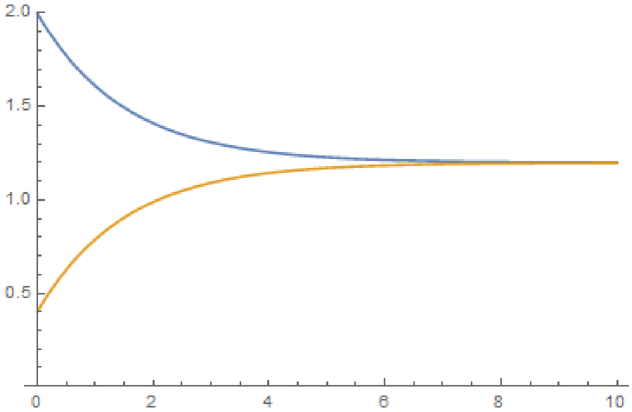

Fig. 9 The dynamics of supplies of the 2-node network with $a_{12} = 1/3$

To exemplify the connectivity between nodes using the second minimum eigenvector, a 14-node example network shown in Fig. 10 is introduced. In the 14-node example network, all link strengths are 1. Fig 11 plots the second minimum eigenvector components divided by the required amount of supplies at the nodes, $v_i = u_{2i}/b_i$ $(i = 1, 2, \cdots, 14)$. As a whole, the 14 plots in Fig. 11 are roiughly classified into 3 clusters: cluster 1 (node 1, 2, 3, 4, 5, and 6), cluster 2 (node 7, 8, 9, and 10) and cluster 3 (node 11, 12, 13, and 14). These are consistent with the network structure in Fig. 10. Within the cluster, the nodes are mutually connected strongly.

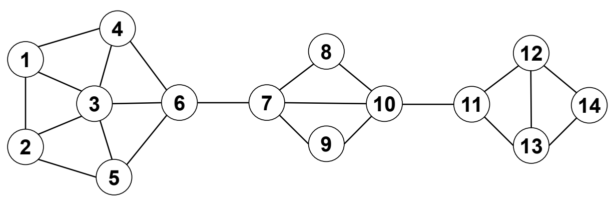

Fig. 10 An example of 14-node network whose link strengths are all 1

In cluster 3, v_{12} is equal to v_{13}. This means that their (asymptotical) supply dynamics are almost same, and the connectivity between nodes 12 and 13 is much stronger within the cluster. Therefore, nodes 12 and 13 can be lumped into a double-size node. Similarly, the pair of nodes 8 and 9 in cluster 2, that of nodes 1 and 2 and group of nodes 3, 4, and 5 in cluster 1 are blocked into two double-size nodes and a triple-size node, respectively.

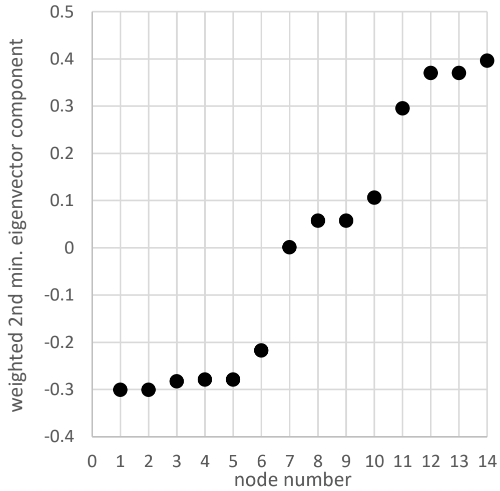

Fig. 11 The values of components of the second minimum eigenvector of the 14-node network

Then, an integrated network shown in Fig 12 is made. In the integrated 9-node network, $b_2 = 3$, $b_1 = b_5 = b_8 = 2$, and $b_3 = b_4 = b_6 = b_7 = b_9 = 1$. In Fig. 10, node 1 is connected with nodes 3 and 4 by two link one by one and node 2 is tied with nodes 3 and 5 by two links one by one. Therefore, a pair of nodes 1 and 2 connect a group of nodes 3, 4, and 5 by 4 links, and $a_{12} = 4$ in the integrated 9-node network of Fig. 12, which means that node 1 is tied to node 2 with 4 links in the integrated 9-node network. Similarly, $a_{23} = 3, a_{45} = 2, a_{56} = 2, a_{78} = 2$, and $a_{89} = 2$.

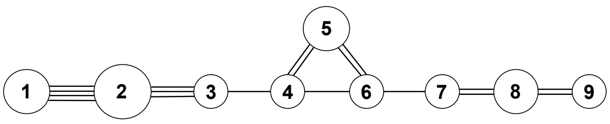

Fig. 12 An integrated 9-node network for the 14-node network

The eigenvalues of the Laplacian matrix of the integrated 9-node network in Fig. 12 are illustrated in Fig. 13. Those of the 14-node network are also shown in Fig. 13. The second minimum eigenvalues of the two networks are almost same. The values of v_i $(i = 1, 2, \cdots, 9)$ are presented in Fig. 14. Simple comparison of $\mathbf{v}$ in the 14-node and integrated 9-node networks cannot be made as the number of nodes is different. For comparison, $\mathbf{v}_{9 \to 14} = (v_1, v_1, v_2, v_2, v_2, v_3, v_4, v_5, v_5, v_6, v_7, v_8, v_8, v_9)$ is

Table 1 The values of the second minimum eigenvectors of 14-node and integrated 9-node networks

node #	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\frac{\mathbf{v}_{9\to14}}{\lVert\mathbf{v}_{9\to14}\rVert}$	-0.300	-0.300	-0.280	-0.280	-0.280	-0.217	0.001	0.058	0.058	0.107	0.296	0.371	0.371	0.397
14-node network	-0.300	-0.300	-0.282	-0.279	-0.279	-0.217	0.002	0.058	0.058	0.107	0.296	0.370	0.370	0.397

introduced, where $v_1, v_2, v_3, \cdots, v_9$ are the values of the integrated 9-node network. As Table 1 indicates, the vector of $\mathbf{v}_{9\to14}/\lVert\mathbf{v}_{9\to14}\rVert$ is almost identical to $\mathbf{v}$ in the 14-node network. Thus, the 14-node network is adequately coarsened to the integrated 9-node network.

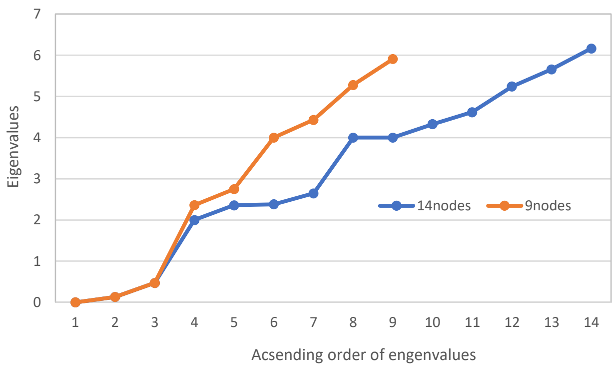

Fig. 13 Eigenvalues of 14-node and its integrated 9-node networks

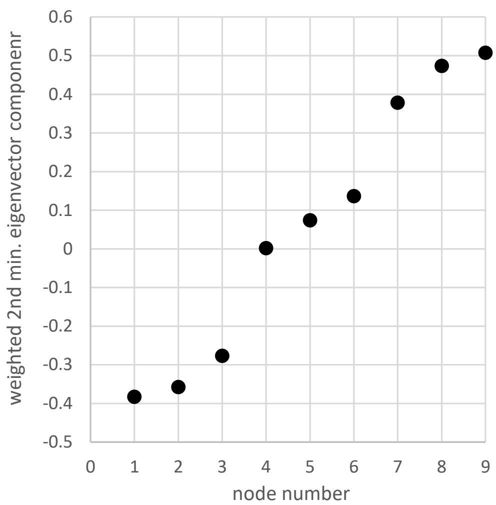

Fig. 14 The values of components of the second minimum eigenvector of the integrated 9-node network

3. REINFORCEMENT OPTIMIZATION PROBLEM FOR NETWORK CONNECTIVITY

From the standpoint of equal distribution problem, the network connectivity is formulated in the previous section. It is natural to consider how to reinforce the network for the fastest equal distribution under the constraints. We shall call this a reinforcement optimization problem.

(1) Formulation of reinforcement optimization problem

The reinforcement optimization problem in this paper is to determine which link is reinforced under the budget. The reinforcement of link (i,j) is to add w_{ij} to component (i,j) of the adjacent matrix $\mathbf{A}$. The component of the reinforced adjacent matrix, $\hat{\mathbf{A}} = \mathbf{A} + \mathbf{W}$, is given by

$$\hat{a}_{ij} = a_{ij} + w_{ij}, \tag{41}$$

where w_{ij} is the reinforcement amount of link (i,j) and $\mathbf{W}$ is the reinforcement matrix, whose component is w_{ij}. Let $\hat{\mathbf{L}}$ denote the reinforced Laplacian matrix, that is, the Laplacian matrix of the reinforced adjacent matrix $\hat{\mathbf{A}}$. In case we emphacize that $\hat{\mathbf{L}}$ is a matrix-valued function of $\mathbf{W}$, we also use $\hat{\mathbf{L}}(\mathbf{W})$ in this paper. A simple budget constraint is as follows:

$$\begin{cases} 0 \le w_{ij} \le \rho_{ij} & \forall(i,j) \\ \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} \le \beta \end{cases} \tag{42}$$

where β is the total budget and ρ_{ij} is the upper limit for component (i,j) of the reinforced adjacent matrix $\hat{\mathbf{A}}$. This constraint is linear, and is convex with respect to $\{w_{ij}\} = \mathbf{W}$. Let $\mathbb{W}$ denote the set of the above budget constraint, and the feasible constraint is expressed as $\mathbf{W} \in \mathbb{W}$.

As stated in the previous section, the exponential convergence speed of the equal distribution problem is principally characterized by the second minimum eigenvalue, λ_2. Therefore, maximizing the second minimum eigenvalue of $\hat{\mathbf{L}}\mathbf{B}^{-1}$ with $\mathbf{W} \in \mathbb{W}$ yields a reinforcement optimization solution. Concludingly, the reinforcement optimization problem for equal distribution problem is formulated as follows:

$$\max_{\mathbf{W}}.\ \lambda_2\left(\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-1}\right) \quad s.t.\ \mathbf{W} \in \mathbb{W} \tag{43}$$

where $\lambda_2\left(\hat{\mathbf{L}}\mathbf{B}^{-1}\right)$ is the second minimum eigenvalue of the reinforced Laplacian matrix $\hat{\mathbf{L}}\mathbf{B}^{-1}$. The eigenvalues of $\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}$ are equal to those of $\mathbf{L}\mathbf{B}^{-1}$, as Eqs. (21), (22), and (23) indicate. Therefore, the maximization problem of Eq. (43) is re-written as

$$\max_{\mathbf{W}}.\ \lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\,\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\right) \quad s.t.\ \mathbf{W} \in \mathbb{W}. \tag{44}$$

(2) Local improvement of the network connectivity

Before examining the globally optimizing the reinforcement problem, local improvement of the network connectivity local is investigated in this sub-section. The reinforce net amount w_{ij} is sufficiently small. In this case, it is written as Δw_{ij}, and $0 \le \Delta w_{ij} \ll 1\ \forall(i,j)$ and $\Delta\mathbf{W} = \{\Delta w_{ij}\}$. The reinforced adjacency matrix is $\mathbf{A} + \Delta\mathbf{W}$ and the Laplacian matrix is $\hat{\mathbf{L}}[\Delta\mathbf{W}] = \mathbf{L} + \Delta\mathbf{L}[\Delta\mathbf{W}]$. In the remainder of this sub-section, $\Delta\mathbf{L}[\Delta\mathbf{W}]$ is denoted simply by $\Delta\mathbf{L}$.

Let $\hat{\lambda}_2$ and $\hat{\mathbf{v}}_2$ denote the second minimum eigenvalue and eigenvector of $\mathbf{B}^{-\frac{1}{2}}(\mathbf{L}+\Delta\mathbf{L})\mathbf{B}^{-\frac{1}{2}}$, respectively. As stated above, $\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$, and $\hat{\mathbf{L}}\mathbf{B}^{-1}\hat{\mathbf{v}}_2 = \mathbf{B}^{-\frac{1}{2}}(\mathbf{L}+\Delta\mathbf{L})\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{v}}_2 = \hat{\lambda}_2\hat{\mathbf{v}}_2$. The difference between λ_2 and $\hat{\lambda}_2$ must be infinitesimal, and $\hat{\lambda}_2 = \lambda_2 + \Delta\lambda_2$. Similarly, $\hat{\mathbf{v}}_2 = \mathbf{v}_2 + \Delta\mathbf{v}_2$. Therefore, $\mathbf{B}^{-\frac{1}{2}}(\mathbf{L}+\Delta\mathbf{L})\mathbf{B}^{-\frac{1}{2}}(\mathbf{v}_2+\Delta\mathbf{v}_2) = (\lambda_2+\Delta\lambda_2)(\mathbf{v}_2+\Delta\mathbf{v}_2)$. Ignoring the second-order infinitesimal terms yields

$$\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{v} + \mathbf{B}^{-\frac{1}{2}}\Delta\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{v}_2 + \mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\Delta\mathbf{v}_2 \cong \lambda_2\mathbf{v}_2 + \Delta\lambda_2\mathbf{v}_2 + \lambda_2\Delta\mathbf{v}_2. \quad (45)$$

Plugging $\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$ into the above equation gives

$$\mathbf{B}^{-\frac{1}{2}}\Delta\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{v}_2 + \mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\Delta\mathbf{v}_2 = \Delta\lambda_2\mathbf{v}_2 + \lambda_2\Delta\mathbf{v}_2. \quad (46)$$

Multiplying the above equation by $\mathbf{v}_2^T$ from the left produces

$$\mathbf{v}_2^T\mathbf{B}^{-\frac{1}{2}}\Delta\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{v}_2 + \mathbf{v}_2^T\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\Delta\mathbf{v}_2 = \Delta\lambda_2\mathbf{v}_2^T\mathbf{v}_2 + \lambda_2\mathbf{v}_2^T\Delta\mathbf{v}_2, \quad (47)$$

because $\|\mathbf{v}_2\| = \mathbf{v}_2^T\mathbf{v}_2 = 1$. Using the $\left(\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{v}_2\right)^T = (\lambda_2\mathbf{v}_2)^T$, we obtain

$$\Delta\lambda_2 = \mathbf{v}_2^T\mathbf{B}^{-\frac{1}{2}}\Delta\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{v}_2 + \left(\mathbf{v}_2^T\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}} - \lambda_2\mathbf{v}_2^T\right)\Delta\mathbf{v}_2 = \mathbf{v}_2^T\mathbf{B}^{-\frac{1}{2}}\Delta\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{v}_2. \quad (48)$$

From $\tilde{\mathbf{u}}_i = \mathbf{B}^{\frac{1}{2}}\mathbf{v}_i$,

$$\Delta\lambda_2 = \tilde{\mathbf{u}}_2^T\mathbf{B}^{-1}\Delta\mathbf{L}\mathbf{B}^{-1}\tilde{\mathbf{u}}_2, \quad (49)$$

where $\tilde{\mathbf{u}}_2$ is the non-normalized second minimum eigenvector of $\mathbf{L}\mathbf{B}^{-1}$ while $\mathbf{v}_2$ is the normalized second minimum eigenvector of $\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}$, as stated above.

In the case that link (i,j) is only reinforced, and w_{ij} is only added to component (i,j) of $\hat{\mathbf{A}}$, that is, $\hat{a}_{ij} = a_{ij} + w_{ij}$, while the other components are not reinforced. In this case, w_{ij}, $-w_{ij}$, $-w_{ij}$, and w_{ij} are added to components (i,i), (i,j), (j,i),and (j,j) in the Laplacian matrix, respectively, and the other components are unchanged. The difference in the Laplacian matrix before and after the reinforcement, $\Delta\mathbf{L}$, is as follows:

$$\Delta\mathbf{L} = \begin{bmatrix} & & & & \\ & w_{ij} & & -w_{ij} & \\ & & & & \\ & -w_{ij} & & w_{ij} & \\ & & & & \end{bmatrix} \quad (50)$$

where blanks denote 0 in the above matrix. Substituting the above $\Delta\mathbf{L}$ into Eq. (48) yields

$$\mathbf{v}_2^T\mathbf{B}^{-\frac{1}{2}}\Delta\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{v}_2 = \mathbf{v}_2^T\mathbf{B}^{-\frac{1}{2}}\begin{bmatrix}\mathbf{0}\\ w_{ij}\left(\frac{v_{2i}}{\sqrt{b_i}}-\frac{v_{2j}}{\sqrt{b_j}}\right)\\ \mathbf{0}\\ -w_{ij}\left(\frac{v_{2i}}{\sqrt{b_i}}-\frac{v_{2j}}{\sqrt{b_j}}\right)\\ \mathbf{0}\end{bmatrix} = w_{ij}\left(\frac{v_{2i}}{\sqrt{b_i}}-\frac{v_{2j}}{\sqrt{b_j}}\right)^2, \quad (51)$$

where $\mathbf{0}$ is zero vector. From $\tilde{\mathbf{u}}_i = \mathbf{B}^{\frac{1}{2}}\mathbf{v}_i$, we also obtain

$$\Delta\lambda_2 = \Delta w_{ij}\left(\frac{v_{2i}}{\sqrt{b_i}}-\frac{v_{2j}}{\sqrt{b_j}}\right)^2 = \Delta w_{ij}\left(\frac{\tilde{u}_{2i}}{b_i}-\frac{\tilde{u}_{2j}}{b_j}\right)^2. \quad (52)$$

Because $\mathbf{u}_i = \tilde{\mathbf{u}}_i/\|\tilde{\mathbf{u}}_i\| = \mathbf{B}^{\frac{1}{2}}\mathbf{v}_i$ and $\tilde{\mathbf{u}}_i = \mathbf{B}^{\frac{1}{2}}\mathbf{v}_i$,

$$\frac{\partial\lambda_2}{\partial w_{ij}} = k\left(\frac{u_{2i}}{b_i}-\frac{u_{2j}}{b_j}\right)^2. \quad (53)$$

where $k = \left\|\mathbf{B}^{\frac{1}{2}}\mathbf{v}_2\right\|^2 = \mathbf{b}^T\mathbf{v}_2$.

(3) Convexity of the problem

The function $\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\,\hat{\mathbf{L}}\,\mathbf{B}^{-\frac{1}{2}}\right)$ looks hard to manipulate, but it is concave as shown below. The constraint is also convex as stated above. Therefore, the reinforcement optimization problem of Eq. (44) is a concave optimization problem.

Substituting $\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}\mathbf{B}^{-\frac{1}{2}}$ into Eq. $(a8)$ in Appendix gives

$$\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}\mathbf{B}^{-\frac{1}{2}}\mathbf{x} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\hat{a}_{ij}\left(\frac{x_i}{\sqrt{b_i}}-\frac{x_j}{\sqrt{b_j}}\right)^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}+w_{ij})\left(\frac{x_i}{\sqrt{b_i}}-\frac{x_j}{\sqrt{b_j}}\right)^2 \quad (54)$$

Let $\mathbf{W}^{mid} = \gamma\mathbf{W}^{\bullet} + (1-\gamma)\mathbf{W}^{\circ}$, where $\mathbf{W}^{\bullet}, \mathbf{W}^{\circ} \in \mathbb{W}$ are different and $0 \le \gamma \le 1$.

$$\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}^{mid}]\mathbf{B}^{-\frac{1}{2}}\mathbf{x} = \frac{\gamma}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij}+w_{ij}^{\bullet}\right)\left(\frac{x_i}{\sqrt{b_i}}-\frac{x_j}{\sqrt{b_j}}\right)^2 = \gamma\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}^{\bullet}]\mathbf{B}^{-\frac{1}{2}}\mathbf{x}+(1-\gamma)\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}^{\circ}]\mathbf{B}^{-\frac{1}{2}}\mathbf{x} \quad (55)$$

where $\hat{\mathbf{L}}[\mathbf{W}^{mid}]$ is the reinforced Laplacian matrix with $\gamma\mathbf{W}^{\bullet}+(1-\gamma)\mathbf{W}^{\circ}$ and $w_{ij}^{\bullet}, w_{ij}^{\circ}$ are components of $\mathbf{W}^{\bullet}, \mathbf{W}^{\circ}$, respectively. From Eq. $(a10)$ in Appendix,

$$\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{mid}\mathbf{B}^{-\frac{1}{2}}\right) = \min_{\mathbf{x}\in\mathbb{X}_2}.\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{mid}\mathbf{B}^{-\frac{1}{2}}\mathbf{x} \quad (56)$$

where $\hat{\mathbf{L}}^{mid} = \hat{\mathbf{L}}[\mathbf{W}^{mid}]$ and $\mathbb{X}_2 = \{\forall\mathbf{x} \mid \mathbf{x}\perp\sqrt{\mathbf{b}},\ \|\mathbf{x}\|=1\}$. Substituting Eq. (55) into the above equation yields

$$\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{mid}\mathbf{B}^{-\frac{1}{2}}\right) = \min_{\mathbf{x}\in\mathbb{X}_2}.\left[\gamma\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{\bullet}\mathbf{B}^{-\frac{1}{2}}\mathbf{x}+(1-\gamma)\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{\circ}\mathbf{B}^{-\frac{1}{2}}\mathbf{x}\right]. \quad (57)$$

Obviously, $\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{\bullet}\mathbf{B}^{-\frac{1}{2}}\mathbf{x} \geq \min_{\mathbf{x}\in\mathbb{X}_2}.\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{\bullet}\mathbf{B}^{-\frac{1}{2}}\mathbf{x} = \lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{\bullet}\mathbf{B}^{-\frac{1}{2}}\right)$ and $\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{\circ}\mathbf{B}^{-\frac{1}{2}}\mathbf{x} \geq \lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{\circ}\mathbf{B}^{-\frac{1}{2}}\right)$. Therefore,

$$\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{mid}\mathbf{B}^{-\frac{1}{2}}\right) \geq \min_{\mathbf{x}\in\mathbb{X}_2}.\left[\gamma\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{\bullet}\mathbf{B}^{-\frac{1}{2}}\right)+(1-\gamma)\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{\circ}\mathbf{B}^{-\frac{1}{2}}\right)\right]. (58) \quad (58)$$

Because $\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{\bullet}\mathbf{B}^{-\frac{1}{2}}\right)$ and $\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{\circ}\mathbf{B}^{-\frac{1}{2}}\right)$ do not includes $\mathbf{x}$, we obtain

$$\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{mid}\mathbf{B}^{-\frac{1}{2}}\right) \geq \gamma\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{\bullet}\mathbf{B}^{-\frac{1}{2}}\right)+(1-\gamma)\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}^{\circ}\mathbf{B}^{-\frac{1}{2}}\right). \quad (59)$$

This indicates that the second minimum eigenvalue of the reinforced Laplacian matrix, $\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\right)$, is concave with respect to the reinforcement matrix $\mathbf{W}$. Concludingly,

$$\min_{\mathbf{W}}.\ -\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\right) \quad s.t.\ \mathbf{W}\in\mathbb{W} \quad (60)$$

is a convex optimization problem (convex programming). Therefore, an ordinary convex optimization algorithm can be applied to Eqs. (44) and (60) and the problem is solved efficiently. Many reinforcement problems are formulated as combinatorial optimization or integer programming problem, but the above reinforcement optimization problem is a convex optimization problem, which is easy to manipulate.

(4) Semi-definite programming for the reinforcement optimization problem

In the previous sub-section, we confirm that the objective function of the reinforcement optimization problem is concave and its constraint is linear.

The reinforcement optimization problem can also be formulated as

$$\max_{\mathbf{W},\xi}.\ \xi \quad s.t.\ \begin{cases}\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\right)\geq\xi \\ \mathbf{W}\in\mathbb{W}\end{cases}. \quad (61)$$

Let $\mathbf{x} = \mathbf{z}+\theta\sqrt{\mathbf{b}}$, where $\mathbf{z}\perp\sqrt{\mathbf{b}}$ and θ is a scalar, that is, $\mathbf{x}$ is decomposed as a vector that is parallel with $\sqrt{\mathbf{b}}$ and an orthogonal vector to $\sqrt{\mathbf{b}}$ without loss of generality. An adjustment constant matrix is introduced. The matrix is $\xi\left[\mathbf{I}-\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T/\sum_{i=1}^n b_i\right]$, and its quadratic form is

$$\xi\mathbf{x}^T\left[\mathbf{I}-\frac{\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T}{\sum_{i=1}^n b_i}\right]\mathbf{x} = \xi\left(\mathbf{x}^T\mathbf{x}-\frac{\mathbf{x}^T\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T\mathbf{x}}{\sum_{i=1}^n b_i}\right) = \xi\mathbf{z}^T\mathbf{z}, \quad (62)$$

because

$$\mathbf{x}^T\mathbf{x} = \left(\mathbf{z}+\theta\sqrt{\mathbf{b}}\right)^T\left(\mathbf{z}+\theta\sqrt{\mathbf{b}}\right) = \mathbf{z}^T\mathbf{z}+\theta^2\sqrt{\mathbf{b}}^T\sqrt{\mathbf{b}} = \mathbf{z}^T\mathbf{z}+\theta^2\sum_{i=1}^n b_i \quad (63)$$

and

$$\mathbf{x}^T\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T\mathbf{x} = \left(\mathbf{z}+\theta\sqrt{\mathbf{b}}\right)^T\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T\left(\mathbf{z}+\theta\sqrt{\mathbf{b}}\right) = \theta^2\sqrt{\mathbf{b}}^T\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T\sqrt{\mathbf{b}} = \theta^2\left(\sum_{i=1}^n b_i\right)^2, \quad (64)$$

due to $\mathbf{z}^T\sqrt{\mathbf{b}} = \sqrt{\mathbf{b}}^T\mathbf{z} = 0$. Therefore, the quadratic form of $\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}$ deduced by the adjustment matrix is as follows:

$$\mathbf{x}^T\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}-\xi\left[\mathbf{I}-\frac{\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T}{\sum_{i=1}^n b_i}\right]\right)\mathbf{x} = \mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\mathbf{x}-\xi\mathbf{z}^T\mathbf{z} \quad (65)$$

In the case of $\mathbf{x}\in\mathbb{X}_2$ ($\mathbf{x}\perp\sqrt{\mathbf{b}},\ \|\mathbf{x}\|=1$), $\mathbf{x}=\mathbf{z}+\theta\sqrt{\mathbf{b}}$, $\theta=0$ and $\|\mathbf{x}\|=\|\mathbf{z}\|=\mathbf{z}^T\mathbf{z}=1$. If $\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\right)\geq\xi$, $\min_{\mathbf{x}\in\mathbb{X}_2}.\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\mathbf{x}\geq\xi$. Therefore,

$$\mathbf{x}^T\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}-\xi\left[\mathbf{I}-\frac{\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T}{\sum_{i=1}^n b_i}\right]\right)\mathbf{x}\geq 0. \quad (66)$$

In the case that $\mathbf{x}$ is parallel to $\sqrt{\mathbf{b}}$ and $\|\mathbf{x}\|=1$, $\mathbf{z}=\mathbf{0}$, and

$$\mathbf{x}^T\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}-\xi\left[\mathbf{I}-\frac{\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T}{\sum_{i=1}^{n}b_i}\right]\right)\mathbf{x}$$
$$=\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\mathbf{x}\geq 0, \qquad (67)$$

because the minimum eigenvalue of $\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}$ is 0. Concludingly, if $\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\right)\geq\xi$, for any $\mathbf{x}\in R^n$,

$$\mathbf{x}^T\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}-\xi\left[\mathbf{I}-\frac{\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T}{\sum_{i=1}^{n}b_i}\right]\right)\mathbf{x}\geq 0. \qquad (68)$$

This indicate that $\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}-\xi\left[\mathbf{I}-\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T/\sum_{i=1}^{n}b_i\right]$ is positive semi-definite if $\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\right)\geq\xi$, that is,

$$\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}-\xi\left[\mathbf{I}-\frac{\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T}{\sum_{i=1}^{n}b_i}\right]\succeq 0, \qquad (69)$$

where $\succeq$ is the matrix inequality.

Next, the reverse is examined. If $\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}-\xi\left[\mathbf{I}-\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T/\sum_{i=1}^{n}b_i\right]$ is positive semi-definite, its quadratic form is non-negative, and, due to Eq. (65),

$$\mathbf{x}^T\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}-\xi\left[\mathbf{I}-\frac{\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T}{\sum_{i=1}^{n}b_i}\right]\right)\mathbf{x}$$
$$=\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\mathbf{x}-\xi\mathbf{z}^T\mathbf{z}\geq 0. \qquad (70)$$

In the case of $\mathbf{x}=\mathbf{z}+\theta\sqrt{\mathbf{b}}\in\mathbb{X}_2$ ($\mathbf{x}\perp\sqrt{\mathbf{b}}$, $\|\mathbf{x}\|=1$),

$$\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\mathbf{x}-\xi\geq 0, \qquad (71)$$

because $\|\mathbf{x}\|=\|\mathbf{z}\|=1$. This indicates

$$\min_{\mathbf{x}\in\mathbb{X}_2}\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\mathbf{x}\geq\xi. \qquad (72)$$

Therefore, $\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\right)\geq\xi$ if $\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}-\xi\left[\mathbf{I}-\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T/\sum_{i=1}^{n}b_i\right]$ is positive semi-definite.

Conclusionally, $\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}\right)\geq\xi$ and $\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}-\xi\left[\mathbf{I}-\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T/\sum_{i=1}^{n}b_i\right]\succeq 0$ are indentical. Therefore, Eq. (61) is re-formulated as the following semi-definite programming problem:

$$\max_{\mathbf{W},\xi}.\ \xi$$
$$s.t.\begin{cases}\mathbf{B}^{-\frac{1}{2}}\hat{\mathbf{L}}[\mathbf{W}]\,\mathbf{B}^{-\frac{1}{2}}-\xi\left[\mathbf{I}-\frac{\sqrt{\mathbf{b}}\sqrt{\mathbf{b}}^T}{\sum_{i=1}^{n}b_i}\right]\succeq 0\\ \mathbf{W}\in\mathbb{W}\end{cases} \qquad (73)$$

(5) Example

In this sub-section, we apply the above reinforcement optimization problem to the 4-node U-shape network in Fig. 1 and the large-scale ETRN (emergency transportation road network) in the Ishikawa and Toyama prefectures in Japan.

The Laplacian matrix of the 4-node U-shape network with $\mathbf{B}=\mathbf{I}$ is given in Eq. (34). The reinforcement optimization problem, whose budget is 1, for this network is expressed as

$$\max_{w_{12},w_{23},w_{34},\xi}.\ \xi$$
$$s.t.\begin{cases}\mathbf{E}\succeq 0\\ w_{12}+w_{23}+w_{34}\leq 1\\ 0\leq w_{12}\leq 1\\ 0\leq w_{23}\leq 1\\ 0\leq w_{34}\leq 1\end{cases}. \qquad (74)$$

$$\mathbf{E}=\begin{bmatrix}1+w_{12}-\frac{3}{4}\xi & -(1+w_{12})+\frac{1}{4}\xi & \frac{1}{4}\xi & \frac{1}{4}\xi\\ -(1+w_{12})+\frac{1}{4}\xi & 2+w_{12}+w_{23}-\frac{3}{4}\xi & -(1+w_{23})+\frac{1}{4}\xi & \frac{1}{4}\xi\\ \frac{1}{4}\xi & -(1+w_{23})+\frac{1}{4}\xi & 2+w_{23}+w_{34}-\frac{3}{4}\xi & -(1+w_{34})+\frac{1}{4}\xi\\ \frac{1}{4}\xi & \frac{1}{4}\xi & -(1+w_{34})+\frac{1}{4}\xi & 1+w_{34}-\frac{3}{4}\xi\end{bmatrix} \qquad (75)$$

The solution of the above problem is $w_{12}=0.2, w_{23}=0.6, w_{34}=0.2, \xi=0.8$. While the second minimum eigenvalue before the reinforcement is $2-\sqrt{2}=0.585\cdots$, it increases to 0.8 by $0.214\cdots$ after the above reinforcement. The reinforcement amount of link (2,3) that is located at the middle of the network is 0.6 and is the largest of all 3 links. From the standpoint of network connectivity, link (2,3) is the most important. On the other hand, those of links (1,2) and (3,4) are 0.2, respectively. These two links are located at the edge of the network, and their importance is less than link (2,3).

From Eq. (53), $\partial\lambda_2/\partial w_{ij}=\left(y_{2i}-y_{2j}\right)^2$ because $k=1$ in this 4-node U-shape network. Therefore, $\partial\lambda_2/\partial w_{12}=\partial\lambda_2/\partial w_{34}=6-4\sqrt{2}=0.343\cdots$, and $\partial\lambda_2/\partial w_{23}=12-8\sqrt{2}=0.686\cdots$. Thus, $\partial\lambda_2/\partial w_{23}$ is exactly twice of $\partial\lambda_2/\partial w_{12}$ and $\partial\lambda_2/\partial w_{34}$. According to this local improvement result, the reinforcement amounts of links (1,2), (2,3), and (3,4) are 0.25, 0.5, and 0.25, respectively, for the situation that the budget is 1. This is slightly different from the global improvement solution of Eq. (74), but they seem the same qualitatively. The local improvement is

The Ishikawa and Toyama ETRN network contains 1058 links and 653 nodes. In this case, each component of the adjacency matrix, namely, link strength, is given by an inverse of link's distance. The budget is set at 10 for a certain convenience. We used the SDPA-M package of MATLAB, and it took 25.20 seconds to solve the problem with a PC with Core-i7-6600U and 8GB RAM. Fig. 15 illustrates the optimum reinforcement results to Ishikawa-Toyama ETRN

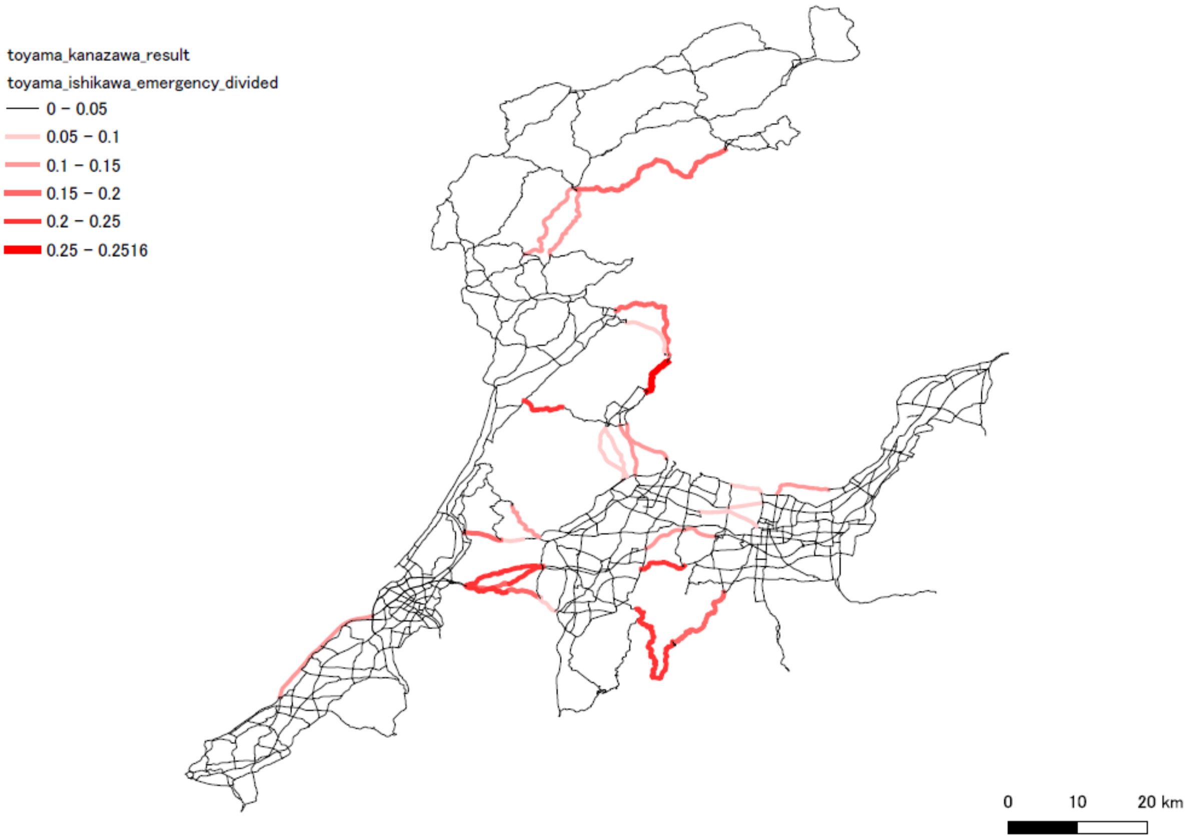

Fig. 15 The results of Ishikawa-Toyama ETRN

4. CONCLUSIONS

In this study, we propose an equal distribution problem for method to evaluate the network con-nectivity using the Laplacian matrix, and discuss why the second minimum eigenvalue of the Laplacian matrix of the network represents an indicator of network connectivity. Then, we develop an optimization problem to reinforce the road network connectivity using the second minimum eigen-value of its Laplacian matrix, and prove that it is formulated as a convex optimization problem. Furthermore, we show that the optimization problem is also formulated as a semi-definite programming, which is an extension of linear programming.

APPENDIX

We can confirm $\frac{d}{dt}\mathbf{x}(t) = \mathbf{B}^{-1}\mathbf{x}(t)$ as follows:

$$\frac{d}{dt}\mathbf{x}(t) = -\begin{bmatrix} \sum_{j=1}^{n} a_{1j} & -a_{12} & \cdots & -a_{1n} \\ -a_{12} & \sum_{j=1}^{n} a_{2j} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & \sum_{j=1}^{n} a_{nj} \end{bmatrix} \begin{bmatrix} \frac{x_1(t)}{b_1} \\ \frac{x_2(t)}{b_2} \\ \vdots \\ \frac{x_n(t)}{b_n} \end{bmatrix} = -\mathbf{L}\mathbf{B}^{-1}\mathbf{x}(t). \quad (a1)$$

Each component of $\mathbf{L}_s = \mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}$, $\mathbf{L}_s$ is ex-pressed as

$$\mathbf{L}_s \coloneqq \mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}} = \mathbf{B}^{-\frac{1}{2}} \begin{bmatrix} \sum_{j=1}^{n} a_{1j} & -a_{12} & \cdots & -a_{1n} \\ -a_{12} & \sum_{j=1}^{n} a_{2j} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & \sum_{j=1}^{n} a_{nj} \end{bmatrix}$$

$$\times\begin{bmatrix}\frac{1}{\sqrt{b_1}} & 0 & \cdots & 0\\ 0 & \frac{1}{\sqrt{b_2}} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{1}{\sqrt{b_n}}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{\sqrt{b_1}} & 0 & \cdots & 0\\ 0 & \frac{1}{\sqrt{b_2}} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{1}{\sqrt{b_n}}\end{bmatrix}\times$$

$$\begin{bmatrix}\frac{1}{\sqrt{b_1}}\sum_{j=1}^{n}a_{1j} & -\frac{a_{12}}{\sqrt{b_2}} & \cdots & -\frac{a_{1n}}{\sqrt{b_n}}\\ -\frac{a_{12}}{\sqrt{b_1}} & \frac{1}{\sqrt{b_2}}\sum_{j=1}^{n}a_{2j} & \cdots & -\frac{a_{2n}}{\sqrt{b_n}}\\ \vdots & \vdots & \ddots & \vdots\\ -\frac{a_{1n}}{\sqrt{b_1}} & -\frac{a_{2n}}{\sqrt{b_2}} & \cdots & \frac{1}{\sqrt{b_n}}\sum_{j=1}^{n}a_{nj}\end{bmatrix}$$

$$=\begin{bmatrix}\sum_{j=1}^{n}\frac{a_{1j}}{b_1} & -\frac{a_{12}}{\sqrt{b_1b_2}} & \cdots & -\frac{a_{1n}}{\sqrt{b_1b_n}}\\ -\frac{a_{12}}{\sqrt{b_1b_2}} & \sum_{j=1}^{n}\frac{a_{2j}}{b_2} & \cdots & -\frac{a_{2n}}{\sqrt{b_2b_n}}\\ \vdots & \vdots & \ddots & \vdots\\ -\frac{a_{1n}}{\sqrt{b_1b_n}} & -\frac{a_{2n}}{\sqrt{b_2b_n}} & \cdots & \sum_{j=1}^{n}\frac{a_{nj}}{b_n}\end{bmatrix}\qquad(a2)$$

The row vector for the ith row of $\mathbf{L}_s$ is denoted by $[\mathbf{L}_s]_{i\cdot}$. Let denote

$$[\mathbf{L}_s]_{i\cdot}\sqrt{\mathbf{b}} = -\frac{a_{1i}}{\sqrt{b_i}} - \cdots - \frac{a_{i-1i}}{\sqrt{b_i}} + \frac{1}{\sqrt{b_i}}\sum_{j=1}^{n}a_{ij} - \frac{a_{i-1i}}{\sqrt{b_i}} - \cdots = 0 \qquad (a3)$$

where

$$\sqrt{\mathbf{b}} = \begin{bmatrix}\sqrt{b_1}\\ \sqrt{b_2}\\ \vdots\\ \sqrt{b_n}\end{bmatrix}. \qquad (a4)$$

The above is applied for each row of $\mathbf{L}_s$. Therefore,

$$\mathbf{L}_s\sqrt{\mathbf{b}} = \mathbf{0}, \qquad (a5)$$

where $\mathbf{0}$ is the null vector whose components are all 0. Thus, 0 is an eigenvalue of $\mathbf{L}_s$ and $\sqrt{\mathbf{b}} = \mathbf{b}^{\frac{1}{2}}$ is its eigenvector.

To derive the quadratic form of $\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}$, the following equation is examined:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\left(\frac{x_i}{\sqrt{b_i}}-\frac{x_j}{\sqrt{b_j}}\right)^2$$
$$=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\left(\frac{{x_i}^2}{b_i}-2\frac{x_ix_j}{\sqrt{b_i}\sqrt{b_j}}+\frac{{x_j}^2}{b_j}\right)$$
$$=2\sum_{i=1}^{n}\frac{{x_i}^2}{b_i}\sum_{j=1}^{n}a_{ij}-2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{x_ix_j}{\sqrt{b_i}\sqrt{b_j}}. \qquad (a6)$$

From Eq. $(a1)$,

$$\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{x}$$
$$=\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\begin{bmatrix}\frac{x_1}{\sqrt{b_1}}\sum_{j=1}^{n}a_{1j}-\frac{x_2}{\sqrt{b_2}}a_{12}-\cdots-\frac{x_n}{\sqrt{b_n}}a_{1n}\\ -\frac{x_1}{\sqrt{b_1}}a_{12}+\frac{x_2}{\sqrt{b_2}}\sum_{j=1}^{n}a_{2j}-\cdots-\frac{x_n}{\sqrt{b_n}}a_{2n}\\ \vdots\\ -\frac{x_1}{\sqrt{b_1}}a_{1n}-\frac{x_2}{\sqrt{b_2}}a_{2n}-\cdots+\frac{x_n}{\sqrt{b_n}}\sum_{j=1}^{n}a_{nj}\end{bmatrix}$$
$$=\mathbf{x}^T\begin{bmatrix}\frac{x_1}{b_1}\sum_{j=1}^{n}a_{1j}-\sum_{j=1}^{n}\frac{x_j}{\sqrt{b_1}\sqrt{b_j}}a_{1j}\\ \frac{x_2}{b_2}\sum_{j=1}^{n}a_{2j}-\sum_{j=1}^{n}\frac{x_j}{\sqrt{b_2}\sqrt{b_j}}a_{2j}\\ \vdots\\ \frac{x_n}{b_n}\sum_{j=1}^{n}a_{nj}-\sum_{j=1}^{n}\frac{x_j}{\sqrt{b_n}\sqrt{b_j}}a_{nj}\end{bmatrix}$$
$$=\sum_{i=1}^{n}\frac{{x_i}^2}{b_i}\sum_{j=1}^{n}a_{ij}-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{x_ix_j}{\sqrt{b_i}\sqrt{b_j}} \qquad (a7)$$

From Eqs. $(b1)$ And $(b2)$, we obtain the quadratic form of $\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}$ as follows:

$$\mathbf{x}^T\mathbf{L}_s\mathbf{x}=\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{x}$$
$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\left(\frac{x_i}{\sqrt{b_i}}-\frac{x_j}{\sqrt{b_j}}\right)^2. \qquad (a8)$$

As the above equation indicates, the quadratic form of $\mathbf{L}_s=\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}$ is non-negative. Therefore, $\mathbf{L}_s=\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}$ is positive semi-definite. A matrix is positive semi-definite if and only if all eigenvalues are non-negative (e.g. Strang, 1976), and all eigenvalues of $\mathbf{L}_s$ are non-negative. Eq. $(a5)$ shows that 0 is an eigenvalue of $\mathbf{L}_s=\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}$, and this is the minimum of all. Thus, we confirm $0=\lambda_1\le\lambda_2\le\cdots\le\lambda_n$ as eigenvalues of $\mathbf{L}_s=\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}$.

The Rayleigh quotient is given by $\mathbf{x}^T\mathbf{A}\mathbf{x}/\mathbf{x}^T\mathbf{x}$ (e.g. Strang, 1976), and minimizing the Rayleigh quotient gives the minimum eigenvector; $\min.\mathbf{x}^T\mathbf{A}\mathbf{x}/\mathbf{x}^T\mathbf{x}$ subject to $\mathbf{x}^T\mathbf{x}=1$, for a real symmetric matrix $\mathbf{A}$. The second minimum eigenvector is given by the following minimization problem:

$$\min_{\mathbf{x}\in R^n}.\mathbf{x}^T\mathbf{A}\mathbf{x} \qquad s.t.\ \ \mathbf{x}\perp\mathbf{y}_1,\ \ \|\mathbf{x}\|=1 \qquad (a9)$$

where $\|\mathbf{x}\|=\sqrt{\mathbf{x}^T\mathbf{x}}$, $\mathbf{y}_1$ is the minimum eigenvector, and $\perp$ is orthogonal. In the problem of equation $(a9)$, $\mathbf{x}\perp\mathbf{y}_1$ is imposed, and it yields the second minimum eigenvector. The minimum eigenvector of $\mathbf{L}_s=\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}$ is $\sqrt{\mathbf{b}}$, as stated in Eq. $(a5)$, and

$$\lambda_2(\mathbf{L}_s)=\lambda_2\left(\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\right)=\min_{\mathbf{x}\in\mathbb{X}_2}.\mathbf{x}^T\mathbf{B}^{-\frac{1}{2}}\mathbf{L}\mathbf{B}^{-\frac{1}{2}}\mathbf{x}\ (a10)$$

where $\lambda_2(\mathbf{L}_s)$ is the second minimum eigenvalue of $\mathbf{L}_s$ and $\mathbb{X}_2=\left\{\forall\mathbf{x}\mid\mathbf{x}\perp\sqrt{\mathbf{b}},\ \|\mathbf{x}\|=1\right\}$.

ACKNOWLEDGMENT: This work was partly supported by JSPS KAKENHI Grant Number 17H03321.

REFERENCES

1) AkbarZadeh, M., Salehi Reihami, SF., Aghababaei Samani, K. (2019) Detecting Critical Links of Urban Networks Using Cluster Detection Methods, Physica A, Vol. 515, pp. 288–298.
2) Bell, M.G.H. and Iida, Y. (1997) Transportation Network Analysis, Wiley, New York.
3) Bell, M.G.H., Kurauchi, F., Perera, S. and Wong, W. (2017) Investigating transport network vulnerability by capacity weighted spectral analysis, Transportation Research, Part B, Vol. 99, pp. 251-266.
4) Demšar, U., O. Špatenková, K. and Virrantaus, K. (2008) Identifying Critical Locations in a Spatial Network with Graph Theory, Transactions in GIS, Vol. 12, pp. 64–82.
5) Fiedler, M. (1973) Algebraic connectivity of graphs, Czechoslovak Mathematical Journal, Vol. 23, pp. 298-305.
6) Grötschel, M., Monma, C. L. and Stoer, M. (1995) Design of Survivable Networks, Handbook in Operation Research and Management Science, Vol. 7, 1995, pp. 617–672.
7) Kurauchi, F., N. Uno, Sumalee, A. and Y. Seto. (2009) Network evaluation based on connectivity vulnerability, Transportation and Traffic Theory 2009: golden Jubilee, pp. 637–649.
8) Nouzard, S.H.H., and Pradhan, A. (2016) Vulnerability of Infrastructure Systems: Macroscopic Analysis of Critical Disruptions on Road Networks, Journal of Infrastructure Systems, Vol. 22, No. 04015014.
9) von Luxburg, U.: A tutorial on spectral clustering, Statistics and Computing, Vol. 17, pp. 395-416, 2007.
10) Strang, G. (1976) Linear Algebra and its Applications, Academic Press, New York.

(Received March 7, 2021)