Link-based Approach for Semi-dynamic Stochastic User Equilibrium Traffic Assignment Model with Sensitivity Analysis

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In semi-dynamic traffic assignment model, if the residual flow for flow propagation is eliminated, not only the travel cost changes but also the inflow changes via network equilibrium. On the other hand, the residual flow, which is the function of the travel time on its link, is added on demand between the end node of that link and original destination in the next period. For solving this so-called bilevel programming problems, sensitivity analysis is chosen as methodology research. The semi-dynamic traffic assignment model with flow propagation based on sensitivity analysis was proposed by Nakayama, S. (2010), Pham Thi Thu Ha et al. (2012) and Itagaki et al. (2014). Applying the proposed method on very large-scale networks is in difficulty, however, it is because of path enumeration in the sensitivity analysis formulations. In fact, the extremely large alternative paths connecting an O-D pair in a real network can cause computer overload. In this study, thus, we will upgrade this method to solve this problem to increase the applicability of the semi-dynamic traffic assignment model with Stochastic User Equilibrium to reality. This research will apply, therefore, link-based procedures which perform only link and node variables.

Key Words: *Stochastic User Equilibrium, semi-dynamic traffic assignment, sensitivity analysis, link-based approach , Dial's algorithm*

1. INTRODUCTIONS

Route assignment, or traffic assignment, the last step of four-step urban transportation planning, provides important information to design of future junctions, propose traffic policies to minimize traffic congestion problems and etc. There are many types of traffic assignment model, including static and dynamic traffic assignment. All of these models aim to represent real transportation networks. In most real networks, because of substantial changes in daily traffic conditions, transportation network analysis is not fully performed by a static traffic assignment model. Static traffic assignment model exhibits only one-time period which cannot depict true network. In contrast, dynamic traffic assignment model shows many time periods in one link, hence, it may exact network flows. In return, nevertheless, it is too complex and requires a heavy computational load. Furthermore, most of these models do not have a unique solution (Iryo 2011). As an effective combination, the semi-dynamic model with flow propagation, based on Fujita et al. (1988, 1989), Miyagi and Makimura (1991) and Akamatsu et al. (1998), is used. In this approach, static network equilibrium is reached in each relatively long period (from about 15 minutes to 90 minutes), thus, most travellers can arrive at their destinations within the period in which they depart. Accordingly, the technique and algorithm of static traffic equilibrium can be utilized in the semi-dynamic model. In addition, the demand modification approach is used with flow propagation which cannot exit the link in a period is propagated to the next period. As a result, real transportation network can be sufficiently expressed in this model. It is, with small computational capacity require, an effective possible choice for illustrating time-varying dynamic network.

In addition, in our study, the selected methodology research is sensitivity analysis for solving so-called bilevel programming problems in semi-dynamic stochastic traffic assignment model. The reason for this is that if the residual flow for flow propagation is eliminated from the present period, not only the travel cost changes but also the inflow changes via network equilibrium. Furthermore, the residual flow, which is the function of the travel time on its link, become the travel demand from the end node of that link to original destination in the next period.

Nakayama, S. (2010) proposed a semi-dynamic traffic assignment model with flow propagation based on sensitivity analysis, Pham Thi Thu Ha et al. (2012) conducted an application of semi-dynamic traffic assignment models to Kanazawa road network and Itagaki et al. (2014) expanded it by considering sensitivity analysis for residual flow. However, the sensitivity analysis formulations based on route approach that could lead to difficulty when it is applied in practice. In fact, the extremely large alternative paths connecting an O-D pair in a real network can cause computer overload. When applying semi-dynamic traffic assignment model with sensitivity analysis, we were faced with the challenge of balancing storage costs and computational time. In this study, thus, we will upgrade this method to solve this problem to increase the applicability of the semi-dynamic traffic assignment model with Stochastic User Equilibrium to reality. As a result, link-based approach which performs only link and node variables for semi-dynamic stochastic user equilibrium traffic assignment model with sensitivity analysis will be proposed in our research.

2. STOCHASTIC USER EQUILIBRIUM WITH LOGIT-BASED TRAFFIC ASSIGNMENT

Within the core of a logit-based model (Sheffi, 1985), static equilibrium is formulated in each time period. A logit-based route choice model can be seen as following

$$f_{h,k} = q_h p_{h,k} = q_h \frac{\exp(-\theta c_{h,k})}{\sum_{k \epsilon K_h} \exp(-\theta c_{h,k})} \quad (1)$$

In this function, while $f_{h,k}$ denotes the flow on the k-th route, $p_{h,k}$ is the probability of choosing the k-th route. q_h is the travel demand and $c_{h,k}$ is the travel cost on the k-th route. Set of routes and positive parameter are denoted by K_h and θ, respectively. h stand for h-th OD pair between r-th node and s-th node and H is set of OD pairs ($h \in H$).

The link flow is given as

$$x_i = \sum_{h \in H} \sum_{k \in K_h} f_{h,k} \delta^h_{i,k} \quad (2)$$

with x_i is the i-th link flow, A is set of links ($i \in A$), $\delta^h_{i,k}$ is the link-route incidence variable. $\delta^h_{i,k}$= 1 if the k-th route between the r-th and s-th nodes includes the i-th link; otherwise, $\delta^h_{i,k}$= 0.

The SUE is worked out by resolving the following minimization function (showed by Daganzo 1982),

$$\mathrm{Z}(\mathbf{t}, \boldsymbol{\epsilon}) = \sum_{i \in A} \int_{t_i(0,\epsilon_i)}^{t_i} x_i\,(w, \epsilon_i) \mathrm{dw} - \sum_{h \in H} q_h S_h(c^h\,(t)) \quad (3)$$

Note that, S_h denotes the expected perceived travel time of a traveler from origin r to destination s. We have

$$S_h\left(c^h(t)\right) = -\frac{1}{\theta} ln \sum_{k \in K_h} \exp(-\theta c_{h,k}) \quad (4)$$

where θ is a positive parameter that depicts the perceived travel time. If θ is large, the perception error is small and the probability of choosing the shortest path of a driver is high (Sheffi, 1985)

And, the derivative of Z with respect to t_i is given by

$$\frac{\partial \mathbf{Z}}{\partial t_i} = x_i(t_i, \epsilon_i) - \sum_{h \in H} \left(q_h \frac{\sum_{k \epsilon K_h} \exp(-\theta c_{h,k}) \delta^h_{i,k}}{\sum_{p \epsilon K_h} \exp(-\theta c_{h,p})} \right) \quad (5)$$

where ϵ_i is a given parameter, t_i is link travel time (cost) of i-th link and t_i is defined as implicit function in free variable ϵ_i. This function equals zero at SUE state that can be achieved by using the method of successive average (MSA method) with Dial's algorithm (Sheffi 1985). The procedure does not assign choice probabilities (and flows) to all routes connecting each OD pair. Instead, many of these routes are supposed unreasonable or ineffective in practice. Consequently, an reasonable or effective route including only links that take the traveler further away from the origin is considered when Dial'salgorithm is put to use in our research.

3. SEMI-DYNAMIC STOCHASTIC USER EQUILIBRIUM TRAFFIC ASSIGNMENT MODEL WITH SENSITIVITY ANALYSIS IN LINK-BASED APPROACH

(1) Semi-dynamic stochastic user equilibrium traffic assignment model

In this study, the framework of a logit-based model (Sheffi 1985) is considered to calculate the static equilibrium in each period of time and to formulate a semi-dynamic traffic assignment model.

A logit-based traffic assignment is assumed as

$$f_{\tau,h,k} = q_{\tau,h}p_{\tau,h,k} = q_{\tau,h}\frac{\exp\left(-\theta c^{\tau}_{h,k}\right)}{\sum_{k\epsilon K_h}\exp\left(-\theta c^{\tau}_{h,k}\right)} \quad (6)$$

The ingredients of this equation are denoted like Eq. (1) and τ is τ-th period.

The above equation can be expressed in vector form as follows

$$\mathbf{f}_{\tau} = \mathbf{Q}_{\tau}\mathbf{p}_{\tau} \quad (7)$$

with $\mathbf{f}_{\tau}$ $(=f_{\tau,1,1},\dots, f_{\tau,2,1},\dots)^T$ is the vector of all route flows, $\mathbf{Q}_{\tau}$ is the diagonal matrix of travel demands, $\mathbf{p}_{\tau}$ $(=p_{\tau,1,1},\dots,p_{\tau,2,1},\dots)^T$ is the vector of all route probabilities and ^{T} is the transpose. The formulation of $\mathbf{Q}_{\tau}$ can be expanded as below diagonal matrices

$$\mathbf{Q}_{\tau,h} = \begin{pmatrix} q_{\tau,h} & & 0 \\ & \ddots & \\ 0 & & q_{\tau,h} \end{pmatrix} \quad (8)$$

$$\mathbf{Q}_{\tau} = \begin{pmatrix} \ddots & & 0 \\ & \mathbf{Q}_{\tau,h} & \\ 0 & & \ddots \end{pmatrix} \quad (9)$$

And, the link travel flow vector is obtained

$$\mathbf{x}_{\tau} = \Delta\mathbf{f}_{\tau} \quad (10)$$

where $\mathbf{x}_{\tau}$ $(= x_{\tau,1}, x_{\tau,2},\dots, x_{\tau,|A|})$ is the vector of all link traffic flows, Δ $(=\{\delta^h_{i,k}\})$ is the link-route incidence matrix.

The route travel cost function is showed as

$$\mathbf{c}(\mathbf{f}_{\tau}) = \Delta^T\mathbf{t}(\Delta\mathbf{f}_{\tau}) \quad (11)$$

where $\mathbf{c}(\mathbf{f}_{\tau})$ is the vector-valued function of route travel cost, and $\mathbf{t}(\Delta\mathbf{f}_{\tau})$ is the vector-valued function of link travel cost. Because the travel cost is a function of its inflow and the probability of route choice is a function of its travel cost, we have

$$\mathbf{f}_{\tau} = \mathbf{Q}_{\tau}\mathbf{p}(\mathbf{c}(\mathbf{f}_{\tau})) \quad (12)$$

In semi-dynamic traffic assignment model, not only is the static equilibrium reached in each period but flow propagation also is considered. Some of the travellers cannot exit the link and are considered as residual flow of that link which is propagated to the next period. And, the present period experiences the remove of residual flow on subsequent links.

In this model, all travellers departing from their origin do not reach their destinations. The residual flow on a link is added on demand between the end node of that link and original destination in the next period. In the present period, this residual flow on subsequent links should be eliminated.

A semi-dynamic traffic assignment model is shown in following Figure

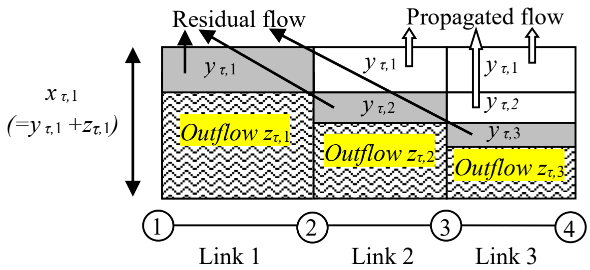


Fig. 1 Semi-dynamic traffic assignment model

Used notations are as follows: $x_{\tau,i}$ denotes the travel flow; $t_{\tau,i}$ $(x_{\tau,i})$ denotes the link travel time; $y_{\tau,i}$ denotes the residual flow; and $z_{\tau,i}$ denotes the link flow after the residual flows are eliminated. All of these are on the i-th link in the τ-th period. Time is divided into the segment of length L. The accuracy of OD and other data decided the length of a segment time period. The more detailed and accurate data is, the shorter the length of the time period is chosen. Therefore, for practical applications, the period length may be from 15 min to 90 min in many cases. Thus, we can determine the length in a much shorter period if the OD demand data are accurate and dynamically detailed. In the given figure, some of the inflow into link 1 cannot exit this link and becomes residual flow, which is propagated to the next period. So, this does not travel on the subsequent links, link 2 and link 3 in the present period. Thus, the travel time on link 2 should be $t_2(x_{\tau,2}) = t_{\tau,2}(x_{\tau,1} - y_{\tau,1})$, and that on link 3 should be $t_3(x_{\tau,3}) = t_{\tau,3}(x_{\tau,1} - y_{\tau,1} - y_{\tau,2})$.

The next considered point is how to compute the residual flows. For simplicity, assumptions are as follows the travellers constantly infiltrate a link (at the constant rate of $x_{\tau,i}/L$); Travel cost is the function of its inflow; Travel cost does not change within each time period; Residual flow is the function of travel time on its link; Residual flow on a link is added to demand between the end node of that link and the original destination in the next period and it may change its original route. The residual flow on a link is determined by the inflow and link travel time, which is a function of its inflow. If the residual flow still travels on the link at the end of the period, it can be simply calculated by the product of the inflow rate and travel time. The residual flow on i-th link in the τ-th period of each h-th OD pair is determined by

$$y_{\tau,k,i}^{h} = \sum_{k\epsilon K_h} \frac{f_{\tau,h,k} t_{\tau,i} \delta_{i,k}^{h}}{L} \quad (13)$$

The residual flow on links in the τ-th period can be written as vector form

$$\mathbf{y}_\tau = \begin{pmatrix} y_{\tau,1} \\ \vdots \\ y_{\tau,|A|} \end{pmatrix} = \frac{1}{L}\mathbf{T}(\Delta\mathbf{f}_\tau)\Delta\mathbf{f}_\tau \quad (14)$$

where $\mathbf{T}$ is diagonal matrix of link travel time

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} t_1(x_1) & & 0 \\ & \ddots & \\ 0 & & t_{|A|}\left(x_{|A|}\right) \end{pmatrix} \quad (15)$$

If we assume that $n_{h,k}$ is the number of links in the k-th route between the r-th and s-th nodes of h-th OD pair. The summary of the residual flow in the k-th route can be written as

$$s_{\tau,k}^{h} = \sum_{m=1}^{n_{h,k}} y_{\tau,k,m}^{h} \quad (16)$$

We consider $\mathbf{B}$ denotes the $|A| \times |A|$ matrix containing the information as if the i-th link is a downstream link of j-th link or i=j, then $b_{i,j} = 1$; otherwise, $b_{i,j} = 0$.

$$\mathbf{B} = \begin{pmatrix} b_{1,1} & & b_{1,|A|} \\ & \ddots & \\ b_{|A|,1} & & b_{|A|,|A|} \end{pmatrix} \quad (17)$$

As a result, we attain the summary of residual flow as

$$\mathbf{s}_\tau = \begin{pmatrix} s_{\tau,1} \\ \vdots \\ s_{\tau,|A|} \end{pmatrix} = \mathbf{B}^T \mathbf{y}_\tau \quad (18)$$

According Eq. (14) we have

$$\mathbf{s}_\tau = \frac{1}{L}\mathbf{B}^T \ \mathbf{T}(\Delta\mathbf{f}_\tau)\Delta\mathbf{f}_\tau \quad (19)$$

The link flow after the elimination of residual flows $\mathbf{z}_\tau$ is given by following

$$\mathbf{z}_\tau = \Delta\mathbf{f}_\tau - \mathbf{s}_\tau \quad (20)$$

The route flow f at t time period is showed as

$$\mathbf{f}_\tau = \mathbf{Q}_\tau \mathbf{p}(\Delta^T \mathbf{t}(\Delta\mathbf{f}_\tau - \mathbf{s}_\tau)) \quad (21)$$

The issue to consider is that if the residual flow is eliminated from the link, not only the link travel time changes but also the inflow changes via network equilibrium. To solve this problem, sensitivity analysis is applied and residual flow is considered as a perturbation parameter. By using sensitivity analysis, the ratios change of link travel flow respect to perturbation of residual flow are considered, *i.e.* the corresponding partial derivatives at SUE state need to be calculated. Pham Thi Thu Ha et al. (2012), Itagaki, Nakayama et al. (2014) solved this by route-based approach. However, if residual flow is eliminated and propagated to the next period, the number of OD pairs will increase dramatically and therefore the number of routes will escalate accordingly and computer may be overloaded. To update this method, hence, we will propose a link-based approach with Dial's algorithm.

(2) Computing derivatives of link traffic flow with respect to residual flow and approximate link traffic flow

The proposed method with sensitivity analysis includes the following four steps.

Step 1: Calculate a static equilibrium in a given time period.

Step 2: Design a sensitivity analysis method and obtain flow propagation to the next time period.

Step 3: Recalculate a static equilibrium assignment by subtracting the propagated flow from the static equilibrium assignment.

Step 4: Deliver the propagated flow to the next time period.

In each time period, Pham Thi Thu Ha et al. (2012) and Itagaki et al. (2014) proposed approximate route traffic flow from static equilibrium as follow

$$\mathbf{f}_\tau = \mathbf{f}_0 + \frac{1}{L}\left[(\mathbf{I} - \mathbf{Q}\nabla_c \mathbf{p}_0 \nabla_x \mathbf{t}_0 \Delta)^{-1} \mathbf{Q}\nabla_c \mathbf{p}_0 \nabla_x \mathbf{t}_0\right]\mathbf{T}(\Delta\mathbf{f}_0)\mathbf{R}^T\mathbf{f}_0 \quad (22)$$

The ingredients of this equation are denoted like above equations with τ is τ-th period, 0 is static equilibrium state and ∇ is the gradient. A reader is referred to Itagaki et al. (2004) for more detail.

The approximation in Eq. (22) can be said to be obtained from the zero order approximation of the residual traffic flow, while Itagaki et al. (2004) introduced the one approximating the residual traffic volume to the first order. A more detailed approximation can be considered, but the computation becomes even more complicated. When applying to Kanazawa city network, Itagaki et al. (2014) also only use Eq. (22). In consequence of existing inverse matrix in these formulations, nevertheless, applying the route-based approach to very large-scale networks is arduous. From the viewpoint of practical use, we use the approximation of Eq. (22) in this study to propose link-based approach.

We start from the equation of logit-based traffic assignment, we have

$$x_{\tau,i} = \sum_{h\in H}\left(q_h \frac{\sum_{k\in K_h}\exp\left(-\theta c^{\tau}_{h,k}\right)\delta^h_{i,k}}{\sum_{p\epsilon K_h}\exp\left(-\theta c^{\tau}_{h,k}\right)}\right) \quad (23)$$

Let $u_{\tau,i}$ denotes the right-hand side of equation (23), $u_{\tau,i}$ is a function of $t_{\tau,i}$ and $t_{\tau,i}$ is a function of $(x_{\tau,i}, s_{\tau,i})$. For simplicity, hereafter, we donot consider τ in following equations.

By using this equation, we can define the following function with vector form

$$\mathbf{d}(\mathbf{x},\mathbf{s}) = \mathbf{x} - \mathbf{u}(\mathbf{t}(\mathbf{x},\mathbf{s})) \quad (24)$$

This equation is assumed as functions with arguments **x** and perturbation residual flow **s**.

The gap at both sides of Eq. (24) should be zero; that is, **d(x,s)** = **0**, under network equilibrium state.

From the general formula for derivative of implicit function. It is clear that

$$\nabla_s\mathbf{x} = -\nabla_x\mathbf{d}^{-1}\nabla_s\mathbf{d} \quad (25)$$

By the chain rule of differentiation, we have $\nabla_x\mathbf{u} = \nabla_t\mathbf{u}\nabla_x\mathbf{t}$ and $\nabla_s\mathbf{u} = \nabla_t\mathbf{u}\nabla_s\mathbf{t}$

Therefore, we obtain the following equation with **I** as a unit matrix

$$\nabla_x\mathbf{d} = \mathbf{I} - \nabla_t\mathbf{u}\nabla_x\mathbf{t} \quad (26)$$

And,

$$\nabla_s\mathbf{d} = -\nabla_t\mathbf{u}\nabla_s\mathbf{t} \quad (27)$$

where:

$\nabla_x\mathbf{t}$ and $\nabla_s\mathbf{t}$ is the diagonal matrix of apparent partial derivatives of t_i as an explicit function with respect to x_i and s_i, respectively.

$$\nabla_x\mathbf{t} = \begin{pmatrix} \frac{dt_1}{dx_1} & & 0 \\ & \ddots & \\ 0 & & \frac{dt_{|A|}}{dx_{|A|}} \end{pmatrix} \quad (28)$$

$$\nabla_s\mathbf{t} = \begin{pmatrix} \frac{dt_1}{ds_1} & & 0 \\ & \ddots & \\ 0 & & \frac{dt_{|A|}}{ds_{|A|}} \end{pmatrix} \quad (29)$$

For example, if t_i is assumed by using a conventional traffic model based on BPR (Bureau of Public Roads) curves. The travel time at each link with perturbation residual flow s_i can be seen as

$$t_i = t_{0,i}\left[1 + \alpha\left(\frac{x_i - s_i}{Capa_i}\right)^{\beta}\right] \quad (30)$$

where t_i is the travel time of the *i*-th link, $t_{0,i}$ is the free-flow travel time, x_i is the link travel flow, α, β are parameters and $Capa_i$ is the capacity of *i*-th the link.

We have apparent partial derivatives of t_i as an explicit function with respect to x_i and s_i

$$\left.\frac{dt_i}{dx_i}\right|_{s_i=0} = \frac{\beta\alpha {x_i}^{\beta-1}}{{Capa_i}^{\beta}} \quad (31)$$

$$\left.\frac{dt_i}{ds_i}\right|_{s_i=0} = -\frac{\beta\alpha {x_i}^{\beta-1}}{{Capa_i}^{\beta}} \quad (32)$$

Only difficult in here is that the calculation of equation $\nabla_t\mathbf{u}$. We have

$$\nabla_t\mathbf{u} = \begin{pmatrix} \frac{\partial u_i}{\partial t_i} & \cdots & \frac{\partial u_i}{\partial t_{|A|}} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_{|A|}}{\partial t_i} & \cdots & \frac{\partial u_{|A|}}{\partial t_{|A|}} \end{pmatrix} \quad (33)$$

It is clear that

$$\frac{\partial u_i}{\partial t_g} = \sum_{h\in H}\left(q_h\left[\frac{-\theta\sum_{k\epsilon K_h}\exp\left(-\theta c_{h,k}\right)\delta^h_{i,k}\delta^h_{g,k}}{\sum_{p\epsilon K_h}\exp\left(-\theta c_{h,p}\right)} - \frac{-\theta\left(\sum_{k\epsilon K_h}\exp(-\theta c_{h,k})\delta^h_{i,k}\right)\left(\sum_{l\epsilon K_h}\exp(-\theta c_{h,l})\delta^h_{g,l}\right)}{\left(\sum_{p\epsilon K_h}\exp(-\theta c_{h,p})\right)^2}\right]\right) \quad (34)$$

Let us expand and explain how it can be efficiently computed by link based approach with Dial's traffic assignment algorithm.

Note that, in the link-based approach, the link flow is total of all link flows of all OD pairs (J.Q.Ying and T.Miyagi 2001)

$$x_i = \sum_{h\in H} x_i^h \tag{35}$$

With $f_{h,k}$ is the flow on the *k-th* route, we have

$$\frac{f_{h,k}}{q_h} = \frac{\exp(-\theta c_{h,k})}{\sum_{k\epsilon K_h}\exp(-\theta c_{h,k})} \tag{36}$$

And,

$$x_i^h = \sum_{k\in K_h} f_{h,k}\,\delta_{i,k}^h \tag{37}$$

From these, we have the proportion of travellers from *r-th* node to *s-th* node who use *i*-th link as follow

$$\frac{x_i^h}{q_h} = \frac{\sum_{k\epsilon K_h}\exp(-\theta c_{h,k})\delta_{i,k}^h}{\sum_{p\epsilon K_h}\exp(-\theta c_{h,p})} \tag{38}$$

And, at SUE state, when the travel time of all links are fixed, the number of travellers from *r-th* node to *s-th* node who use *i*-th link can be calculated by using Dial's algorithm. When x_i^h is easily computed, we can easily attain the right-hand side of equation (38).

After that, with link-based approach, we need to compute

$$\frac{\sum_{k\in K_h}\exp(-\theta c_{h,k})\delta_{i,k}^h\delta_{g,k}^h}{\sum_{p\epsilon K_h}\exp(-\theta c_{h,p})} \tag{39}$$

We suppose that x_{i-g}^h is the number of travellers from *r* to *s* who choose some paths which contain both *i*-th and *g*-th links in such a way that *i*-th link is used prior to *g*-th, the number of travellers from *r* to *s* who use link *g*-th link prior to *i*-th link is denoted by x_{g-i}^h. And $x_{i,g}^h, x_{g,i}^h$ denote the number of travellers who use both *i*-th and *g*-th links without consideration of priority.

It is clear that, the proportion of travellers from *r*-th node to *s*-th node who use both *i*-th and *g*-th links is depicted by

$$\frac{x_{i,g}^h}{q_h} = \frac{\sum_{k\epsilon K_h}\exp(-\theta c_{h,k})\delta_{i,k}^h\delta_{g,k}^h}{\sum_{p\epsilon K_h}\exp(-\theta c_{h,p})} \tag{40}$$

In real large networks, if the number of OD pairs is too large, storage price will be very large for saving both x_{i-g}^h and $x_{i,g}^h$. Furthermore, the fact that either x_{i-g}^h or x_{g-i}^h is zero implies that $x_{i,g}^h = \max\{x_{i-g}^h, x_{g-i}^h\}$ (J.Q.Ying and T.Miyagi 2001). So, we do not need to store $x_{i,g}^h$ and $x_{i,g}^h$ can be queried through x_{i-g}^h

x_{i-g}^h can be computed by the same way as calculating x_i^h by running Dial's algorithm once with assigning x_i^h from origin *j* to destination *s* on each the *g*-th link. There is a small change when using Dial's Algorithm. We do not compute *"link likelihood"* and *"link weight"* (Dial 1971) of all links of network but only for links containing non-zero values of x_i^h. And, we use the same way of storing x_i^h for saving x_{i-g}^h

Accordingly, calculation equation (34) will become calculate below equation.

$$\frac{\partial u_i}{\partial t_g} = \sum_{h\in H}\theta\left[\frac{x_i^h x_g^h}{q_h} - x_{i,g}^h\right] \tag{41}$$

As mentioned above, $x_{i,g}^h$ can be queried through x_{i-g}^h. We will compute this equation for each *h*-th OD pair. We first browse the segment that contains non-zero values of x_i^h, x_g^h. Secondly, we need to query the segment that contains non-zero values of x_{i-g}^h. After all, we will work out the value of this equation in one of the following three cases

$$\frac{\partial u_i}{\partial t_g} = \begin{cases} \sum_{h\in H}\theta\left[\frac{x_i^h x_g^h}{q_h} - x_{i-g}^h\right] \text{ or } \left(\sum_{h\in H}\theta\left[\frac{x_i^h x_g^h}{q_h} - x_{g-i}^h\right]\right) \\ \quad (\,if\ i \neq g\ and\ x_{i-g}^h (or\ x_{g-i}^h) \neq 0) \\ \sum_{h\in H}\theta\left[\frac{x_i^h x_g^h}{q_h}\right] \\ \quad (if\ i \neq g\ and\ (x_{i-g}^h\ \ and\ x_{g-i}^h = 0)) \\ \sum_{h\in H}\theta\left[\frac{x_i^h x_g^h}{q_h} - x_i^h\right] \quad (if\ i = g) \end{cases} \tag{42}$$

After $\nabla_t \mathbf{u}$ is calculated, Eq. (25) become

$$\nabla_s \mathbf{x} = -(\mathbf{I} - \nabla_t \mathbf{u}\nabla_x \mathbf{t})^{-1}\nabla_t \mathbf{u}\nabla_x \mathbf{t} \tag{43}$$

After the derivatives of link traffic flow with respect to residual flow are calculated, we can compute the changes of link traffic flow when the residual flow is eliminated from the present period if we know the residual flow.

Therefore, the approximate link flow the *τ-th* period after the residual flows are eliminated, $\tilde{\mathbf{x}}_\tau$, is given by (according to first order Taylor expansion)

$$\tilde{\mathbf{x}}_\tau = \mathbf{x}_0 - \nabla_s \mathbf{x}\mathbf{s}_0 \tag{44}$$

As a result, the algorithm with sensitivity analysis with link-based approach in this study includes the following four steps.

Step 1: *SUE static computation.* Calculating SUE by using Dial's algorithm with MSA method and obtain the link flows and link cost in time period 1.

Step 2: *Flow propagation*. Computing Residual flow and the flow propagation to the next time period.

Step 3: *Sensitivity analysis.* Computing derivatives of link traffic flow with respect to residual flow and derive the approximate link flow

Step 4: *Recalculation.* Recalculate a static equilibrium assignment and deliver the propagated flow to the next time period.

4. SIMPLE APPLICATION

The following virtual transport network is used as an example to represent the algorithm

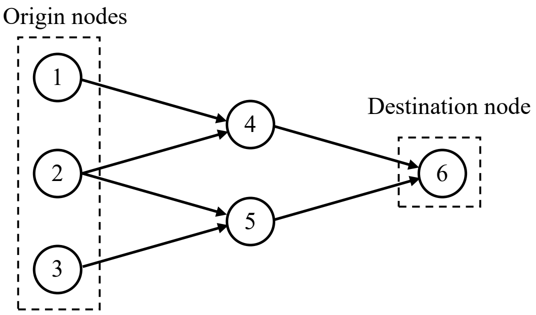


Fig. 2 First virtual network

Table 1 Link parameters in network 1

Link	r ⇨s	Free-flow time	Capacity
1	1 ⇨4	10	150
2	2 ⇨4	10	175
3	2 ⇨5	10	125
4	3 ⇨5	10	150
5	4 ⇨6	10	200
6	5 ⇨6	10	200

Table 2 Travel demand in two time periods

No.	O–D	Time 1	Time 2
1	1 ⇨6	70	60
2	2 ⇨6	350	300
3	3 ⇨6	70	60

A network contains 6 nodes and 6 links, as shown in Figure 2. While Table 1 depicted free-flow time and capacity in each link, Table 2 illustrated travel demand. OD pairs of this network is from node 1 to node 6, node 2 to node 6, node 3 to node 6.

The following parameters are assumed as θ=0.5, α=0.15, β=4 and length of time L=60.

With all the above input data, the Fortran programming language is used for compiling and the output results are attained.

In time period 1, the results of link-based four steps are as follows

Step 1: *SUE static computation*

By applying this example to the Dial's algorithm by the MSA method, the link flows *(x_i)* and link travel time *(t_i)* at the SUE are

Table 3 Results of link flow, link travel time in time period 1 at SUE state

Link	Link flow ($x_{1,i}$)	Link travel time ($t_{1,i}$)
1	70.00	10.07
2	189.78	12.07
3	160.22	14.05
4	70.00	10.07
5	259.78	14.27
6	230.22	12.63

Step 2: *Flow propagation*

The residual flow of links is calculated

Table 4 Results of link residual flow in time period 1 before using sensitivity analysis

Link	Residual flow ($y_{1,i}$)	Total residual flow ($s_{1,i}$)
1	11.75	11.75
2	38.19	38.19
3	37.52	37.52
4	11.75	11.75
5	61.78	111.72
6	48.48	97.74

Step 3: *Sensitivity analysis*

Applying given equations, we have the result of the partial derivatives of the link flows with respect to residual flow

$$\nabla_s \mathbf{x} = \begin{bmatrix} \frac{\partial x_{1,1}}{\partial s_{1,1}} & \cdots & \frac{\partial x_{1,6}}{\partial s_{1,6}} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_{1,6}}{\partial s_{1,1}} & \cdots & \frac{\partial x_{1,6}}{\partial s_{1,6}} \end{bmatrix}$$

$$= \begin{pmatrix} 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & -0.16 & 0.36 & 0.00 & -0.24 & 0.16 \\ 0.00 & 0.16 & -0.36 & 0.00 & 0.24 & -0.16 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & -0.16 & 0.36 & 0.00 & -0.24 & 0.16 \\ 0.00 & 0.16 & -0.36 & 0.00 & 0.24 & -0.16 \end{pmatrix}$$

Step 4: *Recalculation.*

The result of the recalculate static equilibrium assignment is shown as the table

Table 5 Results of link flow, link travel time in time period 1 after using sensitivity analysis

Link	Link flow ($\tilde{x}_{1,i}$)	Link travel time ($\tilde{t}_{1,i}$)
1	70.00	10.07
2	187.09	11.96
3	162.91	14.33
4	70.00	10.07
5	257.09	14.10
6	232.91	12.76

Because in semi-dynamic traffic assignment model, we do not only apply static traffic assignment model but also consider flow propagate to the next time period. So, in the time period 2, there is also flow propagate derived from period 1. Thus, travel demand in time period 2 will be raised in the route from node 4 to node 6 and there is one more OD pair from node 5 to node 6. Travel demand in time period 2 is detailed in the following table

Table 6 Travel demand in time period 2 after flow propagation

No.	O–D	Time 2
1	1 ⇨6	60
2	2 ⇨6	300
3	3 ⇨6	60
4	4 ⇨6	49.94
5	5 ⇨6	49.27

With the same manner, our results of time period 2 are listed in following tables

Table 7 Results of link travel detail in time period 2 before using sensitivity analysis

Link	Link flow ($x_{2,i}$)	Link travel time ($t_{2,i}$)	Residual flow ($y_{2,i}$)	Total residual flow ($s_{2,i}$)
1	60.00	10.04	10.04	10.04
2	158.76	11.02	29.15	29.15
3	141.24	12.45	29.30	29.30
4	60.00	10.04	10.04	10.04
5	268.70	14.89	66.67	105.86
6	250.51	13.69	57.16	96.50

Table 8 The derivatives of the link flows with respect to residual flow in time period 2

Link	1	2	3	4	5	6
1	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	-0.10	0.27	0.00	-0.29	0.23
3	0.00	0.10	-0.27	0.00	0.29	-0.23
4	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	-0.10	0.27	0.00	-0.29	0.23
6	0.00	0.10	-0.27	0.00	0.29	-0.23

And, the perturbed flows corresponding to perturbation s_i is shown as followings

Table 9 Results of link flow, link travel time in time period 2 after using sensitivity analysis

Link	Link flow ($\tilde{x}_{2,i}$)	Link travel time ($\tilde{t}_{2,i}$)
1	60.00	10.04
2	155.88	10.94
3	144.12	12.65
4	60.00	10.04
5	265.82	14.68
6	253.39	13.86

After all, we will show the comparison of results of the route-based semi-DTA (Pham Thi Thu Ha (2012) and Itagaki (2014)) and our link-based semi-DTA. The results of the comparison are demonstrated as following tables

Table 10 Results of route travel of Semi-DTA with MSA method after using Sensitivity Analysis with route-based approach in time period 1

OD	Demand (q_h)	Route	Probability route choice ($p_{h,k}$)	Route travel time ($c_{h,k}$)	Route flow ($f_{h,k}$)
1 ⇨6	70	1 ⇨4⇨6	1.00	24.17	70.00
2 ⇨6	350	2 ⇨4⇨6	0.63	26.06	187.09
		2 ⇨5⇨6	0.37	27.09	162.91
3 ⇨6	70	3 ⇨5⇨6	1.00	22.83	70.00

Table 11 The comparison of the results of link travel of semi-DTA with MSA method after using Sensitivity Analysis in time period 1

Link	Link flow		Link travel time		Residual flow		Total residual flow	
	Link based	Route based	Link based	Route based	Link based	Route based	Link based	Route based
1	70.00	70.00	10.07	10.07	11.75	11.75	11.75	11.75
2	187.09	187.09	11.96	11.96	38.19	38.19	38.19	38.19
3	162.91	162.91	14.33	14.33	37.52	37.52	37.52	37.52
4	70.00	70.00	10.07	10.07	11.75	11.75	11.75	11.75
5	257.09	257.09	14.10	14.10	61.78	61.78	111.72	111.72
6	232.91	232.91	12.76	12.76	48.48	48.48	97.74	97.74

Table 12 Results of route travel of Semi-DTA with MSA method after using Sensitivity Analysis with route-based approach in time period 2

OD	Demand (q_h)	Route	Probability route choice ($p_{h,k}$)	Route travel time ($c_{h,k}$)	Route flow ($f_{h,k}$)
1 ⇨6	60	1 ⇨4⇨6	1.00	24.72	60.00
2 ⇨6	300	2 ⇨4⇨6	0.61	25.62	155.88
	60	2 ⇨5⇨6	0.39	26.52	144.12
3 ⇨6	60	3 ⇨5⇨6	1.00	23.90	60.00
4 ⇨6	49.94	4 ⇨6	1.00	14.68	49.94
5 ⇨6	49.27	5 ⇨6	1.00	13.86	49.27

Table 13 The comparison of the results of link travel of semi-DTA with MSA method after using Sensitivity Analysis in time period 2

Link	Link flow		Link travel time		Residual flow		Total residual flow	
	Link based	Route based	Link based	Route based	Link based	Route based	Link based	Route based
1	60.00	60.00	10.04	10.04	10.04	10.04	10.04	10.04
2	155.88	155.88	10.94	10.94	29.15	29.15	29.15	29.15
3	144.12	144.12	12.65	12.65	29.30	29.30	29.30	29.30
4	60.00	60.00	10.04	10.04	10.04	10.04	10.04	10.04
5	265.82	265.82	14.68	14.68	66.67	66.67	105.86	105.86
6	253.39	253.39	13.86	13.86	57.16	57.16	96.50	96.50

Indeed, the given tables showed that there are no significant differences between the results of route-based approach (model of Pham Thi Thu Ha) and our proposed link-based approach of semi-DTA with sensitivity analysis. Nevertheless, with link-based approach, we can apply the semi-DTA model to real network because of efficient computation.

To demonstrate the effectiveness of link based approach, moreover, in addition, the 2nd virtual network is used.

A more complex network with 20 nodes and 35 links is depicted in figure 3. Parameters remain the same as the first virtual network with θ=0.5, α=0.15, β=4 and L=60. The travel demand of 11 OD pairs is listed in table 14 and free flow travel time and the capacity of 35 links is showed are listed in table 15.

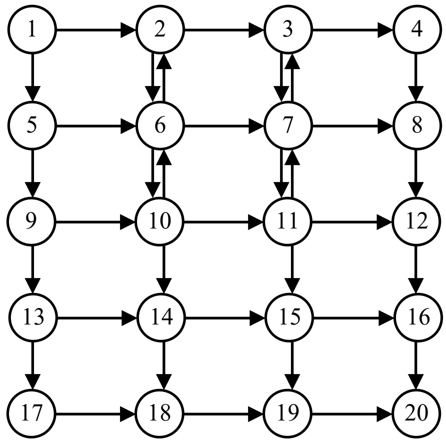


Fig. 3 Second virtual network

Table 14 Travel demand in second virtual network

OD	Travel demand
1⇨12	1000
1⇨13	800
1⇨20	1500
2⇨19	500
3⇨12	450
4⇨16	400
5⇨3	300
6⇨16	400
9⇨4	800
10⇨20	500
13⇨19	400

Table 15 Link parameters in second virtual network

Link	Free-flow time	Capacity
1-2	20	1000
1-5	18	1500
2-3	23	500
2-6	19	500
3-4	17	500
3-7	16	500
4-8	22	500
5-6	14	1000
5-9	24	800
6-2	15	650
6-7	17	1000
6-10	20	500
7-3	18	750
7-8	13	1000
7-11	26	500
8-12	19	1000
9-10	7	800
9-13	20	800
10-6	16	700
10-11	18	800
10-14	14	700
11-7	15	600
11-12	17	800
11-15	30	1000
12-16	38	2000
13-14	15	500
13-17	14	600
14-15	20	700
14-18	30	1800
15-16	25	900
15-19	27	1700
16-20	10	500
17-18	9	500
18-19	20	950
19-20	16	1000

We will look at two criteria. One is how much required memory of each calculation, the other criteria we will use relate to running time of algorithm. Excluding the path enumeration, the differences between the time calculation and required memory of sensitivity analysis with link based and route based approaches follow below figures

Fig. 4 The comparison of time calculation *(second)*

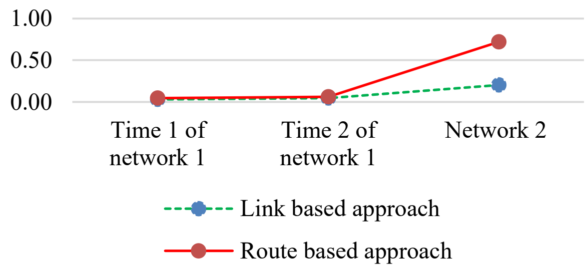
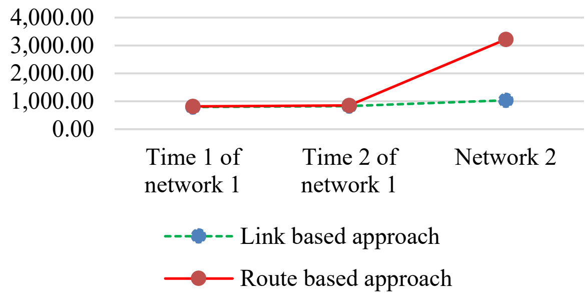


Fig. 5 The comparison of required memory *(Kb)*



The figures provide analyses of the contrast among the link based and route based approach in time calculation and required memory. When comparing the two approaches, it is clear that the more complicated network with more links, nodes and OD pairs, the more link based approach shows the optimum in time calculation and required memory. The difference between the two approaches has increased markedly as the traffic network becomes more complex. While the contrast of virtual network 1 is almost none, then the difference is most noticeable in virtual traffic network 2 with required memory and time calculation of route based approach over 3 times these of link based approach. And, both required memory and time calculation of link based approach tend to increase slowly with the increase of links, nodes and OD pairs. For implementation on large networks, we need to consider balancing the trade-off between storage requirement and computing time and link based approach have shown superiority in both of these criteria.

4. CONCLUSIONS

All thing considered, Semi-dynamic traffic assignment model is an efficient alternative for describing within day traffic dynamics. And, sensitivity analysis is worth to use to reduce computational cost and resolve bilevel programming problem. Furthermore, for the purpose of avoiding path enumeration in semi-dynamic traffic assignment with sensitivity analysis, the link-based approach is an effective alternative. In this study, the semi-dynamic traffic assignment model was proposed using link-based approach and sensitivity analysis. With these approaches, semi-dynamic traffic assignment model can be used to represent traffic flows of large scale network. Sensitivity analysis, however, gives only approximate results of semi-stochastic traffic assignment. In the future, it is also necessary to apply the algorithm on real networks to assess the actual effect.

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