Bifurcation theory of a square lattice economy: Racetrack economy analogy in an economic geography model

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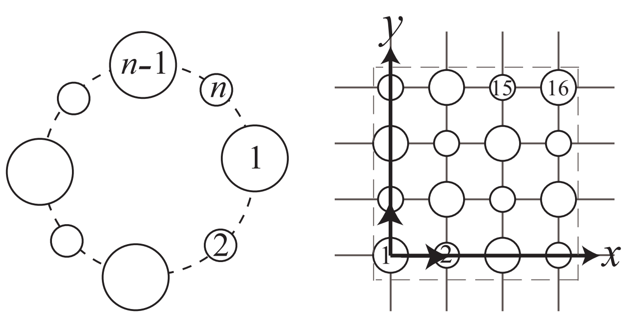
Bifurcation theory for an economic agglomeration in a square lattice economy is presented comparatively with that in a racetrack economy. Existence of a series of equilibria with characteristic geometrical patterns is elucidated. A spatial period doubling bifurcation cascade between these equilibria is advanced as a common mechanism to engender fewer and larger agglomerations in both economies. Analytical formulas for a break point, at which the uniformity is broken under reduced transport costs, are proposed for an economic geography model by synthetically encompassing both economies.

Key Words: *Bifurcation, Economic geography model, Group theory, Replicator dynamics, Spatial period doubling*

1. Introduction



(a) Racetrack economy (b) Square lattice economy

Fig.1 Two economic space models in the state of spatial period doubling.

A proper setting of spatial economy is vital in the investigation of spatial economic agglomerations. A racetrack economy (Fig. 1(a)), which represents a series of places on a circle, is believed to be capable of representing some important agglomeration properties although this economy is essentially one-dimensional. This economy undergoes bifurcations to engender fewer and larger agglomerations (e.g., Krugman, 1993 [20)]). The most characteristic behavior that drew eyes is the "spatial period doubling bifurcation" that leads to alternating core and periphery patterns shown in Fig. 1(a) (see the related studies in Section 2.).

Square lattice economy is often employed as a two-dimensional spatial platform.[1] The spatial period doubling pattern exists also in the lattice economy (Fig. 1(b)). Such coexistence of this pattern implies that the racetrack economy can be interpreted as an idealized one-dimensional counterpart of the agglomerations in two dimensions.

This paper aims to elucidate the mechanism of economic agglomeration in a square lattice, which turns out to be quite complicated (Section 8.). In order to tackle such complexity, a racetrack economy analogy is proposed. The racetrack economy is endowed with a simpler spatial structure that is easier to be treated analytically than the lattice economy. In particular, we would like to answer the following question: "To what qualitative or quantitative extent can the racetrack economy serve as a platform for the agglomerations in two dimensions?" While a qualitative aspect of these agglomerations is described in a general setting by bifurcation theory, a qualitative measure of the

[1] Several studies of spatial agglomeration have been conducted on a square lattice; see, e.g., Clarke and Wilson (1983) [6)], Weidlich and Haag (1987) [32)], Munz and Weidlich (1990) [23)], Brakman et al. (1999) [4)]), and Stelder (2005) [30)].

agglomerations is presented for an economic geography model.

For a qualitative aspect, the progress of agglomeration by repeated bifurcations are studied comparatively in both economies.[2] As a novel contribution of this paper, a bifurcation theory in the square hexagonal lattice is developed and the existence of cascades of spatial period doubling bifurcations leading to fewer and larger agglomerations is elucidated.

For a quantitative aspect, a *break point*[3] is investigated comparatively for the two economies. When investment in transportation infrastructure is committed, the break point indexes the functioning of this investment. Formulas for the square lattice are newly developed and a strong linkage with a tensor structure is found between the racetrack and the lattice economies.

Whereas real economic activities accommodate models of various kinds, we refer to a specific economic geography model, i.e., that of Forslid and Ottaviano (2003) [9] in favor of its analytical tractability. There are unskilled workers who are immobile and equally distributed along places, and skilled ones who are footloose entrepreneurs maximize wages. By numerical comparative static analyses for both economies, progress of agglomeration through successive emergence of spatial period doubling patterns is observed, thereby ensuring the validity of the racetrack economy analogy.

This paper is organized as follows. Related studies are presented in Section 2.. Modeling of a spatial economy for an analytically solvable economic geography model model is presented in Section 3.. Symmetry of racetrack and lattice economies is described in Section 4.. A theory of replicator dynamics is developed in Section 5.. Bifurcating agglomeration patterns are predicted theoretically in Section 6.. Formulas for break points are advanced in Section 7.. Numerical examples are presented in Section 8..

2. Related studies

There are spatial platforms for economic activities of various kinds. The two-place economy has long been employed extensively (e.g., Krugman, 1991 [19]; Fujita, Krugman, and Venables, 1999 [11]; Baldwin et al., 2003 [3]; Oyama, 2009 [25]; Fujishima, 2013 [10]). There are a few study on three places (e.g., Commendatorea et al., 2014 [7]).

Racetrack economy was used to show the evolution of a regular lattice by Krugman (1993) [20] and Fujita, Krugman, and Venables (1999) [11]. Krugman (1996, p.91) [21] regarded the racetrack economy as one-dimensional and inferred its extendibility to a two-dimensional economy to engender hexagonal distributions. Tabuchi and Thisse (2011) [31] examined the racetrack economy for a multi-industry model to show the emergence of central places, which denotes a spatial alternation of a core place with a large population and a peripheral place with a small population. This economy undergoes a sequence of recurrent bifurcations, called the "spatial period doubling cascade," which was observed ubiquitously for NEG models.[4]

A *break point* of the transport cost was introduced for the two-place economy (Fujita, Krugman, and Venables, 1999 [11]). The importance of the break point has come to acknowledged and its formulas have been derived for several spatial economy models in several spatial platforms: a class of footloose-entrepreneur models (Pflüger and Südekum, 2008 [27]), the Pflüger model (2004) [26] in the racetrack economy for logit dynamics (Akamatsu, Takayama, and Ikeda, 2012 [2]), an analytically solvable model (Forslid and Ottaviano, 2003 [9]) in the racetrack economy for the replicator dynamics (Ikeda et al., 2017a [16]), and the same model in the 6×6 hexagonal lattice for the logit dynamics (Ikeda, Murota, and Takayama, 2017b [18]).

Bifurcation mechanism of the square lattice treated in this paper is based that for the hexagonal lattice (Ikeda et al., 2012, 2014 [15),17)]; Ikeda and Murota, 2014 [14]). In comparison with previous studies on the racetrack economy, this paper treats this economy as a one-dimensional counterpart of two-dimensional agglomerations. Synthetic formulas that can encompass both the racetrack and the square lattice economies are proposed, while such formulas for these two economies were derived up to now somewhat independently.

3. Modeling of the spatial economy

Modeling of the spatial economy is presented in this section. As a representative of spatial economy models, an analytically solvable core–periphery model by Forslid

[2] The mechanism of bifurcations in a racetrack economy was elucidated by the group-theoretic bifurcation analysis (Ikeda, Murota, and Akamatsu, 2012 [13]).

[3] The *break point* of the transport cost that produces a core–periphery pattern in a two-place economy was highlighted as a key concept (Fujita, Krugman, and Venables, 1999 [11]).

[4] See, e.g., Picard and Tabuchi (2010) [28], Ikeda, Akamatsu, and Kono (2012) [13], Akamatsu, Takayama, and Ikeda (2012) [2], Akamatsu, Mori, and Takayama (2016) [1], and Osawa, Akamatsu, and Takayama (2017) [24].

and Ottaviano (2003) [9] is used. The fundamental logic and governing equation of a multi-regional version of the model are presented based on work of Akamatsu, Mori, and Takayama (2016) [1], while details are given in APPENDIX I.

(1) Basic assumptions

The economy of this model comprises K places (labeled $i = 1, \dots, K$), two factors of production (skilled and unskilled labor), and two sectors (manufacturing, M, and agriculture, A). Both H skilled and L unskilled workers consume final goods of two types: manufacturing sector goods and agricultural sector goods. Workers supply one unit of each type of labor inelastically. Skilled workers are mobile among places, and the number of skilled workers in place i is denoted by λ_i ($\sum_{i=1}^{K} \lambda_i = H$). The total number H of skilled workers is normalized as $H = 1$. Unskilled workers are immobile and distributed equally across all places with unit density (i.e., $L = 1 \times K$).

Preferences U over the M- and A-sector goods are identical across individuals. The utility of an individual in place i is

$$U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{M}} + (1-\mu) \ln C_i^{\mathrm{A}} \qquad (0 < \mu < 1), \quad (1)$$

where μ is a constant parameter expressing the expenditure share of manufacturing sector goods, C_i^{A} stands for the consumption of the A-sector product in place i and C_i^{M} represents the manufacturing aggregate in place i, which is defined as

$$C_i^{\mathrm{M}} \equiv \left(\sum_{j=1}^{K} \int_0^{n_j} q_{ji}(\ell)^{(\sigma-1)/\sigma} d\ell \right)^{\sigma/(\sigma-1)}, \qquad (2)$$

where $q_{ji}(\ell)$ is the consumption in place i of a variety $\ell \in [0, n_j]$ produced in place j, n_j is the number of produced varieties at place j, and $\sigma > 1$ is the constant elasticity of substitution between any two varieties.

(2) Iceberg form of transport cost

The transportation costs for M-sector goods are assumed to take the iceberg form. That is, for each unit of M-sector goods transported from place i to place j ($\neq i$), only a fraction $1/T_{ij} < 1$ actually arrives ($T_{ii} = 1$). It is assumed that $T_{ij} = T_{ij}(\tau)$ is a function in a transport cost parameter $\tau > 0$ as

$$T_{ij} = \exp(\tau\, m(i, j)\, \tilde{L}), \qquad (3)$$

where $m(i, j)$ is an integer expressing the shortest link between places i and j and $\tilde{L}$ is the distance unit. The spatial discounting factor

$$d_{ji} = T_{ji}^{1-\sigma} \qquad (4)$$

between places j and i represents a distance decaying friction. With the use of

$$r = \exp[-\tau(\sigma - 1)\tilde{L}] \qquad (5)$$

($0 < r < 1$ for $\tau > 0$) expressing trade freeness, the spatial discounting factor $d_{ij} = T_{ij}^{1-\sigma}$ in (4) is expressed as $d_{ij} = r^{\,m(i,j)}$.

(3) Market equilibrium

As worked out in APPENDIX I, the market equilibrium wage vector $\boldsymbol{w}$ is obtained as

$$\boldsymbol{w} = \frac{\mu}{\sigma} \left(I - \frac{\mu}{\sigma} D \Delta^{-1} \Lambda \right)^{-1} D \Delta^{-1} \mathbf{1} \qquad (6)$$

with the notation

$$\begin{cases} \boldsymbol{w} = (w_i), \quad D = (d_{ij}), \quad \Delta = \mathrm{diag}(\Delta_1, \dots, \Delta_K), \\ \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_K), \quad \mathbf{1} = (1, \dots, 1)^{\top}. \end{cases} \qquad (7)$$

The indirect utility v_i is expressed in terms of w_i and $\Delta_i = \sum_{k=1}^{K} d_{ki} \lambda_k$ as

$$v_i = \frac{\mu}{\sigma - 1} \ln \Delta_i + \ln w_i. \qquad (8)$$

(4) Spatial equilibrium

We introduce the spatial equilibrium, for which high skilled workers are allowed to migrate among places. A customary way to define such an equilibrium is to consider the following problem: Find $(\boldsymbol{\lambda}^*, \hat{v})$ satisfying

$$\begin{cases} (v_i - \hat{v})\lambda_i^* = 0, \quad \lambda_i^* \geq 0, \quad v_i - \hat{v} \leq 0, \quad (i = 1, \dots, K), \\ \sum_{i=1}^{K} \lambda_i^* = 1. \end{cases} \qquad (9)$$

For the solution of this problem, $\hat{v}$ serves as the highest (indirect) utility. When the system is in a *spatial equilibrium*, no individual can improve his/her utility by changing his/her location unilaterally.

As guaranteed in Sandholm (2010) [29], it is possible to replace the problem to obtain a set of stable spatial equilibria by another problem to find a set of stable stationary points of the replicator dynamics:

$$\frac{d\boldsymbol{\lambda}}{dt} = \boldsymbol{F}(\boldsymbol{\lambda}, \tau), \qquad (10)$$

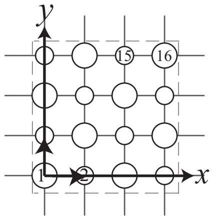
where $\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = (F_i(\boldsymbol{\lambda}, \tau) \mid i = 1, \dots, K)$, and

$$F_i(\boldsymbol{\lambda}, \tau) = (v_i(\boldsymbol{\lambda}, \tau) - \bar{v}(\boldsymbol{\lambda}, \tau))\lambda_i, \quad (i = 1, \dots, K). \qquad (11)$$

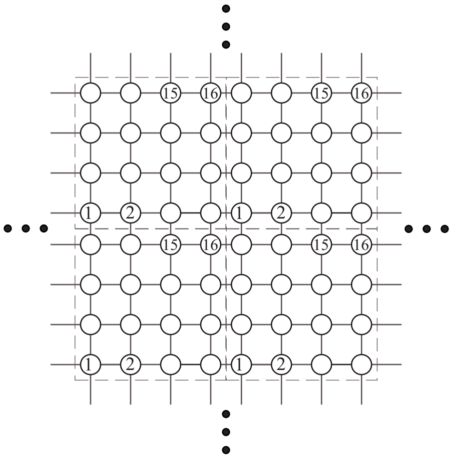
Here, $\bar{v} = \sum_{i=1}^{K} \lambda_i v_i$ is the average utility. Stationary points (rest points) $\boldsymbol{\lambda}^*(\tau)$ of the replicator dynamics (10) are defined as those points which satisfy the static governing equation

$$\boldsymbol{F}(\boldsymbol{\lambda}^*, \tau) = \mathbf{0}. \qquad (12)$$

We classify stability using the eigenvalues of the Jacobian



(a) 4×4 square lattice



(b) Periodically repeated 4×4 square lattice

Fig.2 A system of places on the 4×4 square lattice with periodic boundary conditions.

matrix $J(\lambda^*, \tau) = \partial \boldsymbol{F}/\partial \boldsymbol{\lambda}(\lambda^*, \tau)$ as

$$\begin{cases} \text{linearly stable:} \\ \qquad \text{every eigenvalue has negative real part,} \\ \text{linearly unstable:} \\ \qquad \text{at least one eigenvalue has positive real part.} \end{cases}$$

A stationary point is asymptotically stable or unstable according to whether it is linearly stable or unstable.

4. Symmetry of racetrack and lattice economies

In investigation of bifurcating patterns of a symmetric system, we refer to a group G that labels its symmetry. For the racetrack economy, a series of $K = n$ places (labeled $i = 1, \ldots, n$) is spread equally on the circumference of the circle and these places are connected by roads of the same length $\tilde{L}$. The symmetry of this economy located at the origin of the xy-plane is labeled by the dihedral group $\mathrm{D}_n = \langle s, r \rangle$, where s is the reflection $y \mapsto -y$, r is an $2\pi/n$ anticlockwise rotation around the origin, and $\langle \cdot \rangle$ is a group generated by the elements therein.

An $n \times n$ square lattice with periodic boundary conditions is introduced as a two-dimensional spatial platform. Nodes at a border of this lattice are connected periodically to those on the opposite border to cover an infinite space (Fig. 2(b)). Places of economic activities are located on the nodes, which are connected by roads of the same length $\tilde{L}$ forming a square mesh. The symmetry of the lattice is expressed by the group $\langle r, s, p_1, p_2 \rangle$, which is generated by the following four elements:[5] r: counterclockwise rotation about the origin at an angle of $\pi/2$, s: reflection $y \mapsto -y$, p_1: x-directional periodic translation at the unit length $\tilde{L}$, and p_2: y-directional one.

The symmetry of the governing equation is formulated as the so-called equivariance condition[6]

$$T(g)\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \boldsymbol{F}(T(g)\boldsymbol{\lambda}, \tau), \quad g \in G \tag{13}$$

in terms of a $K \times K$ orthogonal matrix representation[7] T of the group G. Then the Jacobian matrix satisfies the symmetry condition

$$T(g)J = JT(g), \quad g \in G. \tag{14}$$

The flat earth equilibrium (uniform distribution) with $\boldsymbol{\lambda}^* = \frac{1}{K}(1, \ldots, 1)^\top$ exists in both the racetrack and lattice economies. This equilibrium is invariant to $G = \mathrm{D}_n$ in the racetrack economy and to $G = \langle r, s, p_1, p_2 \rangle$ in the lattice economy.

5. Bifurcation theory of replicator dynamics

We introduce a bifurcation theory on the replicator dynamics, which is endowed with a characteristic bifurcation mechanism due to its product form (11). After introducing classifications of stationary points, we formulate a symmetry condition for the existence of trivial solutions and investigate stability and sustainability of trivial solutions as novel contributions of this paper.

(1) Classification of stationary points

Stationary points $(\boldsymbol{\lambda}, \tau)$ of the replicator dynamics are classified in preparation for the description its bifurcation mechanism. First, these points are classified into an *interior solution*, for which all cities have positive population, and a *corner solution*, for which some cities have zero population.

[5] The elements r, s, p_1, and p_2 $r^4 = s^2 = (rs)^2 = {p_1}^n = {p_2}^n = e$, $p_2 p_1 = p_1 p_2$, $rp_1 = p_2 r$, $rp_2 = p_1^{-1} r$, $sp_1 = p_1 s$, $sp_2 = p_2^{-1} s$, where e is the identity element.

[6] This condition was proven for the racetrack economy in Ikeda, Akamatsu, and Kono (2012) [13]. The proof for the lattice economy can be achieved similarly.

[7] Matrix representation means that (i) for each element $g \in G$, $T(g)$ is a $K \times K$ matrix with $T(g)^\top T(g) = I$ (identity matrix), and (ii) $T(g)T(h) = T(gh)$ for all $g, h \in G$.

A solution can be expressed, without loss of generality, by appropriately rearranging the order of independent variables $\boldsymbol{\lambda}$ as

$$\hat{\boldsymbol{\lambda}} = \begin{bmatrix} \boldsymbol{\lambda}_+ \\ \boldsymbol{\lambda}_0 \end{bmatrix} \tag{15}$$

with $\boldsymbol{\lambda}_+ = \{\lambda_i > 0 \mid i = 1, \dots, m\}$ and $\boldsymbol{\lambda}_0 = \mathbf{0}$. Note that $\boldsymbol{\lambda}_0$ is absent for an interior solution. The static governing equation (12) can be rearranged accordingly as

$$\hat{\boldsymbol{F}} = \begin{bmatrix} \boldsymbol{F}_+(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \tau) \\ \boldsymbol{F}_0(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \tau) \end{bmatrix} \tag{16}$$

with the rearranged Jacobian matrix

$$\hat{J} = \begin{bmatrix} J_+ & J_{+0} \\ O & J_0 \end{bmatrix}, \tag{17}$$

where

$$J_+ = \mathrm{diag}(\lambda_1, \dots, \lambda_m)\ \{\partial(v_i - \overline{v})/\partial\lambda_j \mid i, j = 1, \dots, m\},$$
$$J_{+0} = \mathrm{diag}(\lambda_1, \dots, \lambda_m)$$
$$\{\partial(v_i - \overline{v})/\partial\lambda_j \mid i = 1, \dots, m;\ j = m+1, \dots, K\},$$
$$J_0 = \mathrm{diag}(v_{m+1} - \overline{v}, \dots, v_K - \overline{v}).$$

A stable spatial equilibrium is given by a stable stationary solution, for which all eigenvalues of $\hat{J}$ are negative. Such stability condition is decomposed into two conditions:

$$\begin{cases} \text{Stability condition for } \boldsymbol{\lambda}_+\text{:} \\ \qquad \text{all eigenvalues of } J_+ \text{ are negative.} \\ \text{Sustainability condition for } \boldsymbol{\lambda}_0\text{:} \\ \qquad \text{all diagonal entries of } J_0 \text{ are negative.} \end{cases} \tag{18}$$

Next, critical points[8] are classified into a *break bifurcation point*[9] with singular J_+ and a *non-break point* with $v_i - \overline{v} = 0$ for some place i ($i = m+1, \dots, K$); a sustain point is a special kind of non-break point. A bifurcating solution with reduced symmetry branches from a break point, whereas population of some places vanishes at a non-break (sustain) point. A break points is either a *simple bifurcation point*, a *double bifurcation point*, and so on, according to whether the number of zero eigenvalue(s) of the Jacobian matrix $\hat{J}$ is equal to one, two, and so on. A simple bifurcation is either *tomahawk* or *pitchfork*. Bifurcating solutions are unstable for the tomahawk and stable for the pitchfork.

Last, stationary points are classified into a *trivial solution* $(\boldsymbol{\lambda}, \tau)$ with a constant $\boldsymbol{\lambda}$ that exists for any $\tau \in (0, \infty)$ and a *non-trivial solution* $(\boldsymbol{\lambda}, \tau)$ for which $\boldsymbol{\lambda}$ changes with τ. Existence of trivial solutions of various kinds is a special feature of the replicator dynamics.

[8] Critical points are those which have one or more zero eigenvalue(s) of the Jacobian matrix $\hat{J}$.

[9] There is another critical point, a limit point of τ, also with singular J_+ (Ikeda and Murota, 2014 [14]). Yet this kind of point does not play an important role in the discussion in this paper.

Proposition 1. *The flat earth equilibrium* $\boldsymbol{\lambda}^* = \frac{1}{K}(1, \dots, 1)^\top$ *is a trivial equilibrium.*

Proof. Because we have $v_1 = \cdots = v_K = \overline{v}$ for this equilibrium, the conditions (9) for a spatial equilibrium is satisfied for any τ. □

(2) Symmetry condition of a corner solution

A corner solution with m identical agglomerated places, i.e.,

$$\hat{\boldsymbol{\lambda}} = \begin{bmatrix} \boldsymbol{\lambda}_+ \\ \boldsymbol{\lambda}_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{m}\mathbf{1} \\ \mathbf{0} \end{bmatrix} \tag{19}$$

is paid a special attention in this paper. This is a core–periphery pattern with a two-level hierarchy: Population is agglomerated to m core places with identical population, while other peripheral places have no population. An atomic monocenter for $m = 1$ in Fig. 3(a) and twin places for $m = 2$ in (b) serve as simple examples of such a solution.

Assumption 1. *The corner solution with m identical agglomerated places in* (19) *is invariant to a group G and there is a set of permutation matrices* $T_+(g)$ ($g \in G$) *that permutes any two entries of* $\boldsymbol{\lambda}_+$.

Under this assumption, the reduced system $\mathbf{F}_+(\boldsymbol{\lambda}_+, \mathbf{0}, \tau)$ in (16) is endowed with the symmetry conditions:

$$T_+(g)\boldsymbol{F}_+(\boldsymbol{\lambda}_+, \mathbf{0}, \tau) = \boldsymbol{F}_+(T_+(g)\boldsymbol{\lambda}_+, \mathbf{0}, \tau),$$
$$T_+(g)J_+ = J_+T_+(g), \quad g \in G. \tag{20}$$

Trivial solutions have several characteristics as expounded in the following Proposition and Corollary (see APPENDIX II for the proof).

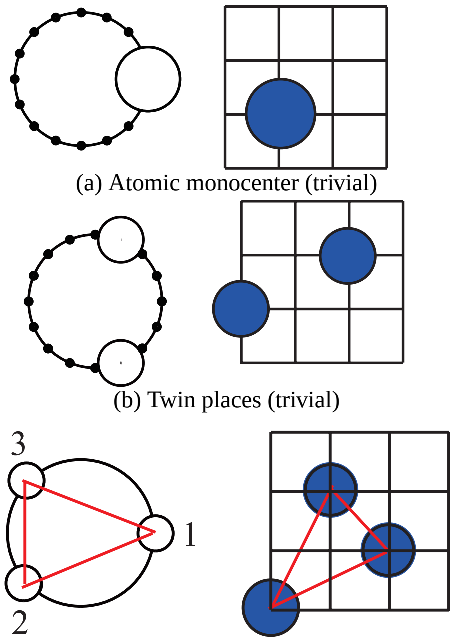
Proposition 2. *A corner solution* $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \tau) = (\frac{1}{m}\mathbf{1}, \mathbf{0}, \tau)$ *satisfying Assumption 1 is a trivial solution.*

Corollary 1. *An atomic monocenter* ($m = 1$) *and twin places* ($m = 2$) *are trivial solutions.*

The corner solutions with m identical agglomerated places in (19) are not always trivial solutions. For example, the spatial patterns shown in Fig. 3(c) are not trivial solutions and there is no guarantee that they are solutions (APPENDIX II).

(3) Stability and sustainability of trivial solutions

Prior to the description of stability and sustainability of trivial solutions, we first refer to those of two places (Fujita, Krugman, and Venables, 1999 [11]): A trivial solution with $\boldsymbol{\lambda} = (1/2, 1/2)^\top$ is stable for $\tau > \tau_{\mathrm{B}}$, where τ_{B} is a



(a) Atomic monocenter (trivial)

(b) Twin places (trivial)

(c) Non-trivial corner solutions

Fig.3 Trivial and non-trivial corner solutions.

break point. On the other hand, the core–periphery pattern $\lambda = (1, 0)^\top$ is sustainable for $\tau < \tau_S$, where τ_S is a sustain point.

In general, a trivial equilibrium possibly has a few non-break points (Section 8.); accordingly, a sustain point is defined as the non-break point with the smallest τ value, which is set as τ_S. We introduce the following assumption, which is in line with the agglomeration behavior (Section 8.) of the economic geography model (Section 3.).

Assumption 2. *For a trivial equilibrium other than the flat earth equilibrium and the atomic monocenter, there are τ_B and τ_S such that the stability condition of the core places in* (18) *is satisfied for $\tau > \tau_B$ and the sustainability condition of the periphery places in* (18) *is satisfied for $\tau < \tau_S$.*

Then we can consider the following classification:

$$\begin{cases} \text{Well-posed trivial solution:} & \tau_B < \tau_S, \\ \text{Ill-posed trivial solution:} & \tau_B > \tau_S. \end{cases} \tag{21}$$

Proposition 3. *A well-posed trivial solution is a stable spatial equilibrium in the range $\tau_B < \tau < \tau_S$, while an ill-posed trivial solution is not a stable spatial equilibrium for any τ.*

Corollary 2. *Under Assumption 2, the flat earth equilibrium is stable for $\tau > \tau_B$.*

Proof. This is apparent since there is no sustain point on this equilibrium. □

6. Bifurcation mechanism of spatial period doubling cascades

Spatial period doubling cascades of the racetrack and the lattice economies are investigated in this section, while these cascades are demonstrated in Section 8. to be predominant in the progress of agglomeration in the economic geography model (Section 3.). It is ensured that spatial period doubling patterns of these economies are always trivial solutions. A bifurcation mechanism of the emergence of these patterns in the lattice economy is newly presented and is meshed consistently with the previous results in the racetrack economy (Ikeda, Akamatsu, and Kono, 2012 [13]).

We are interested in repeated occurrence of *spatial period doubling* engendering *spatial period doubling patterns* (Figs. 4(a) and (b)). These patterns are shown to be trivial solutions[10] in the remainder of this section.

(1) Racetrack economy: spatial period doubling

Bifurcation rules for spatial period doubling cascade starting from the flat earth equilibrium $\lambda^* = \frac{1}{n}(1, \dots, 1)^\top$ en route to an atomic monocenter are presented. When n is even, at a simple break bifurcation point on the flat earth equilibrium, a solution curve bifurcates in the direction of an eigenvector

$$\boldsymbol{\eta}_{\mathrm{Ra}} = (1, -1, \dots, 1, -1)^\top \tag{22}$$

of the Jacobian matrix J. A bifurcating state has a population distribution of the form:

$$\lambda = (1/n + a, 1/n - a, \dots, 1/n + a, 1/n - a)^\top, \quad -1/n \le a \le 1/n. \tag{23}$$

This represents a state in which concentrating places and extinguishing places alternate along the circle and, in turn, to form a chain of spatially repeated core–periphery patterns *a la* Christaller (1933) [5] and Lösch (1940) [22].

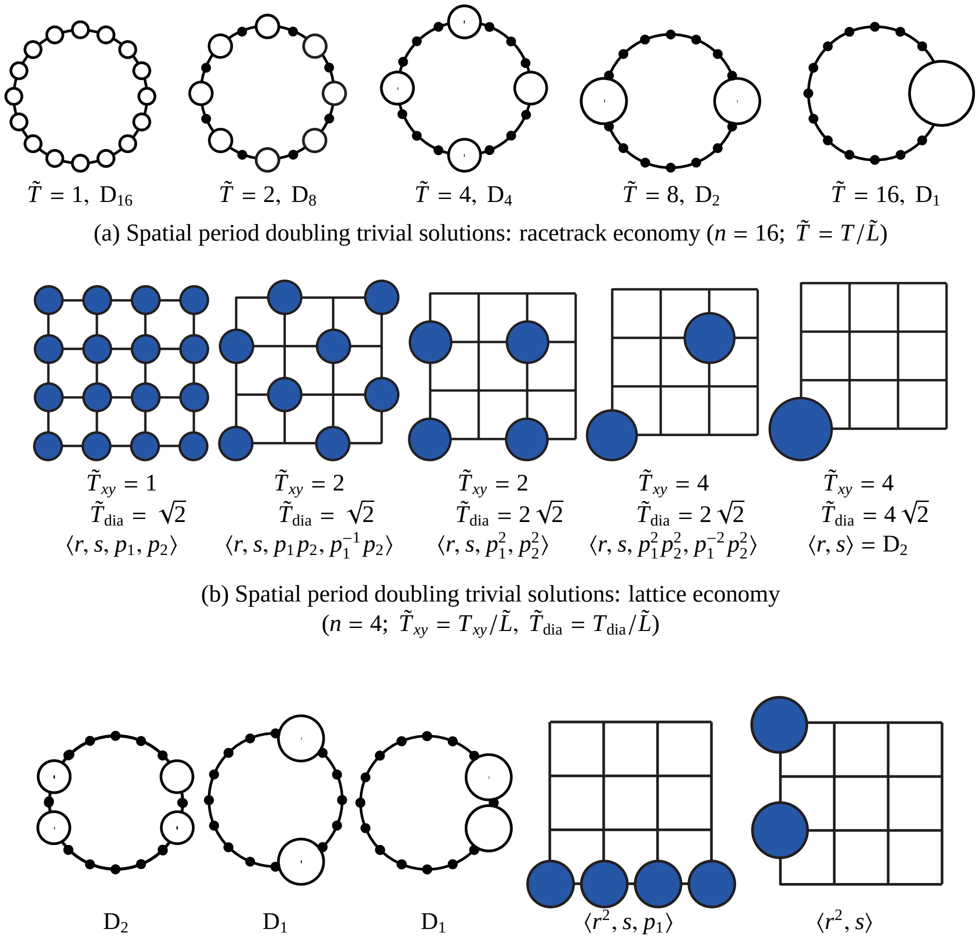
We consider a case where the concentrating and the extinguishing proceed until reaching a non-break (sustain) point with a spatial period doubling pattern:

$$\lambda_{\mathrm{Ra}} = (2/n, 0, \dots, 2/n, 0)^\top, \quad \text{i.e.,} \quad \hat{\lambda} = (2/n, \dots, 2/n; 0, \dots, 0)^\top = \begin{bmatrix} \frac{2}{n}\mathbf{1} \\ \mathbf{0} \end{bmatrix}, \tag{24}$$

which is invariant to a group $D_{n/2}$.

For $n = 2^k$ $(k = 2, 3, \dots)$ places, at a simple break bifurcation point, a secondary bifurcating solution branches in

[10] There are trivial solutions other than spatial period doubling ones as depicted in Fig. 4(c).



(a) Spatial period doubling trivial solutions: racetrack economy ($n = 16$; $\tilde{T} = T/\tilde{L}$)

(b) Spatial period doubling trivial solutions: lattice economy ($n = 4$; $\tilde{T}_{xy} = T_{xy}/\tilde{L}$, $\tilde{T}_{\mathrm{dia}} = T_{\mathrm{dia}}/\tilde{L}$)

(c) Non-doubling trivial solutions

Fig.4 Spatial period doubling and non-doubling trivial solutions.

the direction of an eigenvector

$$\boldsymbol{\eta}_{\mathrm{Rb}} = (1, 0, -1, 0; \ldots; 1, 0, -1, 0)^\top,$$
$$\text{i.e.,} \quad \hat{\boldsymbol{\eta}} = (1, -1, \ldots, 1, -1; 0, 0, \ldots, 0, 0)^\top. \quad (25)$$

Thereafter, a simple break point and a non-break (sustain) point can occur alternatively until reaching an atomic monocenter. There is a series of spatial period doubling patterns associated with a set of groups

$$D_n \to D_{n/2} \to \cdots \to D_1, \quad (26)$$

where ($\to$) denotes an occurrence of spatial period doubling at a simple break bifurcation.

For example, Figure 4(a) depicts spatial period doubling patterns for $n = 16$ places. There are 16, 8, 4, 2, and 1 core (agglomerated) places shown by (◯). The agglomerated places are located equidistantly and the spatial period T between these places is doubled repeatedly.

Proposition 4. *The spatial period doubling patterns of the racetrack economy are trivial solutions.*

Proof. For these patterns, the group G in Proposition 2 is chosen as one of these groups in (26) to ensure the existence of a group permuting any two places with none-zero and identical population. This ensures Assumption 1, and, in turn, Proposition 2, thereby proving that the patterns are trivial solutions. □

(2) Lattice economy I: half spatial period doubling

A bifurcation rule of a spatial period doubling cascade of the lattice economy is presented below, while details of group-theoretic analysis are given in APPENDIX III. When n is even, at a simple break bifurcation point on the flat earth equilibrium, a bifurcating solution with the symmetry of $\langle r, s, p_1 p_2, p_1^{-1} p_2 \rangle$ branches in the direction of an eigenvector

$$\boldsymbol{\eta}_{\mathrm{La}} = \{\cos(\pi(n_1 - n_2)) \mid n_1, n_2 = 1, \ldots, n\} = \boldsymbol{\eta}_{\mathrm{Ra}} \otimes \boldsymbol{\eta}_{\mathrm{Ra}} \quad (27)$$

of the Jacobian matrix J (see (2) for the proof), where $\boldsymbol{\eta}_{\mathrm{R}}$ is the spatial period doubling eigenvector of the racetrack economy in (22). This pattern $\boldsymbol{\eta}_{\mathrm{La}}$ represents period doubling in the horizontal and the vertical directions. The lattice economy is linked to the racetrack economy via the tensor product structure in (27). Such a linkage is called herein a *squared tensor product linkage*.

We consider a case where the concentrating and the extinguishing proceed until reaching a non-break (sustain) point with a spatial period doubling pattern

$$\begin{aligned}\boldsymbol{\lambda}_{\rm La} &= \boldsymbol{\lambda}_{\rm Ra} \otimes \boldsymbol{\lambda}_{\rm Ra}\\ &= (2/n, 0, \ldots, 2/n, 0) \otimes (2/n, 0, \ldots, 2/n, 0), \end{aligned} \tag{28}$$

which is invariant to a group $\langle r, s, p_1 p_2, p_1^{-1} p_2\rangle$.

When $n = 2^m$ ($m = 2, 3, \ldots$), from the spatial period doubling pattern in (28), another doubling pattern branches in the direction:

$$\begin{aligned}\boldsymbol{\eta}_{\rm Lb} &= \boldsymbol{\eta}_{\rm Rb} \otimes \boldsymbol{\eta}_{\rm Rb}\\ &= (1, 0, -1, 0; \ldots; 1, 0, -1, 0)\\ &\quad \otimes (1, 0, -1, 0; \ldots; 1, 0, -1, 0), \end{aligned} \tag{29}$$

which is invariant to a group $\langle r, s, p_1^2, p_2^2\rangle$ (see (2)). In this manner, a series of spatial period doubling trivial solutions is engendered. This, for example, is shown in Fig. 4(b) ($n = 4$), which have 16, 8, 4, 2, and 1 core (agglomerated) places. There is a series of spatial period doubling patterns associated with a set of groups

$$\begin{aligned}&\langle r, s, p_1, p_2\rangle \ \rightarrow\ \langle r, s, p_1 p_2, p_1^{-1} p_2\rangle\\ &\quad \rightarrow\ \langle r, s, p_1^2, p_2^2\rangle \ \rightarrow\ \cdots\ \rightarrow\ {\rm D}_2. \end{aligned} \tag{30}$$

Proposition 5. *The spatial period doubling patterns of the lattice economy are trivial solutions.*

Proof. For these patterns, the group G in Lemma 2 is chosen as one of these groups in (30) to ensure the existence of a group permuting any two places with none-zero and identical population. This proves that the patterns are trivial solutions. □

This lattice economy has a spatial period T_{xy} in the x- and y-directions and $T_{\rm dia}$ in the two diagonal directions.[11] In the spatial period doubling cascade in (30), the spatial period doubling of T_{xy} and that of $T_{\rm dia}$ take place alternatively. This kind of spatial period doubling is called herein *half spatial period doubling* as half of the periods are doubled each time (see, e.g., Fig. 4(b)).

(3) Lattice economy: full spatial period doubling

There are other kinds of bifurcation cascades. When $n = 2^m$ ($m = 2, 3, \ldots$), from a double bifurcation point on the flat earth equilibrium, a bifurcating solution curve branches in the direction of the eigenvector in (29) ((3)):

$$\boldsymbol{\eta}_{\rm Lb} = \boldsymbol{\eta}_{\rm Rb} \otimes \boldsymbol{\eta}_{\rm Rb}. \tag{31}$$

[11] The diagonal distance is not measured by the road distance but by the Euclidean distance.

There are two series of spatial period doubling bifurcation cascades associated with a series of groups

$$\begin{aligned}&\langle r, s, p_1, p_2\rangle \ \Rightarrow\ \langle r, s, p_1^2, p_2^2\rangle\\ &\quad \Rightarrow\ \langle r, s, p_1^4, p_2^4\rangle \ \Rightarrow\ \cdots\ \Rightarrow\ {\rm D}_2, \end{aligned} \tag{32}$$

$$\begin{aligned}&\langle r, s, p_1 p_2, p_1^{-1} p_2\rangle \Rightarrow\ \langle r, s, (p_1 p_2)^2, (p_1^{-1} p_2)^2\rangle \ \Rightarrow\ \cdots\\ &\quad \Rightarrow\ \langle r, s, (p_1 p_2)^{n/2}, (p_1^{-1} p_2)^{n/2}\rangle, \end{aligned} \tag{33}$$

where ($\Rightarrow$) indicates an occurrence of spatial period doubling. This is called *full spatial period doubling* as spatial periods in all four directions are doubled.

Figure 5 depicts mixed occurrence of half and full doubling for $n = 4$. Twice repeated occurrence of half doubling corresponds to single occurrence of full doubling. Such a mixture of half doubling and full doubling makes the progress of agglomeration of the lattice economy more complex than that of the racetrack economy.

In this connection, spatial period doubling patterns are classified into *foursquare patterns* and *oblique patterns* as illustrated in Fig. 6. In foursquare patterns for a sufficiently large n, each first-level center is surrounded by four closest first-level centers located (see the red circle surrounded by four white circles in Fig. 6(a)). In oblique ones, each first-level center is surrounded by as many as eight first-level centers at the same transport distance (Fig. 6(b)). In this sense, the first-level centers of the oblique ones are more densely packed in comparison with those of the foursquare ones, thereby realizing a more favorable environment for inter-place trade. The first cascade in (32) occurs between foursquare ones, while the second cascade in (33) occurs between oblique ones. This classification is vital in the discussion of well-posedness of these patterns for the economic geography model (Section 8.).

7. Break point initiating spatial agglomeration

Formulas for break points for the analytically solvable model (Section 3.) are developed in this section. A break point is defined as the value of τ for the occurrence of a bifurcation that breaks uniformity. When investment in transportation infrastructure is committed, the break point indexes the functioning of this investment. Formulas for the lattice economy are newly developed and are presented in a synthetic manner to encompass the previous result for the racetrack economy (Ikeda, Akamatsu, and Kono, 2012 [13])).

The size n of the economy is chosen as 2 and $4m$ ($m = 1, 2, \ldots$). The total length $\mathcal{L}$ of the road on the racetrack

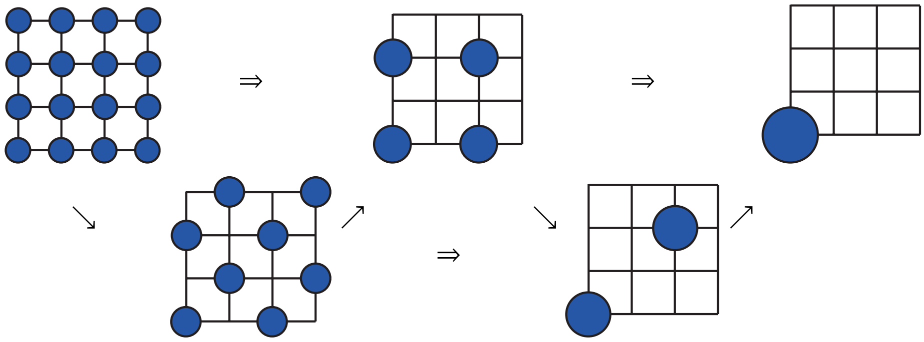
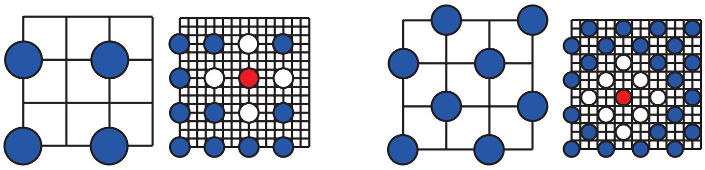


Fig.5 Spatial period doubling cascades for a lattice economy ($n = 4$); ($\Rightarrow$): full doubling; ($\searrow$) and ($\nearrow$): half doubling.



(a) Foursquare patterns (b) Oblique patterns

Fig.6 Foursquare and oblique spatial period doubling patterns.

is chosen as $\mathcal{L} = 1$, the spatial period of the lattice is also chosen as $\mathcal{L} = 1$, and neighboring places are connected by an inter-place road of the length $\tilde{L} = 1/n$.

(1) Fundamentals for deriving the formulas for break point

Breaking uniformity by bifurcation at the flat earth equilibrium $\boldsymbol{\lambda}^*$ is given by a zero eigenvalue of the Jacobian matrix $J(\boldsymbol{\lambda}^*)$. As worked out in (I.14)–(I.16), $J(\boldsymbol{\lambda}^*)$ is related to another Jacobian matrix $V(\boldsymbol{\lambda}^*) = \left(\partial v_i/\partial \lambda_j\right)(\boldsymbol{\lambda}^*)$ as

$$J(\boldsymbol{\lambda}^*) = \left(\frac{1}{K}I - \frac{1}{K^2}\mathbf{1}\mathbf{1}^\top\right)V(\boldsymbol{\lambda}^*) - \frac{\bar{v}}{K}\mathbf{1}\mathbf{1}^\top \tag{34}$$

with

$$V(\boldsymbol{\lambda}^*) = K\left[\kappa'\hat{D} + \left(I - \kappa\hat{D}\right)^{-1} \cdot \hat{D}\left(\kappa I - \hat{D}\right)\right], \tag{35}$$

where $\kappa = \frac{\mu}{\sigma}$, $\kappa' = \frac{\mu}{\sigma-1}$, and $\hat{D} = D/d$ is the normalized spatial discounting matrix. Here $D = (d_{ij})$ is defined by (4) and $d = d(r) = \sum_{j=1}^{K} d_{1j}$ with r being the trade freeness parameter introduced in (5). The spatial discounting matrices for the racetrack and the lattice economies are called D_{R} and D_{L}, respectively, and are given, for example, for $n = 2$ as

$$D_{\mathrm{R}} = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}, \quad D_{\mathrm{L}} = D_{\mathrm{R}} \otimes D_{\mathrm{R}} = \begin{bmatrix} 1 & r & r & r^2 \\ r & 1 & r^2 & r \\ r & r^2 & 1 & r \\ r^2 & r & r & 1 \end{bmatrix}. \tag{36}$$

We have the relation $D_{\mathrm{L}} = D_{\mathrm{R}} \otimes D_{\mathrm{R}}$ that connects the two economies, while the matrices D_{R} for $n = 2^m$ ($m = 2, 3, 4$), for example, are given in (1).

We present the following lemmas for the eigenproblems for the matrices $J(\boldsymbol{\lambda}^*)$, $V(\boldsymbol{\lambda}^*)$, and $\hat{D}$ (see (2) for the proof).

Lemma 1. *The matrices $J(\boldsymbol{\lambda}^*)$, $V(\boldsymbol{\lambda}^*)$, and $\hat{D}$ have the common eigenvector*

$$\boldsymbol{\eta} = \begin{cases} \boldsymbol{\eta}_{\mathrm{Ra}} \text{ in (22)} & \text{for the racetrack economy,} \\ \boldsymbol{\eta}_{\mathrm{La}} \text{ in (27)} & \text{for the lattice economy (half doubling),} \\ \boldsymbol{\eta}_{\mathrm{Lb}} \text{ in (31)} & \text{for the lattice economy (full doubling).} \end{cases} \tag{37}$$

Lemma 2. *The eigenvalues β, γ, and ϵ of the matrices $J(\boldsymbol{\lambda}^*)$, $V(\boldsymbol{\lambda}^*)$, and $\hat{D}$, respectively, for the common eigenvector $\boldsymbol{\eta}$ in (37) are related as*

$$\gamma = K[\kappa'\epsilon + (1 - \kappa\epsilon)^{-1} \cdot \epsilon(\kappa - \epsilon)], \tag{38}$$

$$\beta = \Psi(\epsilon) = \frac{\epsilon\{\kappa + \kappa' - (\kappa\kappa' + 1)\epsilon\}}{1 - \kappa\epsilon}. \tag{39}$$

The break point τ^* can be determined as follows. First, $\epsilon = \epsilon^*$ for the break point[12] satisfying ($\beta =)\Psi(\epsilon^*) = 0$ is given by $\epsilon^* = (\kappa + \kappa')/(\kappa\kappa' + 1)$ and is rewritten using $\kappa = \frac{\mu}{\sigma}$ and $\kappa' = \frac{\mu}{\sigma-1}$ as[13]

$$\epsilon^* = \frac{\mu(2\sigma - 1)}{\sigma(\sigma - 1) + \mu^2}. \tag{40}$$

Next, as shown in the sequel, the parameter for the remoteness r in (5) for the break point is given as a function of ϵ^* as $r^* = \Phi(\epsilon^*)$ with some function Φ. Last, the break point τ^* corresponding to $r = r^*$ can be determined from (5).

Remark 1. *The variable ϵ^* can be interpreted as an index for agglomeration as ϵ^* increases in association with an*

[12] From (39), $\beta = 0$ is satisfied also by $\epsilon = 0$, which represents redispersion. This case, however, is not a major interest of this paper, and is excluded hereafter.

[13] We have a no-black-hole condition $\frac{\mu}{\sigma-1} < 1$ (Forslid and Ottaviano, 2003 [9])) from (40) and $0 < \epsilon < 1$, which arises from (41) and (45) with $0 < r < 1$.

increase in μ or with a decrease of σ, both of which index a few large agglomerations.

(2) Formulas for break point: $n = 2$

As an illustration of basic ideas, formulas for break points are obtained for $n = 2$.[14] For the racetrack (two-place) economy with $D = D_{\rm R}$ in (36), we have

$$\hat{D}\boldsymbol{\eta} = \frac{D}{d}\boldsymbol{\eta} = \frac{1}{1+r}\begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1-r}{1+r}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \epsilon\boldsymbol{\eta}$$

with the eigenvalue $\epsilon = (1-r)/(1+r)$ and the eigenvector $\boldsymbol{\eta} = \boldsymbol{\eta}_{\rm R} = (1, -1)^\top$ for the spatial period doubling. Likewise, for the lattice economy, we have $\epsilon = (1-r)^2/(1+r)^2$ and $\boldsymbol{\eta} = \boldsymbol{\eta}_{\rm R} \otimes \boldsymbol{\eta}_{\rm R} = (1, -1, -1, 1)^\top$. The relation between ϵ and r for the two economies can be expressed in a synthetic manner as

$$\epsilon = \left(\frac{1-r}{1+r}\right)^p, \quad \text{i.e.,} \quad r = \frac{1-\epsilon^{1/p}}{1+\epsilon^{1/p}} \tag{41}$$

using a variable p expressing the squared tensor product linkage as

$$p = \begin{cases} 1 & \text{for the racetrack economy} \\ & \text{and the lattice economy (full doubling),} \\ 2 & \text{for the lattice economy (half doubling).} \end{cases} \tag{42}$$

The break point for $n = 2$ is expressed as

$$\tau^* = \frac{2}{\mathcal{L}(\sigma-1)}\log\left(\frac{1+(\epsilon^*)^{1/p}}{1-(\epsilon^*)^{1/p}}\right) \tag{43}$$

which gives the break point τ^* corresponding to $r = r^*$ with (5) and (41). Under a moderate assumption $\sigma \gg 1$, τ^* can be approximated as

$$\tau^* = \frac{2}{\mathcal{L}(\sigma-1)}\log\left(\frac{1+\epsilon^*}{1-\epsilon^*}\right) \approx \frac{2}{\mathcal{L}(\sigma-1)}2\epsilon^* \approx \frac{8\mu}{\mathcal{L}(\sigma-1)^2}. \tag{44}$$

(3) Formulas for break point: $n = 4m$ $(m = 1, 2, \ldots)$

For $n = 4m$ $(m = 1, 2, \ldots)$, similarly to the case of $n = 2$, we can advance the relation between ϵ and r as

$$\epsilon = \left(\frac{1-r}{1+r}\right)^{2p}, \tag{45}$$

which encompasses both economies via the squared tensor product linkage (42).

Proposition 6. *The break point of the racetrack and the lattice economies for $n = 4m$ $(m = 1, 2, \ldots)$ can be formulated in a synthetic manner as*

$$\tau^* = \frac{n}{\mathcal{L}(\sigma-1)}\log\left(\frac{1+(\epsilon^*)^{1/2p}}{1-(\epsilon^*)^{1/2p}}\right). \tag{46}$$

Proof. The relation (45) is solved for r as $r = \{1 + (\epsilon^*)^{1/2p}\}/\{1-(\epsilon^*)^{1/2p}\}$ and is substituted into (5) to arrive at (46). □

[14] The lattice economy with $n = 2$ is identical with the racetrack economy with $n = 4$.

Proposition 7. *As τ decreases from a large value for the lattice economy, the economic agglomeration is realized earlier for the half spatial doubling than for the full spatial doubling ($\tau^*_{\rm Lb} < \tau^*_{\rm La}$).*

Proof. For a given ϵ^*, (46) gives a larger τ^* for $p = 1$ than that for $p = 2$, which shows $\tau^*_{\rm R} = \tau^*_{\rm Lb} < \tau^*_{\rm La}$. □

Although the synthetic formula (46) is endowed with much desired independency on economic modeling, the influence of the parameter values σ and μ is contained implicitly in ϵ^* and is not transparent (Remark 1). As a remedy of this, we propose the following approximate formulas which make transparent the influence of the values of these parameters on the break point τ^*.

Proposition 8. *Under an assumption $\sigma \gg 1$, the break point τ^* for $n = 4m$ $(m = 1, 2, \ldots)$ are approximated by*

$$\begin{aligned} \tau^*_{\rm R} = \tau^*_{\rm Lb} &\approx 2^{3/2}\frac{n}{\mathcal{L}}\frac{\mu^{1/2}}{(\sigma-1)^{3/2}}, \\ \tau^*_{\rm La} &\approx 2^{5/4}\frac{n}{\mathcal{L}}\frac{\mu^{1/4}}{(\sigma-1)^{5/4}}. \end{aligned} \tag{47}$$

Proof. The proof of these formulas is similar to the proof of (44) for $n = 2$. □

Remark 2. *The formulas for $n = 2$ presented in* (43) *have different forms than the formulas* (46) *for $n \geq 4$. Such a difference, which may be attributable to the influence of far places for $n \geq 4$, demonstrates an insufficiency of the two-place economy as a two-dimensional spatial platform for economic activities.*

8. Comparative static analysis for the emergence of agglomeration

Spatial period doubling cascade of the two economies are studied in this section by comparative static analysis with respect to the transport cost of the economic geography model (Section 3.). The results of this analysis are examined in detail based on an ensemble of theoretical results in the previous sections: the theory of replicator dynamics (Section 5.), the bifurcation mechanism of spatial period doubling (Section 6.), and the formulas for the break point (Section 7.).

The size of the economies was chosen as $n = 2^m$ $(m = 1, 2, 3, 4)$; note that the lattice economy with $n = 2$ is identical with the racetrack economy with $n = 4$. Parameter values were set as $\alpha = 1.0$ and $(\sigma, \mu) = (10.0, 0.4)$, which satisfy the no-black hole condition (Footnote 13).

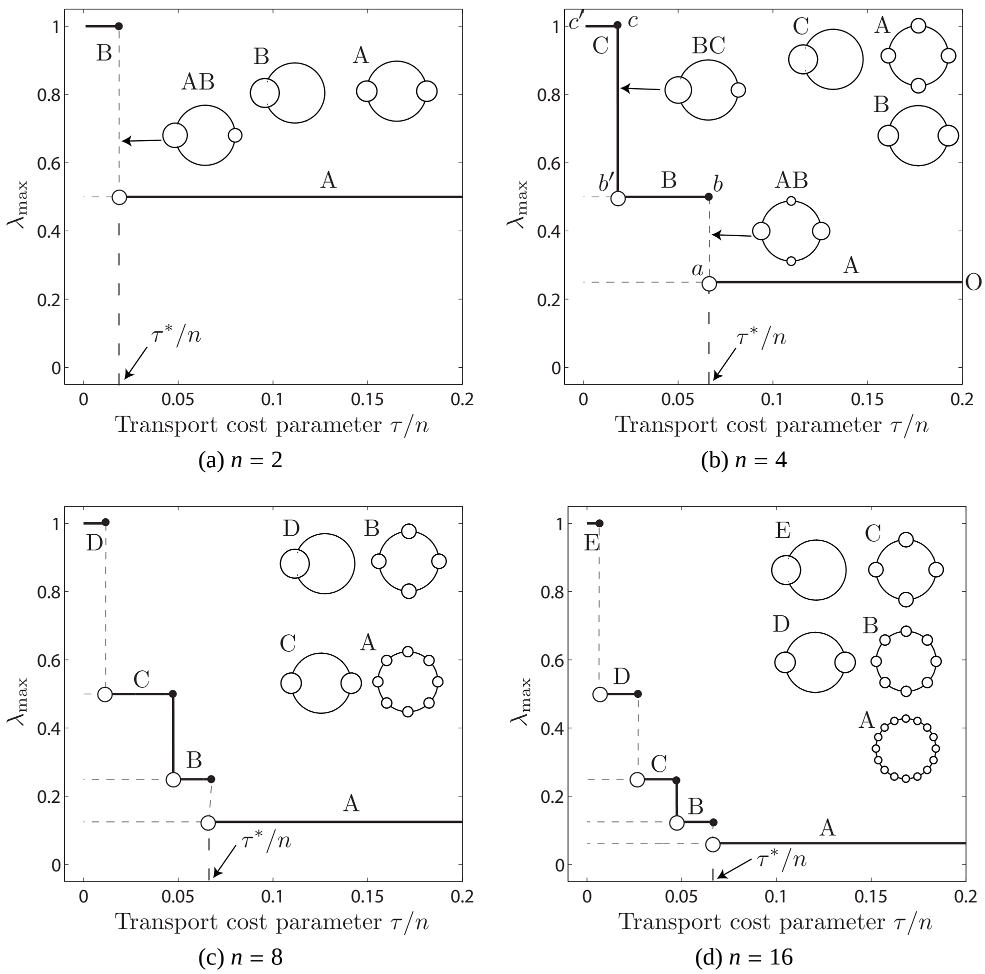


Fig.7 Curves of equilibria for the racetrack economy with n = 2, 4, 8, and 16 (solid lines denote stable equilibia and dashed curves does unstable ones; (○): a simple break bifurcation point; (●): a sustain point; $\lambda_{\max} = \max_{i=1}^{n} \lambda_i$).

(1) Racetrack economy

Curves of equilibria were computed for the racetrack economy (Fig. 7). The horizontal lines A to E denote spatial period doubling trivial equilibria, while non-horizontal curves denote bifurcating equilibria. Stable equilibria are shown by solid lines, and unstable ones by dashed lines. Every trivial solution was well-posed satisfying $\tau_{\rm B} < \tau_{\rm S}$ in (18) and it was possible to find a range of stable spatial equilibria of $\tau_{\rm B} < \tau < \tau_{\rm S}$, which starts from a sustain point and ends with a break point as τ decreases.

For example, for $n = 4$ (Fig. 7(b)), a spatial period doubling cascade between stable equilibria took place as follows. There was a stable flat earth equilibrium for $\tau > \tau^*$ (state A). At the break bifurcation point a at $\tau = \tau^*$ shown by (○), there emerged an unstable transient state AB with two large places and two small places that connect the break point a and the sustain point b. This state regained stability at the point b in the state B of two concentrated places and two extinguished places. Thereafter, at the break point b′, a stable transient state BC emerges en route to a stable atomic monocenter (state C starting from a sustain point c). For n = 8 and 16, there are cascades with more trivial equilibria. As τ decreases, stable equilibria shift to fewer and larger agglomerations. Thus the racetrack economy offer us with an idealistic agglomeration behavior that has been predicted theoretically (Section 6.).

Normalized break points τ^*/n of the flat earth equilibrium A are listed in Table 1(a). Their numerically computed values are in complete agreement with the theoretical ones by (43) or (46) and in good agreement with the approximate ones by (44) or or (47). Such an agreement is also seen (Table 1(b)) for the lattice economy (Section (2)). This suffices to show the validity of the formulas presented in this paper.

(2) Lattice economy

Curves of equilibria for the lattice economy (Fig. 9) displayed spatial period doubling cascade between the trivial equilibria A to I. As τ decreases, stable equilibria shifted to fewer and larger agglomerations.

Table1 Comparison of numerical, theoretical, and approximate break points (underlined values are approximate ones).

(a) Racetrack economy

	Number n of places	2	4	8	16
τ^*/n	Numerically computed	0.019	0.066	0.066	0.066
	Theoretical formula (43) or (46)	0.019	0.066	0.066	0.066
	Approximate formula (44) or (47)	$\underline{0.020}$	$\underline{0.066}$	$\underline{0.066}$	$\underline{0.066}$

(b) Lattice economy

	Number n of places	2	4	8	16
τ^*/n	Numerically computed	0.066	0.134	0.134	0.134
	Theoretical formula (43) or (46)	0.066	0.134	0.134	0.134
	Approximate formula (44) or (47)	$\underline{0.066}$	$\underline{0.121}$	$\underline{0.121}$	$\underline{0.121}$

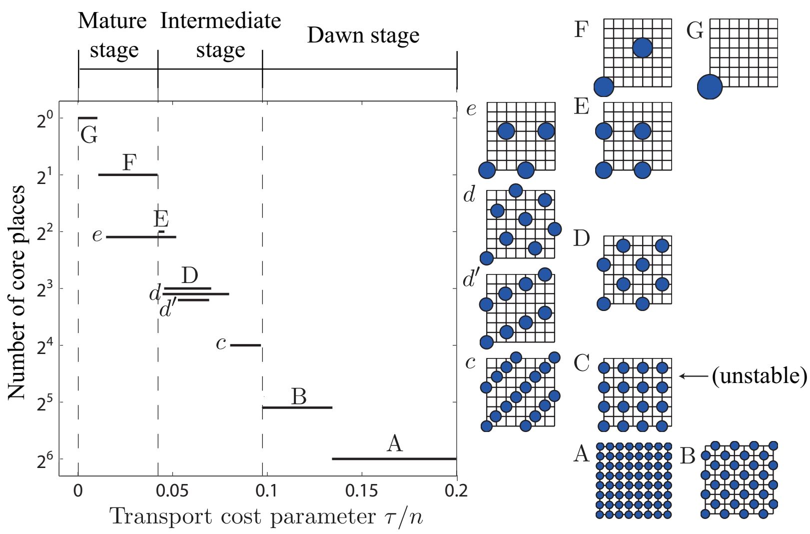


Fig.8 Durations of stable states for n = 8.

Yet, unlike the racetrack economy, not all trivial equilibria were stable. There were several ill-posed solutions, such as C for $n = 4$, 8 and 16 and E for $n = 16$, while most of the solutions were well-posed satisfying $\tau_B < \tau_S$ in (18). All these ill-posed ones were foursquare patterns (cf., Fig. 6(a)), whereas all oblique patterns (cf., Fig. 6(b)) were well-posed. Even well-posed foursquare ones, such as E for $n = 8$ and G for $n = 16$, had very short durations of stable equilibria.

It is possible to classify the progress of agglomeration into three stages: *dawn stage*, *intermediate stage*, and *mature stage*.[15] In the dawn stage, half spatial period doubling between two stable equilibria A and B took place for all cases ($n = 4, 8, 16$). At this stage, the underlying predominance of the market-crowding effect is weakened by an increase in the market-access effect that enlarges the agglomeration force. This reorganizes firms into locations with greater competition, thereby engendering the oblique pattern B. This pattern may be interpreted as a square lattice counterpart of a hexagonal in central place theory.

In the intermediate stage, in which the market-crowding effect gradually decreases, whereas the market-access effect increases, we found that the equilibrium C was ill-posed and there were no stable equilibria for all cases. Full doubling[16] B⇒D took place bypassing C and connecting stable equilibria of B and D. The equilibrium E was also ill-posed for $n = 16$ and full doubling D⇒F took place bypassing E and connecting stable equilibria of D and F.

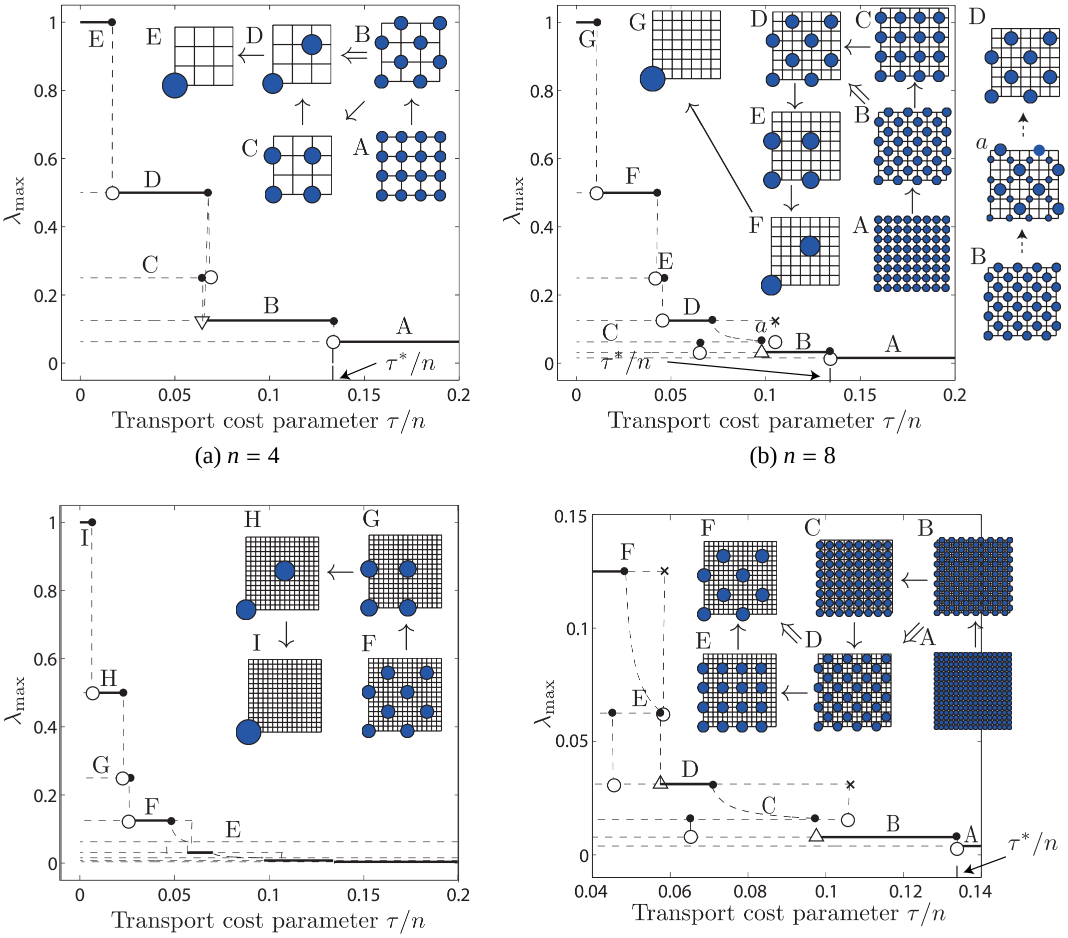
In the mature stage, in which the market-access effect greatly decreases and the dispersion force arising from the taste heterogeneity of workers prevails, stability was regained for all cases and spatial period doubling cascade proceeded stably as

$$\begin{cases} \mathrm{D} \to \mathrm{E} & \text{for } n = 4, \\ \mathrm{D} \to \mathrm{E} \to \mathrm{F} \to \mathrm{G} & \text{for } n = 8, \\ \mathrm{D} \to \mathrm{E} \to \mathrm{F} \to \mathrm{G} \to \mathrm{H} \to \mathrm{I} & \text{for } n = 16. \end{cases}$$

Thus a larger n has entailed more repeated occurrence of

[15] This classification was introduced for the hexagonal lattice economy (Ikeda, Murota, and Takayama, 2017b [18])

[16] For $n = 4$, a break bifurcation in B led directly to D. For $n = 8$ and 16, a break bifurcation in B, followed by a non-break bifurcation, led to D.



(a) n = 4

(b) n = 8

(c) n = 16

Fig.9 Curves of equilibria for the lattice economy with $n = 4$, 8, and 16 (solid lines denote stable solutions and dashed ones does unstable ones; (○): a simple break bifurcation point; (●): a sustain point; (△): a double bifurcation point; (▽): a triple bifurcation point; ×: a non-break point; $\lambda_{\max} = \max_{i=1}^{K} \lambda_i$).

stable half doubling that is quite similar to the spatial period doubling cascade of the racetrack economy. Such similarity ensures the usefulness of the racetrack economy analogy proposed in this paper.

There were several ranges of τ in which stable equlibria are absent in the intermediate stage for $n = 8$ and 16. To supplement such absence, the durations of stable states were investigated for $n = 8$ with also reference to other (non-doubling) equilibria that were obtained based on Proposition 2. Figure 8 depicts these durations comparatively for those of the spatial period doubling equilibria A to G. In the dawn stage, A and B were only stable equilibria. In the intermediate stage and at the beginning of the mature stage, we encountered various kinds of stable trivial equilibria[17] c, d, d′, and e with stripe-like patterns, as well as the spatial period doubling ones D and E. In the last stage of the mature stage, a few and large agglomerations, such as F, G, and e were predominant. Thus we have arrived at a more complete transition of stable equilibria engendering a fewer and larger agglomerations as τ decreases.

[17] Such emergence of various kinds of equilibria was observed also for a hexagonal lattice (Ikeda, Murota, and Takayama, 2017 [18]).

9. Conclusion

A racetrack economy analogy was proposed by highlighting this economy as a one-dimensional counterpart of the two-dimensional economic agglomerations. As a novel contribution of this paper, qualitative aspects of these agglomerations in a lattice economy were described in a general setting by bifurcation theory.

A symmetry condition for the existence of trivial solutions in replicator dynamics was formulated and in turn to advance spatial patterns of various kinds. In particular, spatial period doubling patterns were set forth as important trivial solutions for both economies. Spatial period doubling cascade between these patterns was advanced as a theoretically possible course of the progress of agglomeration and was demonstrated to exist in both economies for an economic geography model. That theory was vital in the understanding of the complicated agglomeration behavior of the lattice economy.

A progress of stable equilibria in association with decreasing transport cost τ in the lattice economy was observed for the economic geography model. In the dawn stage with large τ and in the mature stage with small τ, spatial period doubling cascade was quite predominant, thereby demonstrating the validity and usefulness of the racetrack economy analogy proposed in this paper. In the intermediate stage, however, equilibria of various kinds with stripe-like patterns were fund to be stable.

As a quantitative measure of spatial agglomerations, analytical formulas for the break point were proposed for the economic geography model. In particular, those for the lattice economy were newly developed. The break points of both economies were expressed in a synthetic manner with the aid of the squared tensor product linkage. The validity of all these formulas has been ensured by the comparative static analyses (Section 8.).

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APPENDIX I Details of the modeling of spatial economy

The budget constraint is given as

$$p_i^{\mathrm{A}} C_i^{\mathrm{A}} + \sum_{j=1}^{K} \int_0^{n_j} p_{ji}(\ell) q_{ji}(\ell) d\ell = Y_i, \tag{I.1}$$

where p_i^{A} is the price of A-sector goods in place i, $p_{ji}(\ell)$ is the price of a variety ℓ in place i produced in place j, and Y_i is the income of an individual in place i. The incomes (wages) of skilled workers and unskilled workers are represented, respectively, by w_i and w_i^{L}.

An individual in place i maximizes the utility in (1) subject to the budget constraint in (I.1). This yields the following demand functions of

$$C_i^{\mathrm{A}} = (1-\mu)\frac{Y_i}{p_i^{\mathrm{A}}}, \quad C_i^{\mathrm{M}} = \mu\frac{Y_i}{\rho_i}, \quad q_{ji}(\ell) = \mu\frac{\rho_i^{\sigma-1} Y_i}{p_{ji}(\ell)^{\sigma}}, \tag{I.2}$$

where ρ_i denotes the price index of the differentiated products in place i, which is

$$\rho_i = \left(\sum_{j=1}^{K} \int_0^{n_j} p_{ji}(\ell)^{1-\sigma} d\ell \right)^{1/(1-\sigma)}. \tag{I.3}$$

Because the total income in place i is $w_i\lambda_i + w_i^{\mathrm{L}}$, the total demand $Q_{ji}(\ell)$ in place i for a variety ℓ produced in place j is given as

$$Q_{ji}(\ell) = \mu\frac{\rho_i^{\sigma-1}}{p_{ji}(\ell)^{\sigma}}(w_i\lambda_i + w_i^{\mathrm{L}}). \tag{I.4}$$

The A-sector is perfectly competitive and produces homogeneous goods under constant-returns-to-scale technology, which requires one unit of unskilled labor per unit output. A-sector goods are transported without transportation cost and are chosen as the numéraire. In equilibrium, we have $p_i^{\mathrm{A}} = w_i^{\mathrm{L}} = 1$ for each i.

The M-sector output is produced under increasing-returns-to-scale technology and Dixit-Stiglitz monopolistic competition. A firm incurs a fixed input requirement of α units of skilled labor and a marginal input requirement of β units of unskilled labor. An M-sector firm located in place i chooses $(p_{ij}(\ell) \mid j = 1, \dots, K)$ that maximizes its profit

$$\Pi_i(\ell) = \sum_{j=1}^{K} p_{ij}(\ell) Q_{ij}(\ell) - (\alpha w_i + \beta x_i(\ell)), \tag{I.5}$$

where $x_i(\ell)$ denotes the total supply of variety ℓ produced in place i and $(\alpha w_i + \beta x_i(\ell))$ signifies the cost function introduced by Flam and Helpman (1987).

With the use of the iceberg form of the transport cost, we have

$$x_i(\ell) = \sum_{j=1}^{K} T_{ij} Q_{ij}(\ell). \tag{I.6}$$

Then the profit function of an M-sector firm in place i, given in (I.5) above, can be rewritten as

$$\Pi_i(\ell) = \sum_{j=1}^{K} p_{ij}(\ell) Q_{ij}(\ell) - \left(\alpha w_i + \beta \sum_{j=1}^{K} T_{ij} Q_{ij}(\ell) \right), \tag{I.7}$$

which is maximized by the firm. The first-order condition for this profit maximization yields

$$p_{ij}(\ell) = \frac{\sigma\beta}{\sigma-1} T_{ij}. \tag{I.8}$$

This implies that $p_{ij}(\ell)$, $Q_{ij}(\ell)$, and $x_i(\ell)$ are independent of ℓ. Therefore, argument ℓ is suppressed in the sequel.

(1) Market equilibrium

In the short run, skilled workers are immobile between places, i.e., their spatial distribution $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_K)$ is assumed to be given. The market equilibrium conditions consist of three conditions: the M-sector goods market clearing condition, the zero-profit condition attributable to the free entry and exit of firms, and the skilled labor market clearing condition. The first condition is written as (I.6) above. The second requires that the operating profit of a firm, given in (I.5), be absorbed entirely by the wage bill of its skilled workers. This gives

$$w_i = \frac{1}{\alpha}\left\{ \sum_{j=1}^{K} p_{ij} Q_{ij} - \beta x_i \right\}. \tag{I.9}$$

The third condition is expressed as $\alpha n_i = \lambda_i$ and the price index ρ_i in (I.3) can be rewritten using (I.8) as

$$\rho_i = \frac{\sigma\beta}{\sigma-1}\left(\frac{1}{\alpha} \sum_{j=1}^{K} \lambda_j d_{ji} \right)^{1/(1-\sigma)}. \tag{I.10}$$

The market equilibrium wage w_i in (I.9) can be represented as

$$w_i = \frac{\mu}{\sigma} \sum_{j=1}^{K} \frac{d_{ij}}{\Delta_j}(w_j\lambda_j + 1) \tag{I.11}$$

using (4), (I.4), (I.6), (I.8), and (I.10). Here, $\Delta_j = \sum_{k=1}^{K} d_{kj}\lambda_k$. Equation (I.11) is solvable for w_i as follows. With the notation (7), (I.11) can be written as

$$\boldsymbol{w} = \frac{\mu}{\sigma} D\Delta^{-1}(\Lambda\boldsymbol{w} + \mathbf{1}), \tag{I.12}$$

which is solved for $\boldsymbol{w}$ as

$$\boldsymbol{w} = \frac{\mu}{\sigma}\left(I - \frac{\mu}{\sigma} D\Delta^{-1}\Lambda \right)^{-1} D\Delta^{-1}\mathbf{1}. \tag{I.13}$$

From the equilibrium equation $\boldsymbol{F}$ in (12) with (11), we have

$$\frac{\partial F_i}{\partial \lambda_j} = \left(v_i - \sum_{k=1}^{K} \lambda_k v_k\right)\delta_{ij} + \lambda_i \left(\frac{\partial v_i}{\partial \lambda_j} - v_j - \sum_{k=1}^{K} \lambda_k \frac{\partial v_k}{\partial \lambda_j}\right), \qquad \text{(I.14)}$$

where δ_{ij} is the Kronecker delta. This shows that the Jacobian matrices $J(\boldsymbol{\lambda}) = \partial \boldsymbol{F}/\partial \boldsymbol{\lambda}$ and $V(\boldsymbol{\lambda}) = \partial \boldsymbol{v}/\partial \boldsymbol{\lambda}$ are related as

$$J(\boldsymbol{\lambda}) = \mathrm{diag}(v_1 - \bar{v}, \ldots, v_K - \bar{v}) + (\Lambda - \boldsymbol{\lambda}\boldsymbol{\lambda}^\top)V(\boldsymbol{\lambda}) - \boldsymbol{\lambda}\boldsymbol{v}^\top, \quad \text{(I.15)}$$

where $\bar{v} = \sum_{i=1}^{K} \lambda_i v_i$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_K)$, and $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{\lambda}, \tau) = (v_1(\boldsymbol{\lambda}, \tau), \ldots, v_K(\boldsymbol{\lambda}, \tau))^\top$.

At the flat earth equilibrium with $v_1 = \cdots = v_K = \bar{v}$, (I.15) gives

$$J(\boldsymbol{\lambda}^*) = \left(\frac{1}{K} I - \frac{1}{K^2}\mathbf{1}\mathbf{1}^\top\right) V(\boldsymbol{\lambda}^*) - \frac{\bar{v}}{K}\mathbf{1}\mathbf{1}^\top. \qquad \text{(I.16)}$$

APPENDIX II Details associated with trivial solutions in Section 5.

We present details of trivial solutions. First, the proof of Proposition 2 is given as follows: Since the m places belonging to $\boldsymbol{\lambda}_+$ are permuted each other by $T_+(g)$ ($g \in G$), we have $v_i = \bar{v}$ ($i = 1, \ldots, m$), thereby satisfying $\boldsymbol{F}_+(\frac{1}{m}\mathbf{1}, \mathbf{0}, \tau) = \mathbf{0}$. For $K - m$ places with no population, we have $\lambda_j = 0$, thereby satisfying $\boldsymbol{F}_0(\frac{1}{m}\mathbf{1}, \mathbf{0}, \tau) = \mathbf{0}$. This shows that $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \tau) = (\frac{1}{m}\mathbf{1}, \mathbf{0}, \tau)$ serves as a trivial solution.

Next, the proof of Corollary 1 reads: For an atomic monocenter for $m = 1$, Assumption 1 is satisfied by a group $G = \langle e \rangle$ and $T_+(e) = 1$. Then Proposition 2 guarantees that the corner solution of an atomic monocenter is a trivial solution. For twin places for $m = 2$, Assumption 1 is satisfied by a group $G = \langle h \rangle$ and

$$T_+(h) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where h denotes an exchange symmetry, i.e., $1 \leftrightarrow 2$. Then Proposition 2 guarantees that the corner solution for twin places is a trivial solution.

Last, the pattern in the left of Fig. 3(c), for example, is invariant to $\mathrm{D}_1 = \langle s \rangle$, i.e., the reflection $y \mapsto -y$. This invariance is expressed by the representation matrix

$$T_+(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

which permutes the places 2 and 3 but retains the place 1 unchanged. Since there is no exchange symmetry between place 1 and other places, Assumption 1 is not satisfied. Hence that pattern is not a trivial solution in Proposition 2. The existence of a stationary point with this pattern is conditional on the value of τ.

APPENDIX III Bifurcation of the lattice economy

After a brief introduction of group-theoretic bifurcation theory, bifurcation of the lattice economy is described.

(1) Outline of group-theoretic bifurcation theory

Consider a critical point $(\boldsymbol{\lambda}^*, \tau_c)$ on the flat earth equilibrium curve, which is said to have multiplicity $M (\geq 1)$ if the Jacobian matrix $J = \partial \boldsymbol{F}/\partial \boldsymbol{\lambda}$ of $\boldsymbol{F}$ at $(\boldsymbol{\lambda}^*, \tau_c)$ has M zero eigenvalues. Let $(\boldsymbol{\eta}_i \mid i = 1, \ldots, K)$ be an orthonormal basis of $\mathbb{R}^K$ such that

$$J\boldsymbol{\eta}_i = \mathbf{0}, \quad i = 1, \ldots, M. \qquad \text{(III.1)}$$

We express the variable $\boldsymbol{\lambda}$ as $\boldsymbol{\lambda} = \boldsymbol{\lambda}^* + \sum_{i=1}^{M} \xi_i \boldsymbol{\eta}_i$ and τ as $\tau = \tau_c + \tilde{\tau}$, where $\tilde{\tau}$ denotes an increment of τ.

The full system of equations $\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \mathbf{0}$ in (12) is reduced,[18] in a neighborhood of $(\boldsymbol{\lambda}^*, \tau_c)$, to a system of M equations (called bifurcation equations)

$$\tilde{\boldsymbol{F}}(\boldsymbol{\xi}, \tilde{\tau}) = \mathbf{0} \qquad \text{(III.2)}$$

for some function $\tilde{\boldsymbol{F}}$ in $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_M) \in \mathbb{R}^M$ and $\tilde{\tau} \in \mathbb{R}$ defined above. In this reduction process, the symmetry condition (13) of the full system is inherited by the reduced system (III.2).

The reduced equation (III.2) is to be solved for $\boldsymbol{\xi}$ as $\boldsymbol{\xi} = \boldsymbol{\xi}(\tilde{\tau})$, which is often possible by virtue of the symmetry inherited by $\tilde{\boldsymbol{F}}$. Because $(\boldsymbol{\xi}, \tilde{\tau}) = (\mathbf{0}, 0)$ is a singular (critical) point of (III.2), there can be many solutions $\boldsymbol{\xi} = \boldsymbol{\xi}(\tilde{\tau})$ with $\boldsymbol{\xi}(0) = \mathbf{0}$, which give rise to bifurcation. Each $\boldsymbol{\xi}$ uniquely determines a solution $\boldsymbol{\lambda}$ of the full system (12). Among the (critical) eigenvectors $\sum_{i=1}^{M} \xi_i \boldsymbol{\eta}_i$, only those vectors which satisfy (III.2) are related to bifurcating solutions, whereas those which do not satisfy (III.2) are not. In this way possible bifurcating modes can be predicted using group-theoretic bifurcation analysis.

(2) Half spatial period doubling

A simple break bifurcation point of the lattice is associated with the one-dimensional irreducible representation μ, which exists only when n is even and is given by

$$T^\mu(r) = 1, \quad T^\mu(s) = 1, \quad T^\mu(p_1) = -1, \quad T^\mu(p_2) = -1. \qquad \text{(III.3)}$$

[18] This is a standard procedure called the *Liapunov–Schmidt reduction with symmetry* (Golubitsky, Stewart, and Schaeffer, 1988 [12]).

that satisfy the fundamental relations (Footnote 5). We assume that the variable $\boldsymbol{w} = w$ for the bifurcation equation (III.2) corresponds to the column vectors of

$$\begin{aligned}\boldsymbol{\eta} &= \{\cos(\pi(n_1 - n_2)) \mid n_1, n_2 = 1, \ldots, n\} \\ &= \{1, -1, \ldots, 1, -1;\ -1, 1, \ldots, -1, 1; \\ &\qquad \ldots;\ -1, 1, \ldots, -1, 1\}. \end{aligned} \tag{III.4}$$

As stated in (27), when n is even, a bifurcating solution in the direction of $\boldsymbol{\eta}$ with the symmetry of $\Sigma = \langle r, s, p_1 p_2, p_1^{-1} p_2\rangle$ arises from a critical point of multiplicity 1 associated with the irreducible representation μ. The proof of this statement is given below.

The fixed-point subspace of Σ for T^μ is given by

$$\begin{aligned}\mathrm{Fix}^\mu(\Sigma) &= \{\boldsymbol{w} \in \mathbb{R}^M \mid T^\mu(g)\boldsymbol{w} = \boldsymbol{w} \text{ for all } g \in \Sigma\} \\ &= \{w \in \mathbb{R}\}\end{aligned} \tag{III.5}$$

since $\boldsymbol{w} = w$ and

$$\begin{aligned}&T^\mu(r)w = w, \quad T^\mu(s)w = w, \\ &T^\mu(p_1 p_2)w = (-1)(-1)w = w, \\ &T^\mu(p_1^{-1} p_2)w = (-1)(-1)w = w\end{aligned}$$

by (III.3). Thus the fixed-point subspace $\mathrm{Fix}^\mu(\Sigma)$ of the targeted symmetry Σ is one-dimensional. The equivariant branching lemma then guarantees the existence of a bifurcating path with symmetry Σ (see Chapter 8 of Ikeda and Murota, 2014 [14]) for details of the equivariant branching lemma).

Secondary and further bifurcations for the lattice can be dealt with similarly. For example, for the secondary bifurcation, if we set $P_1 = p_1 p_2$ and $P_2 = p_1^{-1} p_2$, we have the relations

$$\langle r, s, p_1 p_2, p_1^{-1} p_2\rangle = \langle r, s, P_1, P_2\rangle,$$

$$\langle r, s, p_1^2, p_1^2\rangle = \langle r, s, P_1 P_2, P_1^{-1} P_2\rangle.$$

Thus the bifurcation analysis on the groups $\langle r, s, P_1, P_2\rangle$ and $\langle r, s, P_1 P_2, P_1^{-1} P_2\rangle$ is identical with that on the groups $\langle r, s, p_1, p_2\rangle$ and $\langle r, s, p_1 p_2, p_1^{-1} p_2\rangle$.

(3) Full spatial period doubling

A double bifurcation point is associated with the two-dimensional irreducible representation μ, which exists only when n is even, and is given by

$$T^\mu(r) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \ T^\mu(s) = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix},$$
$$T^\mu(p_1) = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, \ T^\mu(p_2) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \tag{III.6}$$

Let us assume that the variable $\boldsymbol{w} = (w_1, w_2)^\top$ for the bifurcation equation (III.2) corresponds to the vectors

$$\{\cos(\pi n_1) \mid n_1, n_2 = 1, \ldots, n\}, \{\cos(\pi n_2) \mid n_1, n_2 = 1, \ldots, n\}. \tag{III.7}$$

When n is even, bifurcating solutions from a critical point of multiplicity 2 associated with the irreducible representation μ exist in the direction: $\boldsymbol{q}_1 + \boldsymbol{q}_2$ with the symmetry of $\langle r, s, p_1^2, p_2^2\rangle$. The existence is shown below. Note

$$\mathrm{Fix}^\mu(\langle r, s, p_1^2, p_2^2\rangle) = \mathrm{Fix}^\mu(\langle r\rangle) \cap \mathrm{Fix}^\mu(\langle s, p_1^2, p_2^2\rangle).$$

Here we have $\mathrm{Fix}^\mu(\langle r\rangle) = \{c(1,1)^\top \mid c \in \mathbb{R}\}$ since $T^\mu(r)(w_1, w_2)^\top = (w_2, w_1)^\top$ by (III.6), whereas $\mathrm{Fix}^\mu(\langle s, p_1^2, p_2^2\rangle) = \mathbb{R}^2$ since $T^\mu(s) = T^\mu(p_1^2) = T^\mu(p_2^2) = I$ by (III.6). Therefore,

$$\mathrm{Fix}^\mu(\Sigma) = \{c(1,1)^\top \mid c \in \mathbb{R}\},$$

that is, $\Sigma = \Sigma^\mu(\boldsymbol{w}_0)$ for $\boldsymbol{w}_0 = (1,1)^\top$. Thus the targeted symmetry Σ is an isotropy subgroup with $\dim \mathrm{Fix}^\mu(\Sigma) = 1$. The equivariant branching lemma then guarantees the existence of a bifurcating path with symmetry Σ.

Secondary and further bifurcations for full spatial period doubling can be dealt with similarly.

APPENDIX IV Details of derivation of formulas for break points

Details of derivation of formulas for break points in Section 7. are presented. In regard to $V(\boldsymbol{\lambda})$ we recall (8):

$$v_i = \frac{\mu}{\sigma - 1} \ln \Delta_i + \ln w_i \tag{IV.1}$$

as well as (I.11):

$$w_i = \frac{\mu}{\sigma} \sum_k \frac{d_{ik}}{\Delta_k}(w_k \lambda_k + 1), \tag{IV.2}$$

where

$$\Delta_k = \Delta_k(\boldsymbol{\lambda}, \tau) = \sum_{j=1}^K d_{jk}\lambda_j.$$

The differentiations of (IV.1) and (IV.2) with respect to λ_j yield, respectively,

$$\frac{\partial v_i}{\partial \lambda_j} = \kappa' \frac{d_{ji}}{\Delta_i} + \frac{1}{w_i}\frac{\partial w_i}{\partial \lambda_j}, \tag{IV.3}$$

$$\frac{\partial w_i}{\partial \lambda_j} = \kappa \sum_{k=1}^K \frac{d_{ik}}{{\Delta_k}^2}\left[\left(\frac{\partial w_k}{\partial \lambda_j}\lambda_k + w_k \delta_{kj}\right)\Delta_k - (w_k\lambda_k + 1)d_{jk}\right] \tag{IV.4}$$

where

$$\kappa = \frac{\mu}{\sigma}, \quad \kappa' = \frac{\mu}{\sigma - 1}. \tag{IV.5}$$

We have $0 < \kappa < 1$ and $\kappa' > 0$ because $\sigma > 1$, $0 < \mu < 1$.

The matrix $V(\boldsymbol{\lambda}^*)$ in (34) can be evaluated as shown below. At $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$, we have

$$\Delta_j = \Delta_j(\boldsymbol{\lambda}^*, \tau) = \sum_{k=1}^K d_{kj}\lambda_k = \frac{d}{K}.$$

Because w_j is independent of j, we may put $w_j = w$; then (IV.2) becomes

$$w = \kappa \sum_{j=1}^{K} \frac{K}{d}\, d_{ij} \left(\frac{w}{K} + 1\right) = \kappa\, (w + K)\,,$$

which yields

$$w = \frac{\kappa K}{1-\kappa}. \tag{IV.6}$$

At $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$, (IV.4) becomes

$$\frac{\partial w_i}{\partial \lambda_j} = \kappa \sum_{k=1}^{K} \frac{K^2}{d^2}\, d_{ik} \left[\left(\frac{1}{K}\frac{\partial w_k}{\partial \lambda_j} + w\delta_{kj}\right)\frac{d}{K} - \left(\frac{w}{K} + 1\right) d_{jk}\right],$$

which in matrix form reads as

$$W = \kappa \frac{K^2}{d^2}\, D\left[\frac{d}{K}\left(\frac{1}{K}W + wI\right) - \frac{w+K}{K} D\right]$$

with $W = (\partial w_i/\partial \lambda_j)$. With the use of (IV.6), this equation can be rewritten as

$$\left(I - \kappa\frac{D}{d}\right) W = Kw\, \frac{D}{d}\left(\kappa I - \frac{D}{d}\right),$$

which can be further rewritten as

$$W = Kw\left(I - \kappa\frac{D}{d}\right)^{-1} \cdot \frac{D}{d}\left(\kappa I - \frac{D}{d}\right).$$

Then the partial derivatives in (IV.3) can be evaluated in matrix form as

$$V(\boldsymbol{\lambda}^*) = K\left[\kappa' \frac{D}{d} + \left(I - \kappa\frac{D}{d}\right)^{-1} \cdot \frac{D}{d}\left(\kappa I - \frac{D}{d}\right)\right]. \tag{IV.7}$$

(1) Spatial discounting matrix

For the racetrack economy, the spatial discounting matrix D for $n = 4$ is given as

$$D_{\mathrm{R}} = \begin{bmatrix} 1 & r & r^2 & r \\ r & 1 & r & r^2 \\ r^2 & r & 1 & r \\ r & r^2 & r & 1 \end{bmatrix}, \tag{IV.8}$$

the matrix for $n = 8$ is given as

$$D_{\mathrm{R}} = R_8 = \begin{bmatrix} \tilde{R}_8 & \hat{R}_8 \\ \hat{R}_8 & \tilde{R}_8 \end{bmatrix} \quad \text{with} \quad \tilde{R}_8 = \begin{bmatrix} 1 & r & r^2 & r^3 \\ r & 1 & r & r^2 \\ r^2 & r & 1 & r \\ r^3 & r^2 & r & 1 \end{bmatrix}, \quad \hat{R}_8 = r^4 \begin{bmatrix} 1 & r^{-1} & r^{-2} & r^{-3} \\ r^{-1} & 1 & r^{-1} & r^{-2} \\ r^{-2} & r^{-1} & 1 & r^{-1} \\ r^{-3} & r^{-2} & r^{-1} & 1 \end{bmatrix},$$

and that for $n = 16$ is given as

$$D_{\mathrm{R}} = R_{16} = \begin{bmatrix} \tilde{R}_8 & \hat{R}_{16} & r^4\hat{R}_8 & \hat{R}_{16}^\top \\ \hat{R}_{16}^\top & \tilde{R}_8 & \hat{R}_{16} & r^4\hat{R}_8 \\ r^4\hat{R}_8 & \hat{R}_{16}^\top & \tilde{R}_8 & \hat{R}_{16} \\ \hat{R}_{16} & r^4\hat{R}_8 & \hat{R}_{16}^\top & \tilde{R}_8 \end{bmatrix} \quad \text{with} \quad \hat{R}_{16} = \begin{bmatrix} r^4 & r^5 & r^6 & r^7 \\ r^3 & r^4 & r^5 & r^6 \\ r^2 & r^3 & r^4 & r^5 \\ r & r^2 & r^3 & r^4 \end{bmatrix}.$$

(2) Proof of Lemmas 1 and 2

First, (14) gives a commutability $T(g)J(\boldsymbol{\lambda}^*) = J(\boldsymbol{\lambda}^*)T(g)$ $(g \in G)$ holds for the group G that labels the symmetry of each economy. Next, from (34), we have

$$T(g)\left(\frac{1}{K}I - \frac{1}{K^2}\mathbf{1}\mathbf{1}^\top\right)V(\boldsymbol{\lambda}^*) - T(g)\frac{\bar{v}}{K}\mathbf{1}\mathbf{1}^\top = \left(\frac{1}{K}I - \frac{1}{K^2}\mathbf{1}\mathbf{1}^\top\right)V(\boldsymbol{\lambda}^*)T(g) - \frac{\bar{v}}{K}\mathbf{1}\mathbf{1}^\top T(g),$$

which gives a commutability $T(g)V(\boldsymbol{\lambda}^*) = V(T(g)\boldsymbol{\lambda}^*)$ by $T(g)\mathbf{1}\mathbf{1}^\top = \mathbf{1}\mathbf{1}^\top T(g) = \mathbf{1}\mathbf{1}^\top$ and $\mathbf{1}\mathbf{1}^\top V(\boldsymbol{\lambda}^*) = V(\boldsymbol{\lambda}^*)\mathbf{1}\mathbf{1}^\top = \hat{V}$, where $\hat{V}$ is the sum of the entries of a column of $V(\boldsymbol{\lambda}^*)$ that is identical for all the columns by the symmetry of the system. Last, from (35), we have a commutability $T(g)\hat{D} = \hat{D}T(g)$. These three commutabilities guarantee the existence of the common eigenvector $\boldsymbol{\eta}$. A concrete form of $\boldsymbol{\eta}$ can be determined uniquely by adapting the method for the hexagonal lattice (Ikeda and Murota, 2014, Section 7.5 [14]).

Multiplying $\boldsymbol{\eta}$ to $V(\boldsymbol{\lambda}^*)$ in (35) from the right and using $\hat{D}\boldsymbol{\eta} = \epsilon\boldsymbol{\eta}$, we obtain $V(\boldsymbol{\lambda}^*) \cdot \boldsymbol{\eta} = \gamma\boldsymbol{\eta}$ with $\gamma = K[\kappa'\epsilon + (1 - \kappa\epsilon)^{-1} \cdot \epsilon(\kappa - \epsilon)]$. Multiplying (34) by $\boldsymbol{\eta}$ from the right and using $\mathbf{1}^\top\boldsymbol{\eta} = 0$ and $\mathbf{1}^\top V(\boldsymbol{\lambda}^*) \cdot \boldsymbol{\eta} = \gamma\mathbf{1}^\top\boldsymbol{\eta} = 0$, we obtain $J(\boldsymbol{\lambda}^*) \cdot \boldsymbol{\eta} = \frac{\gamma}{K}\boldsymbol{\eta}$. Then the eigenvalue β of the Jacobian matrix $J(\boldsymbol{\lambda}^*)$ for the eigenvector $\boldsymbol{\eta}$ is expressed in terms of ϵ as $\beta = \Psi(\epsilon)$ in (39).

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