Stochastic Stability Analysis of a Model of Endogenous Urban Subcenter Formation

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We present a detailed analysis of the Fujita and Ogawa (1982) model,[1)] *the* reference model for endogenous formation of polycentric intra-urban configuration. Stability of its equilibria, however, has not yet been studied; it thus has been unclear whether polycentric patterns actually emerge from the model. We show that the model is a *potential game*. We apply *stochastic stability analysis* in evolutionary game theory literature to select globally stable equilibria. We demonstrate that polycentric spatial patterns in the FO model are actually (stochastically) stable.

Key Words: *agglomeration, urban subcenter formation, local stability, stochastic stability*

1. Introduction

We present a stochastic stability analysis of the equilibrium solutions for the Fujita and Ogawa[1)] model (henceforth "the FO model"). The model is the reference model for endogenous formation of *polycentric* urban configurations. It describes the processes of suburbanization via interactions between two types of mobile agents: firms and households. The centripetal force in the model is *technological externalities* between firms, which is interpreted as business communications; for the centrifugal force, there is *competition over land* between agents. In this sense, the model is similar to the social interaction models à la Beckmann.[2)]

An additional feature of the FO model is that there are households who are assumed to commute to job locations, thereby producing another cost of firms' agglomeration; since a larger agglomeration of firms result in longer commuting length of households, they should be compensated by higher wage. Thus, the basic trade-off in the model can be captured by two parameters. One is the commuting cost parameter t of households and the other is the communication cost parameter τ of firms.

A striking characteristic of the FO model is that the model admits multiple equilibria, typically characterized by *polycentric patterns*.[1] The well-known classification of possible equilibrium patterns on (τ, t)-plane is shown in **Fig.1**, highlighting emergence of multiple equilibrium spatial configurations at some parameter values. In **Fig.1**, below the curve C_1 lies the monocentric pattern; inside C_2 the duocentric pattern; above C_C a complete dispersion.[2]

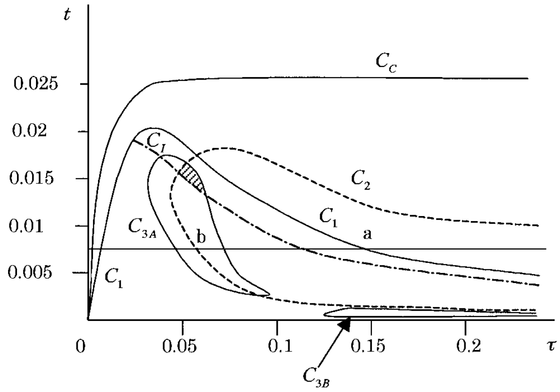


Fig.1 Emergence of multiple equilibria and polycentric patterns (taken from Fujita and Thisse[3)]).

Even though the FO paper discusses multiple equilibria and structural transitions, *stability* of solution has been long-standing issue, as Fujita and Thisse[3)] puts it:

> Note, however, that we do not know anything about the stability of these equilibria.

[1] In fact, the model was designed to demonstrate the insufficiency of classical monocentric models.

[2] The completely integrated pattern; see the next section.

> Performing such an analysis when the unknowns are continuous curves is a hard task left for future research

or as Fujita and Ogawa implicitly discusses it (emphasis by the present author):

> Once a catastrophic structural transition of the urban configuration has been recognized, the ensuing problem is to understand the underlying *dynamics* which generate it.

Under the existence of agglomeration economies, a lack of stability analysis can result in a theoretical artifact; namely, one can select non-sustainable equilibrium. The situation around the FO model need some resolution.

Do stable polycentric patterns emerge from the FO model? Takayama and Akamatsu[4] argues that a source of polycentricity is existence of a global type of dispersion force. Here, a global type of agglomeration force is a dispersion force that explicitly depend the distance between locations. A typical example of such dispersion force is spatial competition among firms over spatially fixed demands. For instance, in Krugman,[5] distance-dependent demands from immobile firmers produce a global type of dispersion force.

The conclusion of Takayama and Akamatsu throws some doubts on actual emergence of polycentric patterns from the FO model, because *the model does not seem to have any global type of dispersion force*. On the other hand, since the conclusion is drawn from models with a single-type of agents, its adaptability to the FO model—a model with multiple types of agents—is ambiguous. That said, we have to conduct explicit stability analysis of equilibrium patterns in the FO model to assess weather polycentricity results or not.

The analysis in this paper shall provide an answer to the stability issue of the model. In doing so, we employ *stochastic stability analysis* in evolutionary game theory[3] also employing the fact that the FO model is a *potential game*. The facts we use in the present paper is that:

Fact 1. In a potential game, global maximizers of the potential function is stochastically stable.

Fact 2. In a potential game, local isolated maximizers of the potential function is locally stable under a wide range of dynamics.

The two facts together posit that by simplly comparing potential values for possible equilibrium patterns, we can select locally and stochastically stable equilibria.

[3] Readers should consult Sandholm[6] for a survey. For stochastic stability, a recent survey by Wallace and Young[7] should be a good inroduction.

Our conclusion is that *polycentric patterns in the FO model are (locally or stochastically) stable*, thereby once again highlighting the theoretical importance of the FO model. We will also show that evolution of spatial structure follows the so-called *bell-shaped development*, which is reminiscent of implications from inter-regional models in the previous literature. To achieve these goals, we first introduce a discrete analogue of the FO model to avoid difficulties arising from continuous space. Then existence of an associated potential function is proved. Equipped with the potential function, we conduct stochastic stability analysis to show emergence of polycentricity. We also conduct an approximated local stability analysis to reveal economic mechanisms behind emergence of polycentricity.

2. Fujita and Ogawa Model

In this section, we formulate a discrete version of the FO model and analyze its basic properties. We first show existence of the equivalent optimization problem. We consider a decomposition of the equilibrium problem into the short- and long-run components, the latter of which can be naturally interpreted as a potential game. The interpretation allows us to apply stochastic stability analysis.

(1) Basic Assumptions and Agents

a) Basic Assumptions

We consider a one-dimensional space $\mathcal{K}$ with K discrete locations, which is interpreted as a single *city*. Land endowment at each location $x \in \mathcal{K}$ is fixed to be $\bar{S} \equiv S/K$, so that total land endowment in the economy is given by S. As usual, land is owned by absentee landlords. For notational simplicity, the opportunity cost of land is normalized to be zero.

There are *two types of mobile agents* in the economy: households and business firms; there are a continuum N of identical households, as well as a continuum M of identical firms. Each household supplies one unit of labor to a firm. The only income of a household is the wage earned in compensation of the labor. Firms are owned by absentee shareholders and export the same good to the outside world. Firms produce the good under *technological externalities* which prefer proximity to other firms. Both of the agents consume land for residential or production use. Thus, in addition to the interactions in the job-market, the two types of actors interact also in the land market. Both markets are assumed to be perfectly competitive.

b) Household

Households consume land and composite good. The identical preference for land and composite commodity is expressed by $U(s,z)$, where $U : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a strictly increasing function of land consumption s and composite good consumption z. In the FO model, for tractability, lot size s of a household is assumed to be a fixed constant S_h. Each household supplies one unit of labor at some job location $j \in \mathcal{K}$, and compensated by the market wage W_j prevailing at j. The budget constraint of a household, who reside at $i \in \mathcal{K}$ and commute to $j \in \mathcal{K}$, is given by

$$W_j - t \cdot T_{ij} = z + R_i \cdot S_\mathrm{h}, \tag{1}$$

where R_i is the *market land rent* at $i \in \mathcal{K}$, T_{ij} is the Euclid distance between i, j (or commuting length), and t is the unit commuting cost of households. Note that z is taken as the numéraire. The left hand side of (1) is the net income of a household that locate at i and commute to j.

Maximization of utility U in our setting (fixed land consumption s) is equivalent to maximization of composite good consumption. Under fixed residential location i and fixed job location j, the utility-maximizing consumption level of the composite good z_{ij} is

$$z_{ij} \equiv \arg\max_{z \geq 0} U(s = S_\mathrm{h}, z) \tag{2}$$

$$= W_j - R_i \cdot S_\mathrm{h} - t \cdot T_{ij}. \tag{3}$$

Given the residential location $i \in \mathcal{K}$ of a household, for later use we introduce the notion of *commuting function*, $C : \mathcal{K} \to \mathcal{K}$, which represents optimal job choice of households. It is a function that satisfies the following property:

$$C(i) = \arg\max_{j \in \mathcal{K}} \{W_j - t \cdot T_{ij}\} \tag{4}$$

In other words, $C(i) \in \mathcal{K}$ is the job location that maximizes net income of household at i.

c) Business Firm

Firms produce good using a fixed amount of land and fixed unit of labor, the value of which given by S_f and L, respectively. There are technological externalities; the output level of a firm depends on the amount of *business communications* such as face-to-face contacts between the other firms. In reduced-form, the profit function of a firm at location $i \in \mathcal{K}$ may be expressed as

$$\Pi_i(\boldsymbol{m}) = A_i(\boldsymbol{m}) - R_i \cdot S_\mathrm{f} - W_j \cdot L, \tag{5}$$

where $A_i(\boldsymbol{m})$ is the *accessibility field* defined by[4]

$$A_i(\boldsymbol{m}) = \sum_{j \in \mathcal{K}} d_{ij} \cdot m_j, \tag{6}$$

in which m_j is the number of firms at location, $\boldsymbol{m}$ is the vector of m_i, and $j \in \mathcal{K}$ and d_{ij} is accessibility between locations. In the following, we concentrate on exponential accessibility function, or the *spatially discounted accessibility*, as it is known that linear accessibility yields only monocentric spatial equilibrium patterns (Ogawa and Fujita[8]):

[4] See Fujita and Thisse,[3] Chapter 6, for possible micro-foundations of $A_i(\boldsymbol{m})$.

Assumption 1. $\boldsymbol{D} = [d_{ij}]$ is given by the following exponential formula with the communication cost parameter $\tau > 0$:

$$d_{ij} = \exp\left[-\tau \cdot T_{ij}\right]. \tag{7}$$

In sum, there are two fundamental parameters: the communication cost parameter τ for firms and the commuting cost parameter t for households.

(2) Equilibrium Conditions

Let n_{ij} denote the number of households that choose residential location $i \in \mathcal{K}$ and job location $j \in \mathcal{K}$. At an equilibrium, no household has an incentive to change its location or job. The condition is expressed by the following:

$$\begin{cases} z^* = z_{ij} & \text{if } n_{ij} > 0, \\ z^* \geq z_{ij} & \text{if } n_{ij} = 0. \end{cases} \quad \forall i, j \in \mathcal{K} \tag{8}$$

where z^* is the equilibrium level of the composite good consumption. There should be no incentive for a firm to change its location:

$$\begin{cases} \Pi^* = \Pi_i(\boldsymbol{m}) & \text{if } m_i > 0, \\ \Pi^* \geq \Pi_i(\boldsymbol{m}) & \text{if } m_i = 0. \end{cases} \quad \forall i \in \mathcal{K} \tag{9}$$

where Π^* is the equilibrium profit.

The market land rent profile $\{R_i\}$ and wage profile $\{W_i\}$ should be determined by equilibrium conditions. The land market equilibrium condition is

$$\begin{cases} S_\mathrm{h} \cdot \sum_{j \in \mathcal{K}} n_{ij} + S_\mathrm{f} \cdot m_i = \bar{S} & \text{if } R_i > 0 \\ S_\mathrm{h} \cdot \sum_{j \in \mathcal{K}} n_{ij} + S_\mathrm{f} \cdot m_i \leq \bar{S} & \text{if } R_i = 0 \end{cases} \quad \forall i \in \mathcal{K} \tag{10}$$

Note that the left hand sides are the total land demand at location x, whereas $\bar{S}$ is the land endowment at each location. Similarly, the job market equilibrium condition is

$$\begin{cases} \sum_{i \in \mathcal{K}} n_{ij} = L \cdot m_j & \text{if } W_j > 0, \\ \sum_{i \in \mathcal{K}} n_{ij} \geq L \cdot m_j & \text{if } W_j = 0. \end{cases} \quad \forall j \in \mathcal{K} \tag{11}$$

The number of agents should be conserved:

$$\sum_{i \in \mathcal{K}} \sum_{j \in \mathcal{K}} n_{ij} = N, \tag{12}$$

$$\sum_{i \in \mathcal{K}} m_i = M. \tag{13}$$

An equilibrium in the FO model is defined as a set of variables that satisfies all of the above conditions.

Definition 1 (Spatial equilibrium in the FO model)**.** A spatial equilibrium in the FO model is a collection of variables $(\{m_i\}, \{n_{ij}\}, \{R_i\}, \{W_i\}, \Pi^*, z^*)$ that satisfies the no-arbitrage conditions (8), (9), the market clearing conditions (10), (11), and the conservation conditions (12), (13).

Note in passing that the equilibrium conditions are mathematically equivalent to those formulated by Fujita and Ogawa[1)] based on the *bid-rent approach* of Alonso.[9)]

(3) Equivalent Optimization Problem and Potential Game Interpretation

We have the following proposition via standard complementarity problem arguments.[5]

Proposition 1 (Equivalent optimization problem for the FO model)**.** Spatial equilibria in the FO model coincide with the solutions for the following optimization problem:

$$\max_{\boldsymbol{m},\boldsymbol{n}} Z(\boldsymbol{m},\boldsymbol{n}) \equiv Z^1(\boldsymbol{m}) - Z^2(\boldsymbol{n}) \quad (14)$$

$$\text{s.t.} \quad S_{\mathrm{h}} \cdot \sum_{j\in\mathcal{K}} n_{ij} + S_{\mathrm{f}} \cdot m_i \le \bar{S} \qquad \forall i \in \mathcal{K} \qquad (R_i) \quad (15)$$

$$\sum_{i\in\mathcal{K}} n_{ij} \ge L \cdot m_j \qquad \forall j \in \mathcal{K} \qquad (W_j) \quad (16)$$

$$\sum_{i\in\mathcal{K}}\sum_{j\in\mathcal{K}} n_{ij} = N \qquad (z^*) \quad (17)$$

$$\sum_{i\in\mathcal{K}} m_i = M \qquad (\Pi^*) \quad (18)$$

where the functions $Z^1(\boldsymbol{m})$ and $Z^2(\boldsymbol{n})$ are defined by

$$Z^1(\boldsymbol{m}) \equiv \frac{1}{2}\sum_{i\in\mathcal{K}}\sum_{j\in\mathcal{K}} d_{ij} \cdot m_i m_j, \quad (19)$$

$$Z^2(\boldsymbol{n}) \equiv t\sum_{i\in\mathcal{K}}\sum_{j\in\mathcal{K}} T_{ij} \cdot n_{ij}. \quad (20)$$

Here $\boldsymbol{n}$ is a K^2-dimensional vector of n_{ij}. One can easily verify that the Karush–Kuhn–Tacker (KKT) conditions for the above problem coincides with the equilibrium conditions in the previous section.

As in the social interaction models, one can observe the fundamental trade-off between agglomeration force and dispersion force in the above objective function $Z(\boldsymbol{m},\boldsymbol{n})$. The first term $Z^1(\boldsymbol{m})$ represents the tendency of firms to agglomerate, while the second term $Z^2(\boldsymbol{n})$ prefers dispersion of firms. Note also that $Z^2(\boldsymbol{n})$ is exactly the total commuting costs experienced by households.

We interpret that the optimization problem consists of two components: the short-run and the long-run. In the short-run, the location of firms are given (i.e., $\boldsymbol{m}$ is given). Only households choose their residential and job locations. The short-run component of the equivalent optimization problem may be written as

$$\min_{\boldsymbol{n}} Z^2(\boldsymbol{n} \mid \boldsymbol{m}) \quad \text{s.t.} \quad (15), (16), (17) \quad (21)$$

with a fixed spatial pattern of firms $\boldsymbol{m}$. Let the solution for (21) be $\boldsymbol{n}^*$, and the optimal value function of (21) be $Z^2_*(\boldsymbol{m}) \equiv Z^2(\boldsymbol{n}^* \mid \boldsymbol{m})$, as (21) can be regarded as a parametric optimization problem with parameter $\boldsymbol{m}$.

The other component is the long-run. In the long-run, firms are allowed to relocate; firms' location choice pattern $\boldsymbol{m}$, which is the parameter of (21), can vary. Then, one may formulate the following problem that characterize the *long-run* location choice of *firms*:

$$\max_{\boldsymbol{m}} Z^1(\boldsymbol{m}) - Z^2_*(\boldsymbol{m}) \quad \text{s.t.} \quad (18) \quad (22)$$

Conceptually,[6] letting

$$\hat{\Pi}_i(\boldsymbol{m}) \equiv \frac{\partial Z^1(\boldsymbol{m})}{\partial m_i} - \frac{\partial Z^2_*(\boldsymbol{m})}{\partial m_i}, \quad (23)$$

and $\mathcal{M} \equiv \{\boldsymbol{m} \in \mathbb{K}^K_+ \mid \sum_{i\in\mathcal{K}} m_i = M\}$, the long-run problem can be regarded as an equivalent optimization problem for a potential game $\mathcal{G} \equiv (\mathcal{S}, \mathcal{M}, \hat{\boldsymbol{\Pi}})$ with potential function $Z^1(\boldsymbol{m}) - Z^2_*(\boldsymbol{m})$. Thus, stochastically stable equilibria are global maximizers of $Z^1(\boldsymbol{m}) - Z^2_*(\boldsymbol{m})$, or equivalently, $Z(\boldsymbol{m},\boldsymbol{n})$.

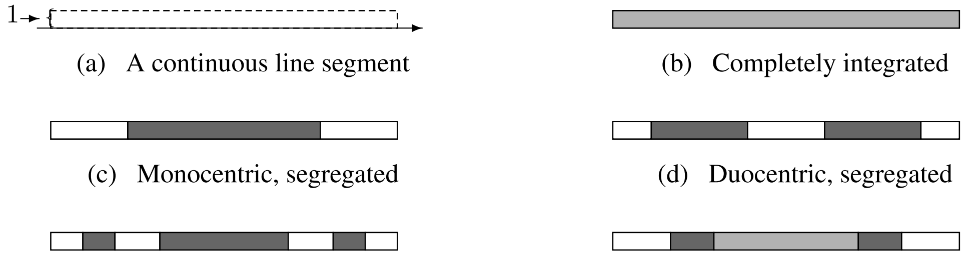
3. Equilibrium Patterns

(1) Equilibrium Patterns on a Line Segment and Difficulties

We shall first discuss equilibrium patterns and their properties. The analysis in the original Fujita and Ogawa[1)] paper considers a *continuous line segment* as depicted in **Fig.2a**. The original article showed that there are three essential types of equilibria. The first type is the *completely integrated pattern*. The second is *segregated patterns*, in which segregation between firms and households, or endogenous formation of business and residential districts, occur (**Fig.2c**, **Fig.2d**, **Fig.2e**). Showing emergence of *polycentric patterns* such as **Fig.2d** or **Fig.2e** is a major contribution of Fujita and Ogawa.[1)] The last type is *incompletely integrated patterns* in which complete segregation does not occur (**Fig.2f**). The original paper does not analyze polycentric incompletely integrated patterns to avoid complication.

[5] The existence of equivalent optimization problem for the FO model is demonstrated by Akamatsu.[10)]

[6] Strictly speaking, $Z^2_*(\boldsymbol{m})$ might not always be differentiable with respect to $\boldsymbol{m}$.



(a) A continuous line segment
(b) Completely integrated
(c) Monocentric, segregated
(d) Duocentric, segregated
(e) Tricentric, segregated
(f) Monocentric, incompletely integrated

Fig.2 Example of spatial equilibrium patterns in a line segment derived by Fujita and Ogawa.[1)] Dark gray regions depict pure business district; light gray regions depict integrated regions; white are pure residential districts.

On a line segment, polycentric equilibrium patterns such as duocentric and tricentric cannot be analytically derived due to asymmetry. To determine the loci of boundaries of business districts, we should solve highly nonlinear equations; in fact, there is no hope for obtaining analytical solutions. Although numerical approaches such employed by the original FO paper is possible, the intractability has long been a major weakness of the FO model.

(2) Simplification: Continuous Approximation and Normalizations

It is the effect of boundaries of space, or more concretely, the locational advantage in the central portion of a line segment, that cause the above difficulty. In the following analysis, for simplification and abstraction from the first-nature advantage, we shall consider a racetrack economy in line with Akamatsu et al.:[11)]

Assumption 2 (A racetrack economy)**.** The underlying space $\mathcal{S} \equiv (\mathcal{K}, \boldsymbol{T})$ is a racetrack economy whose total land endowment is S.

It is immediate that polycentric segregated patterns with equidistantly placed, same-sized business districts are trivial equilibria on a racetrack economy, thereby overcoming difficulties in obtaining relevant equilibrium patterns for our analysis.

In addition, for simplicity[7] of computation we shall approximate our discrete space by a continuous one, as we did for the SI model:

Assumption 3 (Continuous approximation)**.** Throughout our analysis of the FO model, a continuous racetrack economy $\mathcal{S}$ is assumed as an approximation. Locations are assumed to be indexed by $x \in [-S/2, S/2)$, with a periodic boundary condition.

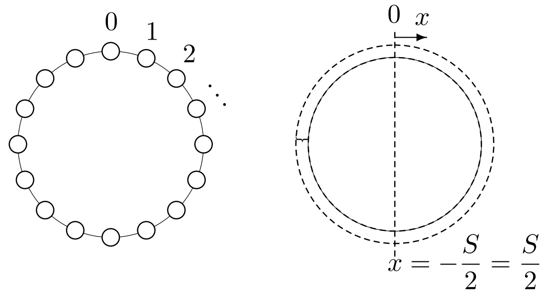


Fig.3 Discrete ($K = 16$) and continuous racetrack economies. In both geographies, the total land endowment is assumed to be S. For the continuous one, space is indexed as $\mathcal{S} = [-S/2, S/2)$.

At this point, the land endowment at each location $x \in \mathcal{S}$ is assumed to be 1, so that total land endowment equals the original discrete setting. See **Fig.3** for a comparison between discrete and continuous racetrack.

Now that space is continuous, spatial patterns and job choice patterns $\boldsymbol{m}, \boldsymbol{n}$ are represented by continuous density functions $m : \mathcal{S} \to \mathbb{R}_+$ and $n : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$. Also, we shall use $z(x, y)$, $\Pi(x)$, $C(x)$, $R(x)$ and $W(x)$ in the places of z_{ij}, $\Pi_i(\boldsymbol{m})$, $C(i)$, R_i, and W_i. In particular, the accessibility function $A_i(\boldsymbol{m})$ is now computed as

$$A(x) = \int_{\mathcal{S}} D(x, y) m(y) \mathrm{d}y \qquad \forall x \in \mathcal{S} \tag{24}$$

where $D : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$ is defined by

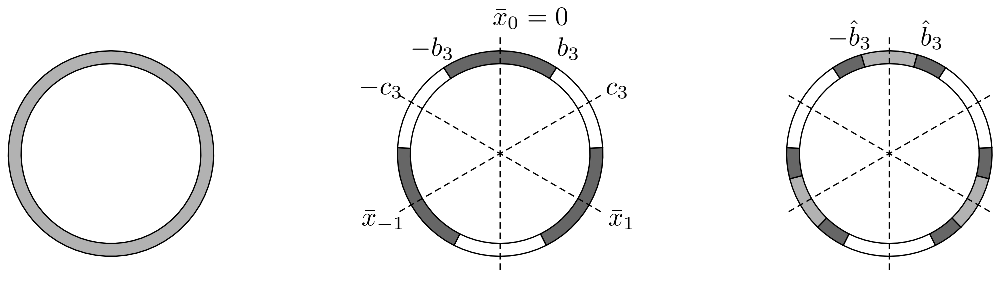
$$D(x, y) \equiv \exp[-\tau |x - y|] \tag{25}$$

Here, $|x - y|$ is loosely interpreted as the shortest path length between x and y on the continuous racetrack. The continuous-space analogue of the potential function is given by:

$$Z[m, n] \equiv Z^1[m] - Z^2[n] \tag{26}$$

$$Z^1[m] \equiv \frac{1}{2} \iint_{\mathcal{S} \times \mathcal{S}} D(x, y) m(x) m(y) \mathrm{d}x \mathrm{d}y, \tag{27}$$

[7] On a discrete space, boundaries between business and residential districts such depicted in **Fig.2** might not be uniquely determined due to discretization errors. However, such indeterminacies are irrelevant for analyzing essential properties of equilibrium patterns. In the present thesis, we shall avoid such unnecessary complications by employing the continuous approximation.



(a) Completely integrated (b) Segregated (c) Incompletely integrated

Fig.4 Example of spatial equilibrium patterns. Dark gray regions depict pure business district; light gray regions depict integrated regions; white are pure residential districts. For segregated or incompletely integrated patterns, the $J = 3$ (tricentric) case are taken as examples.

$$Z^2[n] \equiv t \iint_{\mathcal{S}\times\mathcal{S}} |x - y| n(x, y)\mathrm{d}x\mathrm{d}y. \tag{28}$$

where (27) and (28) corresponds to (19) and (20), respectively. Similarly, the constraints (15) and (16) are rewritten as

$$S_{\mathrm{h}} \cdot \int_{\mathcal{S}} n(x, y)\mathrm{d}y + S_{\mathrm{f}} \cdot m(x) \leq 1 \quad \forall x \in \mathcal{S}, \tag{29}$$

$$\int_{\mathcal{S}} n(x, y)\mathrm{d}x \geq L \cdot m(y) \quad \forall y \in \mathcal{S} \tag{30}$$

In addition to Assumption 2 and Assumption 3, we make some normalization for simplicity of presentation. Integrating (29) and (30) over the continuous racetrack, it follows that

$$S_{\mathrm{h}} N + S_{\mathrm{f}} M \leq S \tag{31}$$

$$N \geq LM \tag{32}$$

In the following, we shall assume *full employment* (i.e., $N = LM$) as the economy is closed. Without loss of generality, we further assume that there are *no vacant land* in the racetrack economy (i.e., $S_{\mathrm{h}} N + S_{\mathrm{f}} M = S$), since firm's preference for proximity to other firms and household's desire to shorten commuting length together generically result in a connected support (i.e., $\mathrm{supp}[m] \cup \mathrm{supp}[n]$ must be a convex set in an equilibrium). Under the above two natural assumptions, we can use the following normalization by choosing appropriate unit for measuring the number of mobile agents N and M:[8]

Assumption 4 (Normalizations)**.** $S_{\mathrm{f}} = 1$, $S_{\mathrm{h}} = 1$.

In the following, we shall summarize possible equilibrium patterns under the above normalization. Note that as $N = LM$, by Assumption 4 the only independent parameters other than τ and t are M and L; the two parameters together determine the size of the city by the relation $S = (1 + L)M$.

[8] Specifically, we shall set $M := S_{\mathrm{f}} M$, $N := S_{\mathrm{h}} N$, $L := (S_{\mathrm{h}}/S_{\mathrm{f}})L$ using the original constants. See also Fujita and Thisse,[3)] Chapter 6.

(3) Equilibrium Patterns and Their Properties

Assuming the continuous racetrack economy, we consider three kinds of possible spatial patterns, analogous to those in a line segment (**Fig.2**).

a) Completely Integrated Pattern

As usual on the racetrack, the uniform distribution of mobile agents is *always* an equilibrium (**Fig.4a**). We shall call the pattern the *completely integrated pattern*, in line with that on a line segment (**Fig.2b**). The completely integrated pattern is

$$m(x) = \bar{m} \equiv \frac{M}{S} = \frac{1}{1+L} \quad \forall x \in \mathcal{S} \tag{33}$$

$$n(x) = \bar{n} \equiv \frac{N}{S} = \frac{L}{1+L} \quad \forall x \in \mathcal{S} \tag{34}$$

where

$$n(x) \equiv \int_{\mathcal{S}} n(x, y)\mathrm{d}y \quad \forall x \in \mathcal{S} \tag{35}$$

is the residential density function of households. Note that $\bar{n} = L\bar{m}$ as $N = LM$.

In the completely integrated pattern, there is no costly commuting. The corresponding commuting function $C(x)$ is the identity map: $C(x) = x$ for all $x \in \mathcal{S}$. Let the uniform level of market wage and land rent be $\bar{W}$ and $\bar{R}$, we have

$$z^* = \bar{W} - \bar{R}, \tag{36}$$

$$\Pi^* = \bar{A} - \bar{W}L - \bar{R}, \tag{37}$$

where

$$\bar{A} = \frac{2M}{S\tau} \cdot \left(1 - \exp\left[-\tau \cdot \frac{S}{2}\right]\right) \tag{38}$$

is the uniform level of firms' accessibility.[9] Letting a normalization $\bar{R} = 0$ and assuming zero-profit of firms in order to avoid indeterminacy, we have $\bar{W} = \bar{A}/L$. It is thus obvious that the completely integrated pattern is *always* an equilibrium.[10]

[9] Observe that $\exp[-\tau(S/2)]$ is the accessibility between antipodal points on the circumference (i.e., $D(0, S/2)$).

[10] Note that, on a line segment, the completely integrated pattern cannot be an equilibrium for specific regions of the transport cost parameter pairs (τ, t).

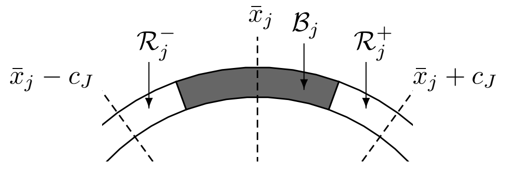


Fig.5 A business district $\mathcal{B}_j$ and the associated residential districts $\mathcal{R}_j^+, \mathcal{R}_j^-$

b) Segregated Patterns

Consider J-centric *segregated patterns* in which $J \in \mathbb{Z}_{++}$ same-sized business districts (BDs) are equidistantly placed over the racetrack, and the rest are used for residential districts (RDs). **Fig.4b** illustrates the 3-centric segregated equilibrium. These patterns are symmetric versions for **Fig.2d** or **Fig.2e**. We assume that there is a BD whose center is located at the origin $x = 0$, and call it the central BD. From the central BD, other BDs are numbered by j where $j = J^-, \ldots, -1, 0, 1, \ldots, J^+$ with $J^- \equiv \lceil -(J-1)/2 \rceil$, $J^+ \equiv \lceil -(J-1)/2 \rceil$. The center of the jth business district is denoted by $\bar{x}_j$, the actual value of which being $\bar{x}_j = (S/J)j$.

For each J-centric segregated pattern, we shall denote jth BD by $\mathcal{B}_j \subset \mathcal{S}$, and the union of all the BDs by $\mathcal{B} \equiv \cup_j \mathcal{B}_j \subset \mathcal{S}$. As the number of firms at each BD (= the width of the BD, as the land density is unity) is given by M/J, the "radius" of a single BD is given by

$$b_J \equiv \frac{M}{2J}. \tag{39}$$

Using b_J, jth BD may be expressed as $\mathcal{B}_j \equiv [\bar{x}_j - b_J, \bar{x}_j + b_J]$. The J-centric segregated pattern is thus expressed as

$$m(x) = 1_{\mathcal{B}} \tag{40}$$

$$n(x) = 1_{\mathcal{S}\setminus\mathcal{B}} \tag{41}$$

where for a given set $\mathcal{X}$, $1_{\mathcal{X}}$ denotes the indicator function for it.

Polycentric segregated patterns includes commuting of households. It is immediate that households commute to the nearest business district. Define $c_J \equiv S/(2J)$; also define residential district $\mathcal{R}_j$ associated to $\mathcal{B}_j$ by

$$\mathcal{R}_j \equiv \mathcal{R}_j^- \cup \mathcal{R}_j^+, \tag{42}$$

$$\mathcal{R}_j^- \equiv [\bar{x}_j - c_J, \bar{x}_j - b_J), \tag{43}$$

$$\mathcal{R}_j^+ \equiv (\bar{x}_j + b_J, \bar{x}_j + c_J). \tag{44}$$

Then, households in $\mathcal{R}_j$ commute to $\mathcal{B}_j$. For an illustration of $\mathcal{R}_j^{\pm}$, see **Fig.5**. Then, commuting function $C(x)$ for $\mathcal{R}_j$ may be expressed as[11]

$$C(x) = \begin{cases} \bar{x}_j + \dfrac{b_J}{l_J}(x - \bar{x}_j + b_J) & \text{if } x \in \mathcal{R}_j^- \\ \bar{x}_j + \dfrac{b_J}{l_J}(x - \bar{x}_j - b_J) & \text{if } x \in \mathcal{R}_j^+ \end{cases} \tag{45}$$

where $l_J \equiv c_J - b_J = N/(2J)$ is the width of the residential districts $\mathcal{R}_j^-$, $\mathcal{R}_j^+$. Let $\mathcal{E}_j \equiv \mathcal{B}_j \cup \mathcal{R}_j$. We shall call $\mathcal{E}_j$ "j-th *employment area*".

We can easily check the J-centric segregated spatial configurations are potential equilibria. We shall first derive the accessibility profile $A(x)$ for J-centric pattern. Because of symmetry, we shall only consider the accessibility function in the central employment area ($\mathcal{E}_0$). Recalling $m(x) = 1$ in $\mathcal{B}$,

$$A(x) = \int_{\mathcal{S}} D(x,y)\mathrm{d}y = \sum_j \int_{\mathcal{B}_j} D(x,y)\mathrm{d}y \tag{46}$$

To provide some intuitions, we shall compute a part of $A(x)$. For $j > 0$, the accessibility to $\mathcal{B}_j$, $A_j(x)$ is computed as

$$A_j(x) \equiv \int_{\mathcal{B}_j} D(x,y)\mathrm{d}y \tag{47}$$

$$= \int_{\bar{x}_j - b_J}^{\bar{x}_j + b_J} \exp[-\tau(y-x)]\mathrm{d}y \tag{48}$$

$$= \frac{2}{\tau} \cdot \underbrace{\exp[-\tau(\bar{x}_j - x)]}_{\text{(a)}} \cdot \underbrace{\sinh[\tau b_J]}_{\text{(b)}}. \tag{49}$$

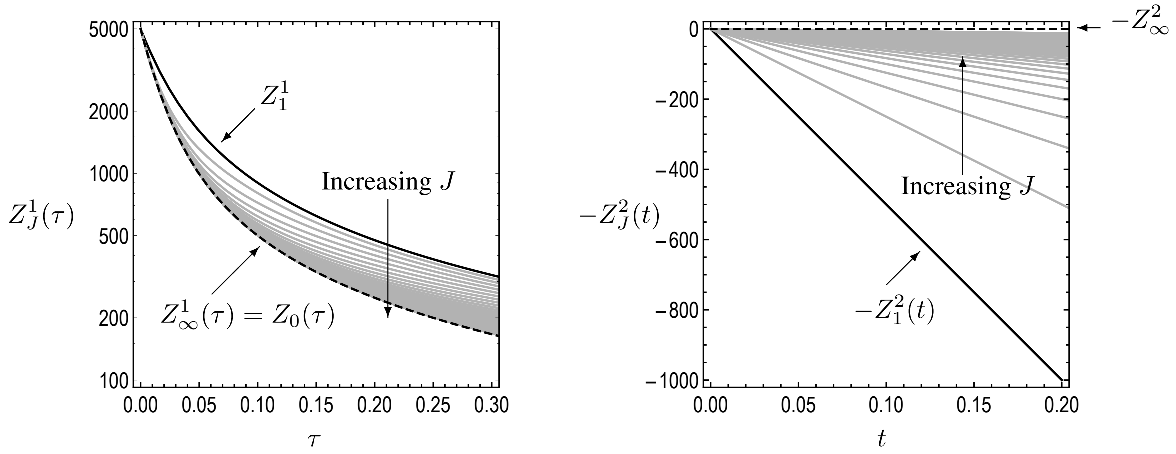
for all $x \in \mathcal{B}_0$. In $A_j(x)$, (a) can be interpreted as the effect of the average distance to the destination support $\mathcal{B}_j$ from location $x \in \mathcal{B}_0$, while (b) the effect of "spread" of $\mathcal{B}_j$. Via a similar straightforward computation, we show that $A(x)$ is

$$A(x) = \begin{cases} \dfrac{2}{\tau} \cdot \left(\hat{\alpha}_J - \hat{\beta}_J \cosh[\tau(x + c_J)]\right), \\ \qquad \forall x \in \mathcal{R}_0^- \\ \dfrac{2}{\tau} \cdot \left(\alpha_J - \beta_J \cosh[\tau x]\right), \\ \qquad \forall x \in \mathcal{B}_0 \\ \dfrac{2}{\tau} \cdot \left(\hat{\alpha}_J - \hat{\beta}_J \cosh[\tau(x - c_J)]\right), \\ \qquad \forall x \in \mathcal{R}_0^+ \end{cases} \tag{50}$$

where α_J and β_J are defined by the following

$$\alpha_J \equiv \begin{cases} 1 & (J\text{: odd}) \\ 1 - r_J^{J/2} & (J\text{: even}) \end{cases} \tag{51}$$

11 We assume *parallel commuting* in Fujita and Ogawa.[1)] Berliant and Tabuchi[12)] give a more detailed scrutiny on commuting patterns in the FO model when convexity of commuting cost change.



(a) $Z_J(\tau, 0) = Z_J^1(\tau)$ (b) $Z_J(0, t) = -Z_J^2(t)$

Fig.6 The two terms in the potential values in equilibria ($Z_J^1(\tau)$ and $-Z_J^2(t)$). $Z_J^1(\tau)$ is strictly decreasing in J, where as $-Z_J^2(t)$ is strictly increasing in J.

$$\beta_J \equiv \begin{cases} \exp[\tau b_J] + 2\sinh[-\tau b_J] \cdot d_J & (J\text{: odd}) \\ \exp[\tau b_J] - r_J^{J/2} \exp[-\tau b_J] \\ \quad +2\sinh[-\tau b_J] \cdot d_J \\ \qquad (J\text{: even}) \end{cases}. \tag{52}$$

$$\hat{\alpha}_J \equiv \begin{cases} -r_J^{J/2} & (J\text{: odd}) \\ 0 & (J\text{: even}) \end{cases} \tag{53}$$

$$\hat{\beta}_J \equiv \begin{cases} -r_J^{J/2} \exp[-\tau b_J] \\ \quad +2 \cdot r_J^{-1/2} \cdot \sinh[-\tau b_J] \cdot \hat{d}_J \\ \qquad (J\text{: odd}) \\ 2 \cdot r_J^{-1/2} \cdot \sinh[-\tau b_J] \cdot \hat{d}_J \\ \qquad (J\text{: even}) \end{cases}. \tag{54}$$

with $r_J \equiv \exp[-\tau(S/J)]$ being accessibility between two consecutive centers $\bar{x}_j, \bar{x}_{j+1}$. Using r_J, d_J and $\hat{d}_J$ are defined by

$$d_J \equiv \sum_{j=0}^{\lceil (J-1)/2 \rceil} r_J^j, \quad \hat{d}_J \equiv \sum_{j=1}^{\lfloor J/2 \rfloor} r_J^j. \tag{55}$$

Observe that d_J and $\hat{d}_J$ reflect a *global* accessibility structure, analogous to the row-sum of spatial discounting matrix $\boldsymbol{D}$. In other words, accessibility function in the FO model has implicit global structure. Also observe that $A(x)$ is a unimodal function, as readily expected from its meaning.

c) Incompletely Integrated Patterns

A polycentric incompletely integrated pattern with $J = 3$ is illustrated by **Fig.4c**. Such incompletely integrated patterns are here considered to be "transitional" patterns. For instance, on a line segment, monocentric incompletely integrated pattern can emerge between the completely integrated equilibrium and segregated equilibria (Fujita and Ogawa[1)]). Sufficiently large values of t induces emergence of the completely integrated pattern (complete dispersion), whereas small values of t fosters that of segregated patterns. Incompletely integrated patterns emerge, if any, for intermediate values of the commuting cost parameter t. Even on the symmetric racetrack, however, we encounter difficulties in obtaining analytical formula for incompletely integrated patterns. For instance, boundaries of integrated districts (i.e., $\hat{b}_3$ in **Fig.4c**) should be obtained by solving nonlinear equations.

We shall recall that the main objective of the present thesis is the contrast between polycentricity and monocentricity. In light of this, our scrutiny in the present paper shall focus on the simple question of: *do stable polycentric patterns actually emerge from the FO model?* To this end, assuming sufficiently small t, we shall refrain here from analyzing incompletely integrated patterns, as they can be regarded as transitional ones. Curiously enough, supplementary numerical assessments have shown that incompletely integrated patterns *can never emerge* if we take into account polycentric segregated equilibria with large J.

4. Globally Stable Equilibria in the FO Model

(1) Potential Values

To analyze stochastic stability of equilibria, we shall first compute the potential values at relevant equilibrium patterns: the completely integrated pattern and segregated patterns.

For the completely integrated pattern, there are no commuting of households. Using the accessibility formula (38), it is straightforward to compute:

Lemma 1 (The potential value for the completely integrated equilibrium). Let Z_0 be the potential value at the completely integrated pattern. Then,

$$Z_0(\tau) = \frac{M^2}{S\tau}\left(1 - \exp\left[-\tau \cdot \frac{S}{2}\right]\right). \tag{56}$$

As the completely integrated pattern does not cause households' commuting, the potential value does not depend on the commuting cost parameter t.

For the segregated patterns, using commuting function described in the previous section, we show the following lemma.

Lemma 2 (The potential values for J-centric segregated patterns)**.** Let Z_J be the potential value at J-centric segregated pattern. Then, Z_J is given by

$$Z_J(\tau, t) = Z_J^1(\tau) - Z_J^2(t), \tag{57}$$

$$Z_J^1(\tau) \equiv \frac{M}{\tau}\left(\alpha_J - \beta_J \frac{\sinh[\tau b_J]}{\tau b_J}\right), \tag{58}$$

$$Z_J^2(t) \equiv \frac{(1+L)LM^2}{4} \cdot \frac{t}{J}. \tag{59}$$

where α_J and β_J are defined by (51) and (52), respectively.

It can be shown that when the number of BDs goes to infinity for the segregated patterns, the associated potential value converges to that of the completely integrated pattern:

$$\lim_{J\to\infty} Z_J(\tau, t) = Z_0(\tau). \tag{60}$$

The convergence result is natural, since as J increase, commuting lengths of households become shorter and shorter, asymptotically approaching zero; the limiting pattern can be roughly interpreted as a "completely integrated" pattern in an average sense.

Fig.6 depicts the two terms ($Z_J^1(\tau)$ and $-Z_J^2(t)$) in the potential values for segregated patterns (Lemma 2). **Fig.6a** illustrates the first term $Z_J^1(\tau)$ of the potential value. Observe that for segregated pattern, $Z_J^1(\tau)$ is strictly decreasing in J and converge to $Z_0(\tau)$ as J goes to infinity. On the other hand, the second term $-Z_J^2(t)$ is strictly increasing in J as illustrated in **Fig.6b**, reflecting smaller commuting costs. The opposite properties concretely reveals the fundamental trade-off between commuting costs of households and communication level of firms. For instance, we can immediately show the following intuitively straightforward propositions by just comparing the potential values:

Proposition 2 (Monocentricity without commuting)**.** If $t = 0$, then the monocentric segregated equilibrium is stochastically stable.

Proposition 3 (Dispersion without communication costs)**.** If $\tau = 0$, then the completely integrated equilibrium is stochastically stable.

(2) Emergence of Polycentricity in Globally Stable Equilibria

For deriving stochastically stable equilibrium patterns, we have to compare potential values obtained in Lemma 1 and Lemma 2. Even though apparently simple at the first glance, the potential values for J-centric segregated equilibria $Z_J(\tau, t)$ have quite complicated form, particularly due to the first term $Z_J^1(\tau)$ and does not seem to facilitate simple comparison in general.[12] We shall look into some simple cases ($J = 1, 2$), and then proceed to a numerical assessment for larger Js.

We shall first compare the monocentric pattern ($J = 1$) and the duocentric pattern ($J = 2$), as well as the completely integrated pattern. In order to assess model's ability to produce polycentric patterns, it is sufficient to show that there are parameter values such that the monocentric and the completely integrated patterns are dominated by some polycentric patterns. We show the following proposition:

Proposition 4. Assume that the communication cost parameter τ is sufficiently large. Then, there exist a range of the commuting cost parameter t such that $Z_2(\tau, t) > \max\{Z_0(\tau), Z_1(\tau, t)\}$ holds.

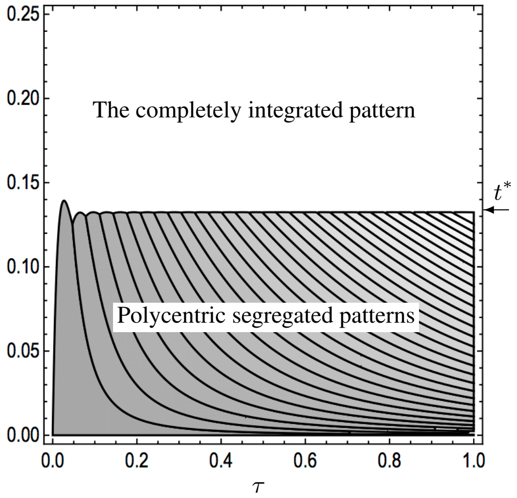
Corollary 1 (Emergence of polycentricity)**.** There are pairs (τ, t) of the transport cost parameters such that neither of the monocentric segregated pattern nor the completely integrated pattern is stochastically stable.

In sum, the above statements formally demonstrate that the FO model can produce polycentric patterns in *stochastically stable equilibria*, as neither of monocentricity nor complete dispersion will result.

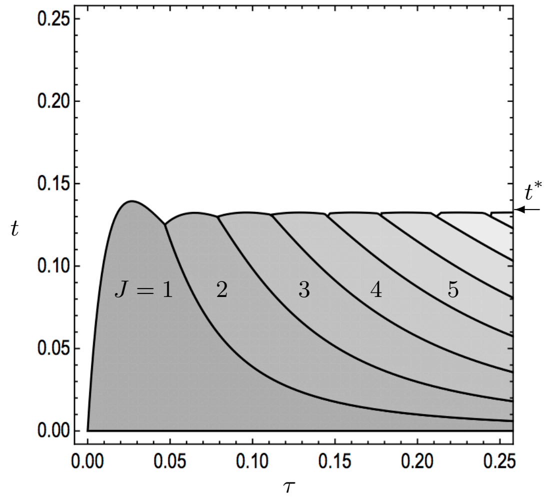
Numerically comparing all the potential values $\{Z_J\}$ ($J = 0, 1, 2, \ldots$), we obtain more concrete intuitions regarding Proposition 4 and Corollary 1. **Fig.7a** shows a whole picture of the partition of (τ, t)-plane based on the stochastically stable equilibrium pattern. The gray areas are where segregated patterns dominate the completely integrated pattern, whereas the white region the opposite. Each gray region corresponds to one of J-centric segregated patterns. In τ-axis, each gray region are aligned in the increasing order of J (see **Fig.7b**). Also observe that (at least for relatively large values of τ) there appears to be a threshold value, t^*, of the commuting cost parameter below which segregated patterns emerge. Observe that our partition in **Fig.7** is qualitatively consistent with the results presented in the FO paper (see **Fig.1** in the introduction).

A major implication of the classification illustrated inf **Fig.7** is the effect of changing the communication cost parameter τ of firms. **Fig.8** depicts a evolutionary path of stochastically stable spatial struc-

[12] For instance, see the exact analytical formula for β_J in (52).



(a) Whole picture



(b) Part of (a) enlarged

Fig.7 Classification of stochastically stable equilibria on the (τ, t)-plane. Independent parameters are set to be $(M, L) = (100, 1)$, in agreement with Fujita and Thisse.[3)] Gray regions are where J-centric equilibrium patterns are stochastically stable. (a) Polycentric segregated equilibria are stochastically stable for small t, especially below a threshold value t^* for large τs. (b) J is aligned in the increasing order in τ-direction.

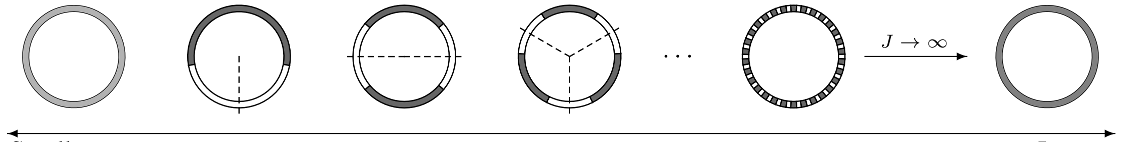


Fig.8 The communication cost τ vs. stochastically stable equilibria in the FO model.

ture in line with changing τ.[13] A striking property here is that *a steady decrease in τ induces a repetitive emergence of polycentric patterns with decreasing J which converges to monocentricity, and then at the final stage re-dispersion from the monocentric pattern*. Curiously enough, we observe such behavior in *inter-regional* models with both local and global dispersion forces (e.g., Tabuch,[13)] Puga;[14)] in spatial economic literature, such two-stage (agglomeration and re-dispersion) behavior is called a "bell-shaped development" of spatial structure (Fujita and Thisse[3)]). It may be interpreted that the FO model exhibits a bell-shaped development in *intra-urban scale*. It should be noted that, however, in the FO model the agglomeration and dispersion behaviors occur in a *single city*, i.e., a single convex connected support which corresponds to a single location in inter-regional models. Although qualitatively similar, the evolutionary behavior should be regarded as a distinct one compared to those in inter-regional models.[14]

[13] Here, the commuting cost parameter t is assumed to be fixed at a level below t^*, say, $t = 0.1$.

[14] To explicitly demonstrate the difference, we must consider possibility of *vacant land*. Due to a lack of symmetry, however, analytical study would become a hard task.

(3) Mechanism Behind Polycentricity in the FO Model

We have a partition of the (τ, t)-plane which suggests emergence of polycentricity from the FO model. Yet, it is hard to tell what actually is the key for emergence of polycentricity in the FO model only from the analysis in the previous sections. We discuss how polycentricity is formed in the FO model.

First, we note that since households and firms are evenly spread at the completely integrated pattern, emergence of some agglomeration from the pattern can be approximated by a model *without location choice of households*—as land market acts towards only dispersion. As land market competition can work as local scale of dispersion force, what matters for the emergence of polycentricity from uniformity in the FO model would be *job choice behavior* of households. To elaborate on this point, abstracting location choice of households, we here analyze the effect of job choice of households via some local stability analysis arguments and try to explain how the partition of (τ, t)-plane in **Fig.7** is formed.

To analyze local stability, we again consider the discrete version of the FO model with K locations with K being a multiple of four. For simplicity, we also assume that $L = 1$ (i.e., the number of households equals to the number of firms) and $N = M = K$. Then, at the completely integrated pattern, $h_i \equiv \sum_j n_{ij} = 1$ and $m_i = 1$ for all $i \in \mathcal{K}$. We denote the spatial pattern of households by a vector $\boldsymbol{h}$.

We are to see if polycentricity can emerge from the completely integrated pattern by inspecting the properties of the eigenvalues of the profit function $\boldsymbol{\Pi}(\boldsymbol{m})$ of firms. When we abstract from land market and set L to unity, it is given by

$$\boldsymbol{\Pi}(\boldsymbol{m}) = \boldsymbol{D}\boldsymbol{m} - \boldsymbol{W}, \tag{61}$$

where $\boldsymbol{W} = [W_i]$ is the wage vector and $\boldsymbol{D}$ is the spatial discounting matrix of interactions between firms with a single *communication freeness* parameter

$$r \equiv \exp\left[-\tau \frac{S}{K}\right]. \tag{62}$$

where S is the circumferential length of the racetrack economy. To inspect the eigenvalues of $\boldsymbol{\Pi}(\boldsymbol{m})$ at the completely integrated pattern, we should evaluate the Jacobian matrix

$$\nabla\boldsymbol{\Pi}(\bar{\boldsymbol{m}}) = \boldsymbol{D} - \nabla\boldsymbol{W} \tag{63}$$

at the completely integrated pattern $\bar{\boldsymbol{m}} \equiv \boldsymbol{1}$. However, the second term, $\nabla\boldsymbol{W}$ cannot simply evaluated since the job-choice behavior of households is assumed to be deterministic in the FO model; $\boldsymbol{W}$ is not differentiable.

In order to ensure differentiability of consumer's choice, we assume a household chooses its job location through *logit probability*, analogous to consumption behavior in the SISC model. Specifically, we assume that a household in location i choose job location j according to the following logit probability

$$P_{ij}(\boldsymbol{W}) = \frac{\exp[\theta(W_j - t \cdot T_{ij})]}{\sum_{k \in \mathcal{K}} \exp[\theta(W_k - t \cdot T_{ik})]}, \tag{64}$$

where θ is, as usual, a positive constant that expresses the (inverse) intensity of randomness. When $\theta \to \infty$, the choice probability converges to deterministic maximization of wage net of commuting cost, i.e., $\lim_{\theta\to\infty} P_{ij} > 0$ if and only if $j \in \arg\max_j W_j - t \cdot T_{ij}$. We assume θ is relatively large so that we can approximate the original FO model.[15]

In the following, we define

$$w_i \equiv \exp[\theta W_i], \tag{65}$$

$$d_{ij}^{\mathrm{H}} \equiv \exp[-\theta t T_{ij}], \tag{66}$$

and *the freeness of commuting*

$$r^{\mathrm{H}} \equiv \exp\left[-\theta t \frac{S}{K}\right] \tag{67}$$

so that $\boldsymbol{D}_{\mathrm{H}} = [d_{ij}^{\mathrm{H}}]$ is the spatial discounting matrix *for households' commuting* with a single parameter r^{H}. Using $\boldsymbol{D}_{\mathrm{H}}$ and $\boldsymbol{w} = [w_i]$, the choice probability matrix $\boldsymbol{P} = [P_{ij}]$ can be expressed as

$$\boldsymbol{P} = \mathrm{diag}[\boldsymbol{D}_{\mathrm{H}}\boldsymbol{w}]^{-1}\boldsymbol{D}_{\mathrm{H}}\,\mathrm{diag}[\boldsymbol{w}]. \tag{68}$$

The job market clearing condition yields that job demand at i, $Lm_i = m_i$ should met by the above commuting behavior:

$$m_i = \sum_{j\in\mathcal{K}} n_{ij} = \sum_{j\in\mathcal{K}} h_j P_{ji} \tag{69}$$

or in the vector-matrix form

$$\boldsymbol{m} = \boldsymbol{P}^{\top}\boldsymbol{h} \tag{70}$$

The condition is analogous to wage equations of typical NEG model à la Krugman;[5] a subtle difference is that the above equation is a quantity-based equation, in contrast to price-based ones in the NEG literature. In effect, the above equation yields that the Jacobian matrix $\nabla\boldsymbol{W}$ of the wage with respect to $\boldsymbol{m}$ is given by

$$\nabla_{\boldsymbol{m}}\boldsymbol{W} = \frac{1}{\theta}\left(\mathrm{diag}[\boldsymbol{P}^{\top}\boldsymbol{1}] - \boldsymbol{P}^{\top}\boldsymbol{P}\right)^{-1} \tag{71}$$

for our case ($\boldsymbol{h} = \boldsymbol{1}$). From symmetry, wage is constant across locations at the completely integrated pattern; we then have $\boldsymbol{P} = \bar{\boldsymbol{D}}_{\mathrm{H}}$ at the completely integrated pattern, where $\bar{\boldsymbol{D}}_{\mathrm{H}}$ is the row-normalized version of $\bar{\boldsymbol{D}}_{\mathrm{H}}$. It thus follows that

$$\nabla_{\boldsymbol{m}}\boldsymbol{W} = \frac{1}{\theta}\left[\boldsymbol{I} - \bar{\boldsymbol{D}}_{\mathrm{H}}^2\right]^{-1} \tag{72}$$

We thus have obtained desired analytical expression of $\nabla\boldsymbol{\Pi}(\bar{\boldsymbol{m}})$:

$$\nabla\boldsymbol{\Pi}(\boldsymbol{m}) = \boldsymbol{D} - \theta^{-1}(\boldsymbol{I} - \bar{\boldsymbol{D}}_{\mathrm{H}}^2)^{-1}. \tag{73}$$

We show the following lemma by applying the method of Akamatsu et al.:[11]

Lemma 3 (The eigenvalues of $\nabla\boldsymbol{\Pi}(\bar{\boldsymbol{m}})$). For the modified FO model with (i) logit commuting choice of households and (ii) no land market, the eigenvalues of $\nabla\boldsymbol{\Pi}(\bar{\boldsymbol{m}})$ at the completely integrated pattern is given by

$$e_k = \frac{1}{1-(f_k^{\mathrm{H}})^2} \cdot G(f_k^{\mathrm{H}}, \hat{f}_k) \tag{74}$$

$$G(f_k^{\mathrm{H}}, \hat{f}_k) \equiv \{1 - (f_k^{\mathrm{H}})^2\} \cdot \hat{f}_k - \frac{1}{\theta}. \tag{75}$$

where $\hat{\boldsymbol{f}}$ is the eigenvalues of $\boldsymbol{D}$ and $\boldsymbol{f}^{\mathrm{H}}$ is the eigenvalues of $\bar{\boldsymbol{D}}_{\mathrm{H}}$. $\hat{\boldsymbol{f}}$ and $\boldsymbol{f}^{\mathrm{H}}$ are obtained as analytic functions of the freeness of firm communication, r, and the freeness of household commuting, r^{H}, respectively.

Note that the range of $\hat{\boldsymbol{f}}$ is not normalized to $(0, 1)$ as $\boldsymbol{D}$ is not normalized by $d(r)$, the row-sum of $\boldsymbol{D}$. As discussed in Akamatsu et al.[15] or Osawa et al.[16] The eigenvalues e_k or the function $G(f_k^{\mathrm{H}}, \hat{f}_k)$ can be interpreted as the net agglomeration force in the k-th direction.

[15] Whether a concrete value of θ is "large" or "small" depends on scale of $\boldsymbol{W}$ or $\boldsymbol{T}$. In the following, we assume θ is at least larger than one.

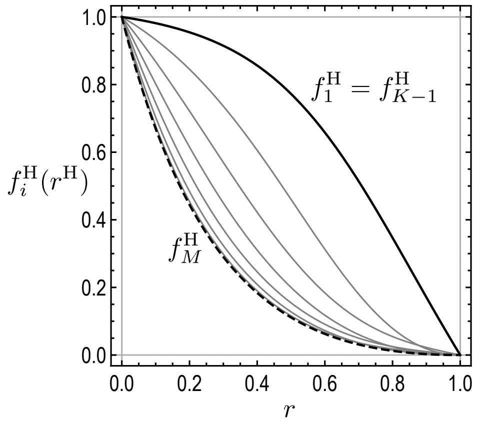


Fig.9 The eigenvalues $\{f_i(r)\}_{i=1}^{K-1}$ of $\bar{\boldsymbol{D}}^{\mathrm{H}}$ for $K = 16$. The thick black curve is the maximal eigenvalues $f_1 = f_{K-1}$, while the dashed black curve is the minimal eigenvalue f_M. The gray curves between them are the other eigenvalues.

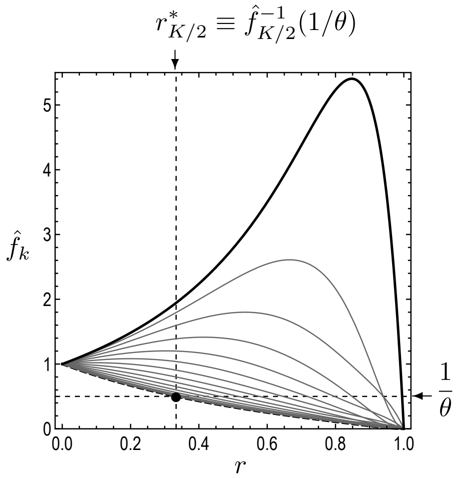


Fig.10 Eigenvalues $\hat{\boldsymbol{f}}$ of $\boldsymbol{D}$ and $1/\theta$. If $r > r_{K/2}^*$ is satisfied, $K/2$-th eigenvector e_k is always negative and the period-doubling bifurcation does not occur from the completely integrated pattern.

There are many implication from the functional form of the eigenvalues $\{e_k\}$. Even though f_k^{H} and f_k are both monotone in k for $1 \leq k \leq K/2$ (see **Fig.9** and **Fig.10**), the maximal among e_k is ambiguous. This ambiguity arises because there are two transport costs (i.e., commuting and communication), one prefers unimodal agglomeration and the other prefers period-doubling (polycentric) agglomeration.

First, the need to communicate with the other firms prefer unimodal agglomeration. For instance, fix the freeness of commuting r^{H} at very *high* level. Then, $f_k^{\mathrm{H}} \approx 0$ for all k (see **Fig.9**) and $1 - (f_k^{\mathrm{H}})^2 \approx 1$ for all k. Then, e_k would be written as

$$G(f_k^{\mathrm{H}}, \hat{f}_k) \approx \hat{f}_k - \frac{1}{\theta}. \tag{76}$$

The above formula is typical one for models with monocentric agglomeration; a *decrease* of the freeness of communication r results in emergence of monocentricity.

On the other hand, polycentric agglomeration is preferred if commuting behavior of households dominates. For example, fix the freeness of communication r at very *low* level ($r \approx 0$) so that communication between distant firms is impossible. Then, $\hat{f}_k \approx 1$ for all k and thus

$$G(f_k^{\mathrm{H}}, \hat{f}_k) \approx -(f_k^{\mathrm{H}})^2 + 1 - \frac{1}{\theta}. \tag{77}$$

The above formula is typical one for models with polycentric agglomeration. Even though firms hope to agglomerate due to the low freeness of communication, job market clearing does not allow firms to produce unimodal agglomeration because firms must compensate commuting costs of households by setting higher wage so that it can meet its own (fixed) labor demand L.

The relative strengths of the above two opposing forces toward monocentricity and polycentricity depend heavily upon exact values of communication/commuting freeness pair (r, r^{H}). Actually, depending on the freeness of communication r, possible numbers of peaks in emergent spatial pattern decrease. The following lemma gives the critical value of $r^{\mathrm{H}}(r)$ at which the bifurcation from the completely integrated pattern emerge:

Lemma 4 (The critical value of r^{H} as a function of r)**.** Consider a fixed r. Let $r^{\mathrm{H}*}(r)$ be the critical value of r^{H} at which the bifurcation from the completely integrated pattern emerge. Then, it is given by

$$r^{\mathrm{H}*}(r) \equiv \min_{k \in \mathcal{K}(r)} r_k^{\mathrm{H}*}(r), \tag{78}$$

$$r_k^{\mathrm{H}*}(r) \equiv f_k^{\mathrm{H}\,-1}(f_k^{\mathrm{H}*}(r)), \tag{79}$$

$$f_k^{\mathrm{H}*}(r) \equiv \sqrt{1 - \frac{1}{\theta \hat{f}_k(r)}}, \tag{80}$$

with $\mathcal{K}(r) \equiv \{k \mid \hat{f}_k(r) > 1/\theta\}$.

Observe that the critical value for $f_k^{\mathrm{H}}(r^{\mathrm{H}})$ is k-*dependent* and the actual critical point for r^{H} is the minimal among $\{r_k^{\mathrm{H}*}\}$. It is also noted that when one increase the agglomeration force of firms (i.e., increase r, or equivalently, decrease τ), the number of emergent peaks decrease. As depicted by **Fig.10**, depending on the value of r, the condition for the existence of (80) might not be satisfied for some k. For instance, for a k such that

$$r > r_k^* \equiv \hat{f}_k(1/\theta), \tag{81}$$

bifurcation in the k-th direction cannot occur since e_k is negative for all values of the freeness of commuting r^{H}. This means that, even though households prefer as much polycentricity as possible to shorten their commuting distance, the merit of agglomeration overcomes tendency toward polycentricity.

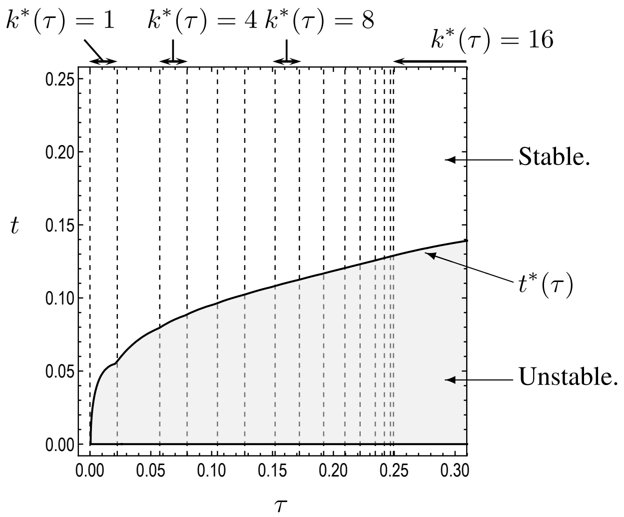


Fig.11 Local stability and bifurcation directions from the completely integrated pattern in line of decrease in the commuting cost parameter t ($K = 2^5$, $\theta = 3$). The figure is obtained by a suitable change of coordinate $(r, r^{\mathrm{H}}) \mapsto (\tau, t)$. In the light gray region, the completely integrated pattern is unstable. As t decreases, the completely integrated pattern becomes unstable at a τ-dependent critical point $t^*(\tau)$. Dashed lines depicts critical values of τ where $k^*(\tau)$ switch.

We denote the bifurcation direction at $r^{\mathrm{H}*}$ by $k^*(r)$; that is,

$$k^*(r) \equiv \underset{k \in \mathcal{K}(r)}{\arg\min}\ r_k^{\mathrm{H}*}(r). \tag{82}$$

We have following characterization for the bifurcation from the completely integrated pattern after Lemma 4:

Corollary 2 (The value of r and monocentricity/polycentricity)**.** Consider a bifurcation from the completely integrated pattern due to increase of the freeness of commuting r^{H}. Then, emergent patterns are characterized as follows (see **Fig.12** for $K = 32$):

- For $r \approx 0$, $k^*(r) = K/2$. The associated eigenvector is $\boldsymbol{\eta}_{K/2} \equiv [(-1)^j]$ where alternate locations increase its inhabitants. That is, spatial period-doubling patterns[17)] emerge.
- For $r \approx 1$, $k^*(r) = 1$. That is, monocentric patterns emerge.

Fig.11 illustrates the above discussions on (τ, t)-plane via an appropriate change of coordinate $(r, r^{\mathrm{H}}) \mapsto (\tau, t)$ so that one can compare the figure with **Fig.7**.[16] The number of locations is set to $K = 32$. When r is sufficiently small, i.e., the communication cost τ is large, $k^*(\tau) = K/2 = 16$; a steady decrease of the commuting cost t result in a period-doubling bifurcation. On the contrary, if r is large, i.e., τ is small, $k^*(\tau) = 1$; a decrease of t thus result in an emergence of monocentric pattern. **Fig.11** further illustrates that k^* is monotonically decreasing if we decrease τ. In particular, $k^*(\tau) = 16, 8, 4, 1$ are highlighted in the figure, as they are analogous to the spatial period-doubling cascade path discussed in Akamatsu et al.[11)] and Ikeda et al.[17)] Associated bifurcation modes $\boldsymbol{\eta}_1, \boldsymbol{\eta}_4, \boldsymbol{\eta}_8$ and $\boldsymbol{\eta}_{16}$ are illustrated in **Fig.12**.

[16] Qualitative difference between the shapes of regions in these figures should be understood as a consequence of (i) discretization of the underlying space and (ii) logit approximation of commuting behavior.

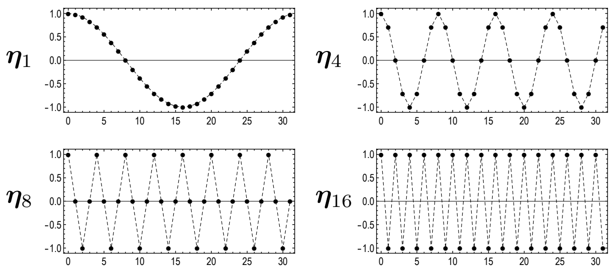


Fig.12 Eigenvectors $\boldsymbol{\eta}_1, \boldsymbol{\eta}_4, \boldsymbol{\eta}_8, \boldsymbol{\eta}_{16}$ in a $K = 32$ location racetrack. Observe that the eigenvectors are given by cosine curves.

In sum, the section showed that the distance-dependent job-choice behavior of households in the FO model can produce a *global* class of dispersion force *on the side of firms*. An agglomeration of firms at a location calls for larger number of household commuting to the location to meet its labor demand; this in turn requires a higher wage so that the location can be attractive compared with other locations, despite (possibly) longer commuting distance on the side of households. In effect, if the commuting cost t is sufficiently high, firms cannot attract distant households and global dispersion occur so that commuting distance becomes shorter. The mechanism behind polycentricity in the FO model is thus basically consistent with the conclusions of Takayama and Akamatsu.[4)] Explicitly considering location choice of households will not affect the basic mechanism, as competition over land can produce only local dispersion force.

5. Concluding Remarks

This paper has applied the global stability analysis technique to analyze the FO model. Although the model has been the reference model for urban polycentricity, it has been also criticized for its intractability, and also for resulting lack of stability analysis of equilibrium patterns. This paper has introduced a discrete version of the FO model to recast the model into an instance of potential games, so that stochastic stability analysis is possible. Via stochastic stability analysis on a symmetric racetrack economy, it is shown that: (a) the FO model admits *stochastically stable polycentric patterns*, and (b) the evolutionary path of spatial structure exhibits a specific pattern analogous to the so-called "bell-shaped developments." These results together provide an answer to the stability issue of equilibrium patterns for the FO model, as well as clarifying the effect of the communication cost parameter τ on resulting spatial structures. In addition, we have found that job-choice behavior of households is the key for emergence of polycentric patterns in the FO model. Since the job-choice of commuters are distance-dependent, it in turn produces a distance-dependent, *global* dispersion force. The result shows a fundamental importance of existence of a dispersion force that acts in global scale in producing polycentric patterns. We also

As our analysis in the paper has focused solely on showing emergence of *stable* polycentric patterns in the FO model, there remains a number of interesting research directions. First, it should be interesting to dig deeply into properties of spatial equilibrium patterns in the racetrack, e.g., wage profile, land rent profile, or equilibrium payoffs, and their comparative statics, e.g., dynamic transitions between stable spatial patterns. Also, making a concrete comparison between the original spatial setting (a line segment) and the present setting (a racetrack) should be an interesting task. In particular, it is probable that incompletely integrated patterns or polycentric segregated patterns with different-sized supports (cf. **Fig.2**) are consequences of the first nature of line segment.

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