Dynamic User Equilibrium with Side Constraints: Interpretation and Application to Traffic Management*

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1. Introduction

In managing a traffic network, it is desirable in some case to control the traffic such that the traffic volume on a certain link does not exceed a predefined threshold (e.g. its capacity). One method is to introduce physical capacity constraints for those links. Another method is to use an asymptotic travel time function that forces the travel time of the link to approach infinity when the flow approaches the link's threshold. The main focus in this paper is on the latter approach, i.e. introducing explicit capacity constraint on the link volume. This type of constraint is widely referred to as 'side constraint' in traffic assignment field. It has been shown that, under the static user equilibrium (UE), the traffic assignment problem with side-constraints generates a flow pattern which satisfied the modified Wardrop UE principle in the sense that the equilibrium condition holds in terms of generalized travel costs: the original Wardrop equilibrium travel costs plus link queuing delays. In addition, the Lagrange multiplier as derived from the side-constraint of the assignment problem can be viewed as the traffic control parameter (delay or adjusted travel cost function) on that link to ensure that the UE link flow satisfy the side-constraint[4].

This paper aims to extend the static side-constraint traffic assignment to the dynamic framework. The motivation is based on the application of this approach to the real-world traffic management scheme which involves a highly dynamical system with temporal demand and supply interaction. The dynamic framework allows a time-dependent control scheme to be derived from the proposed model. In this paper, we focus on problem formulation and analysis of such side-constraint problem with the Dynamic User Equilibrium (DUE) using linear exit-flow dynamic link model[1]. Under the dynamic framework, the inflow and outflow of each link are implicitly restricted by the dynamic link model already. Thus, the side-constraint will be related to the 'length' of queue to ensure the desired congestion level on different links.

In this paper, we formulate this problem as a continuous time optimal control problem with state dependent time delays and saturation, where the queue length is taken as a state variable and the inflow to a link is taken as a control variable. The state dependent formulation avoids the pitfalls of improper flow propagation and violation of causality of the original exit-flow dynamic link model[3]. However, the induced state dependent time delays by such formulation complicates the control problem. The Pontryagin Minimum Principle is applied to our analysis to obtain the necessary condition. The Lagrange multipliers is obtained as the costate variable. After obtaining the optimality condition, we provide the interpretation of the costate variable as compared to the Lagrange multiplier in the static case.

2. Problem formulation

To simplify the problem, we only consider the network with a number of parallel links connecting between each OD pairs in which each link has one bottleneck. Under the DUE condition, the dynamic equilibrium assignment problem can be defined as an optimal control problem with saturated state as follows:

$$J = \min_{e} \sum_{\forall a} \int_0^T \Psi_a(s, x_a^*) e_a(s) ds, \tag{1}$$

subject to,

$$\frac{dG_a[\tau_a(s)]}{ds} = e_a(s), \forall a, \forall s, \tag{2}$$

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$$\frac{dx_a(s)}{ds} = e_a(s) - g_a(s), \forall a, \forall s, \tag{3}$$

$$\frac{dE_a(s)}{ds} = e_a(s), \forall a, \forall s, \tag{4}$$

$$\sum_{\forall a} E_a(T) = J_{od}, \tag{5}$$

$$e_a(s) \geq 0, \forall a, \forall s, \tag{6}$$

$$x_a(s) \leq Q_a(s) \Leftrightarrow \int_0^s e_a(u)du - \int_0^s g_a(u)du \leq Q_a(s), \tag{7}$$

where the performance index as shown in (1) is proposed in Friesz *et al.*[2], $e_a(s)$ and $g_a(s)$ denotes the rates at which traffic enters and leaves the link a at time s respectively. Equation (2) ensures the proper flow propagation along each route, in the sense that the cumulative traffic that enter the link up to time s must exit from the link by time $\tau_a(s)$. Equation (3) is the state equations that govern the evolution of link traffic $x_a(s)$. Equation (4) defines the cumulative inflow $E_a(s)$. Equation (5) specifies the amount of total throughput J_{od} generated in the system within the time horizon T. Condition (6) ensures the non-negativity of the control variable. Given a positive inflow $e_a(s)$, the corresponding outflow $g_a(s)$ and link traffic volume $x_a(s)$ a is guaranteed to be positive. Hence, we do not add explicit constraints to ensure the non-negativity of $g_a(s)$ and $x_a(s)$. Finally, (7) is the capacity constraint proposed to restrict the length of the queue.

The travel time model we adopt here is proposed by Friesz *et al.* [1], the whole link linear travel time model,

$$\tau_a(s) = s + \nu_a + \frac{x_a(s)}{C_a},$$

It can be show that with this model the proper flow propagation, FIFO queuing discipline, and causality condition are hold.

The Lagrangian can be defined as:

$$\begin{aligned} Z^* = \min_{e_a^*(s)} \sum_{\forall a} \int_0^T \{ & \Psi_a(s, x_a^*) e_a(s) + \lambda_a(s)\left(e_a(s) - g_a(s) - \frac{dx_a(s)}{ds} \right) \\ & + \mu_a(s)\left(e_a(s) - \frac{dE_a(s)}{ds} \right) + \gamma_a(s)\left(e_a(s) - \frac{dG_a(\tau_a(s))}{ds} \right) - \rho_a(s) e_a(s) \\ & + \eta_a(s)(x_a(s) - Q_a(s)) \} ds + \phi_{od}\left(J_{od} - \sum_{\forall a} E_a(T) \right). \end{aligned} \tag{8}$$

With the zero initial state, we have

$$\begin{aligned} Z^* = & \sum_{\forall a} \left(-\lambda_a(T) x_a(T) - \mu_a(T) E_a(T) \right) + \phi_{od}\left(J_{od} - \sum_{\forall a} E_a(T) \right) \\ & + \sum_{\forall a} \int_0^T \{ \Psi_a(s, x_a^*) e_a(s) + \lambda_a(s)\left(e_a(s) - g_a(s) \right) + \frac{d\lambda_a(s)}{ds} x_a(s) \\ & + \mu_a(s) e_a(s) + \frac{d\mu_a(s)}{ds} E_a(s) + \gamma_a(s)\left(e_a(s) - g_a(\tau_a(s)) \frac{d\tau_a(s)}{ds} \right) \\ & - \rho_a(s) e_a(s) + \eta_a(s)(x_a(s) - Q_a(s)) \} ds. \end{aligned} \tag{9}$$

The Hamiltonian function can be defined as

$$\begin{aligned} H_a(s) = & \Psi_a(s, x_a^*) e_a(s) + \lambda_a(s)\left(e_a(s) - g_a(s) \right) + \mu_a(s) e_a(s) \\ & + \gamma_a(s)\left(e_a(s) - g_a(\tau_a(s)) \frac{d\tau_a(s)}{ds} \right) - \rho_a(s) e_a(s) + \eta_a(s)(x_a(s) - Q_a(s)), \end{aligned} \tag{10}$$

the Lagrangian is then given as

$$Z^* = \sum_{\forall a}\left(-\lambda_a(T)x_a(T) - \mu_a(T)E_a(T)\right) + \phi_{od}\left(J_{od} - \sum_{\forall a}E_a(T)\right)$$
$$+\sum_{\forall a}\int_0^T\left(H_a(s) + \frac{d\mu_a(s)}{ds}E_a(s) + \frac{d\lambda_a(s)}{ds}x_a(s)\right)ds.$$

The set of first order necessary condition of this optimal control problem can then be defined:

$$\begin{aligned}
&\lambda_a(T) = 0,\ \ \mu_a(T) = -\phi_{od},\\
&\frac{\partial H_a(s)}{\partial e_a(s)} = \Psi_a(s, x_a^*) + \lambda_a(s) + \mu_a(s) + \gamma_a(s) - \rho_a(s) = 0,\\
&\frac{\partial H_a(s)}{\partial x_a(s)} = \eta_a(s) = -\frac{d\lambda_a(s)}{ds},\\
&\frac{d\mu_a(s)}{ds} = 0 \Rightarrow \mu_a(s) = -\phi_{od},\\
&\frac{\partial H_a(s)}{\partial g_a(s)} + \frac{\partial H_a(s)}{\partial g_a\left(\tau(\sigma_a(s))\right)}\frac{1}{\dot{\tau}(\sigma_a(s))} = -\lambda_a(s) - \gamma_a\left(\sigma_a(s)\right) = 0 \Rightarrow \gamma_a(s) = -\lambda_a(\tau_a(s)).
\end{aligned} \tag{11}$$

where $\sigma_a(s)$ is the time of entry to the link with the exit time s.

Also, we have another set of KKT condition holds for the constraints:

$$\begin{aligned}
&e_a(s) \geq 0,\ e_a(s)\rho_a(s) = 0,\ \rho_a(s) \geq 0,\\
&x_a(s) - Q_a(s) \leq 0,\ \eta_a(s) \geq 0,\\
&\eta_a(s)\left(x_a(s) - Q_a(s)\right) = 0.
\end{aligned} \tag{12}$$

From the Hamiltonian function, we also have the conditions related to the side-constraint: (*i*) if $x_a(s) < Q_a(s)$, the Lagrange multiplier $\eta_a(s) = 0$, and (*ii*) if $x_a(s) = Q_a(s)$, the Lagrange multiplier $\eta_a(s) \geq 0$. The value of ϕ_{od} is determined by the total amount of traffic (J_{od}). $\psi_a(s, x^*)$ can be viewed as the generalized cost travel for the traffic departing at time s on route p under travel condition x^*. Evaluating the co-state equation together with the transversarity condition we have

$$\lambda_a(s) = -\int_T^s \eta_a(t)dt = \int_s^T \eta_a(t)dt. \tag{13}$$

Thus we have

$$\lambda_a(s) - \lambda_a(\tau_a(s)) = \int_s^{\tau_a(s)} \eta_a(t)dt. \tag{14}$$

The stationary condition for the DUE with side constraints can then be defined as:

$$\Psi_a(s, x_a^*) + \int_s^{\tau_a(s)} \eta_a(t)dt = \varphi_{od}. \tag{15}$$

We summarize the above necessary condition for DUE with side constraints as follows:

Proposition 2.1 The necessary condition for the DUE with side constraints can be given as follows:

$$e_a(s)\begin{cases} > 0 \Rightarrow \Psi_a(s, x_a^*) + \int_s^{\tau_a(s)} \eta_a(t)dt = \varphi_{od},\\ = 0 \Rightarrow \Psi_a(s, x_a^*) + \int_s^{\tau_a(s)} \eta_a(t)dt = \varphi_{od},\end{cases} \tag{16}$$

where the value of φ_{od} is determined by the total amount of traffic J_{od}.

Remark 2.1 As cited in Friesz *et al.*[2)], the Optimal Control Problem cannot be used for computational purpose to obtain the dynamic user equilibrium departure rates. In our case, the optimal control problem provides a mathematical platform for analyzing the necessary conditions of the DUE with side constraints, allowing us to use the Pontryagin Minimum Principle and/or other necessary conditions of optimal control theory.

3. Interpretation of DUE with side constraints

(3.1) Comparison of DUE and static UE with and without side constraints

For the used route at instant time s, we have $e_a(s) > 0$ which implies $\rho_a(s) = 0$. Furthermore if there is no side constraint, *i.e.* $x_a(s) < Q_a(s)$ hold for any time instant, we have $\eta_a(s) = 0$. The third equation of (11) implies $\frac{d\lambda_a(s)}{ds} = 0$, which implies $\lambda_a(s)$ be a constant. In this case we have $\lambda_a(s) + \gamma_a(s) = \lambda_a(s) - \lambda_a(\tau_a(s)) = 0$. The case reduces to $\Psi_a(s, x_a^*) = \varphi_{od}$, which is consistent with that of Friesz et al[2], *i.e.* the normal DUE condition. In fact, we can show that the DUE with side constraints can also reduce to UE with side constraints as well. Since in the static case the system is operating in the steady state, $e_a(s) = e_a$, $g_a(s) = g_a$, and $\dot{x}_a(s) = 0$. By defining the performance index (1) to be that of static case, the necessary conditions for DUE with side constraints reduces to static UE with side constraints.

(3.2) Interpretation of DUE with side constraints

For convenience, we first define both necessary conditions for DUE with side constraints and UE with side constraints for our interpretation of DUE with side constraints.

Lemma 3.1 The necessary condition for UE with side constraints is

$$f_{rw} \begin{cases} > 0 \Rightarrow c_{rw} = \mu_w, \\ = 0 \Rightarrow c_{rw} \geq \mu_w, \end{cases} \tag{17}$$

where f_{rw} is the flow on path $r \in R_w$ between OD pair $w \in W$, $t_a(v_a)$ is the normal travel time on link $a \in A$, λ_a is the Lagrange multiplier associated with the side-constraint, the following generalized link travel time is defined:

$$c_{rw} = \sum_{a \in A} \hat{t}_a(v_a)\delta_{ar}, r \in R_w, w \in W,$$
$$\hat{t}_a(v_a) = t_a(v_a) + \lambda_a, a \in A.$$

Notice that the formulations of these two necessary conditions are similar in the sense that if we consider their similarities:

$$t_a(v_a) \triangleq \Psi_a(s, x_a^*), \lambda_a \triangleq \int_s^{\tau_a(s)} \eta_a(t)dt. \tag{18}$$

We can consider the term $\int_s^{\tau_a(s)} \eta_a(t)dt$ (as in static UE with side-constraints) as an additional time penalty that users traveling on this saturated link are willing to wait for using this link. We can thus interpret the UE with side-constraints as a special case of the DUE with side-constraints (*i.e.* the steady state case) in which $\dot{x}_a(s) = 0$. In the static case, λ_a is a constant whereas the additional time penalty in the DUE with side-constraints, $\int_s^{\tau_a(s)} \eta_a(t)dt$, is dynamic.

The DUE with side-constraints has at least has two extreme solutions, namely the normal DUE solution and static UE with side constraints. In fact, the general existence of a solution to the DUE with side-constraints can be shown. Since the whole link linear travel time model is adopted, the strong FIFO[5] is satisfied and the side constraints provide a priori boundedness of path flows, the existence condition of such DUE is satisfied following Zhu and Marcotte[5].

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