AN APPLICATION OF ROSEN GRADIENT PROJECTION METHOD TO UE ASSIGNMENT WITH CAPACITY CONSTRAINTS

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1. Introduction

In the procedure of evaluating travel time reliability of congestion network, we aim to build the sensitivity analysis and solve the capacitated user equilibrium problem. This paper is one part in the research of network reliability evaluation.

The conventional user equilibrium model (Sheffi, 1985) is pervasively used for solving traffic problem in both theory and practice, because of simplicity and understandable interpretation of the problem. But in the congested network, it is demanded to estimate the waiting time due to congestion. To obtain a more reasonable description of network flow, upper bounds on the link flows have been suggested to add in the conventional model. Such capacitated equilibrium model has been studied (Inoue, 1986; Yang and Yagar, 1994; Larsson and Patriksson, 1995; Bell and Iida, 1997), but received not much attention. One of the important reasons is the underdevelopment of efficient algorithms.

This paper adapts the Rosen gradient projection method for the capacitated UE, which is formulated in general nonlinear optimization problem with explicit link capacity constraints. The model can improve the knowledge about a congested network. We exploit the characteristics of Rosen gradient projection method to combine it with the interior penalty method, for efficiently achieving the capacitated equilibrium solution. This paper is composed of six sections. The capacitated equilibrium model and its properties are briefly described in Section 2. In Section 3 a penalty function is used to translate the capacitated model into an analogue of the conventional UE, in order to utilize Rosen method. Section 4 presents the Rosen gradient projection algorithm for the capacitated model. Section 5 displays a numerical example solved by the proposed method. In section 6, we draw some conclusions.

The notations used throughout this article are summarized below.

N	the set of nodes of the network
A	the set of links of the network
W	the set of origin-destination pairs
K^w	the set of paths with OD pair, $w\in W$
r^w	the number of paths in path set, K^w
$z[\mathbf{x}(\mathbf{f})]$	the objective function
$\bar{z}[\mathbf{x}(\mathbf{f})]$	the penalty function
$\mathbf{A}=[\delta_{ak}{}^w]$	the link/path incidence matrix
$\mathbf{\Lambda}=[\Lambda_k{}^w]$	the origin-destination/path incidence matrix
$\mathbf{q}=[q^w]$	the vector of OD demands, $w\in W$
$\mathbf{u}=[u^w]$	the dual vector of conservation equations
$\mathbf{x}=[x_a]$	the vector of link flows, $a\in A$
$\Delta\mathbf{x}=[\Delta x_k{}^w]$	the vector of link flow direction
$\mathbf{t}=[t_a]$	the vector of link travel times,
$\bar{\mathbf{t}}=[\bar{t}_a]$	the vector of generalized link travel costs
$\mathbf{c}=[c_a]$	the vector of link capacity, $a\in A$
$\mathbf{f}=[f_k{}^w]$	the vector of path flows, $k\in K^w\subset K$
$\Delta\mathbf{f}=[\Delta f_k{}^w]$	the vector of path flow direction
$\mathbf{d}=[d_a]$	the dual vector of capacity constraints, $a\in A$
$\boldsymbol{\tau}=[\tau_k{}^w]$	the vector of path travel times, $k\in K^w\subset K$
$\bar{\boldsymbol{\tau}}=[\bar{\tau}_k{}^w]$	the vector of generalized path travel costs
$\bar{k}^w(n)$	the shortest path within w, at iteration n
γ^m	the penalty parameter at penalty iteration m
σ	the reduction rate of γ
λ	the step of the flow moving
λ^*	the optimal step of the flow moving

2. Capacitated User Equilibrium

The capacitated user equilibrium assignment problem is equivalent to the following minimization problem:

$$\text{minimize} \quad z(\mathbf{x})=\sum_a \int_0^{x_a} t_a(x,c_a)\,dx \qquad (1a)$$

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subject to $\sum_k f_k^w = q_w \quad \forall w \quad (u^w)$ (1b)

$x_a = \sum_w \sum_k f_k^w \delta_{ak}^w \quad \forall a$ (1c)

$x_a \le c_a \quad \forall a \quad (d_a)$ (1d)

$f_k^w \ge 0$ (1e)

This problem becomes conventional UE if dropping the link capacity constraints. Since all path flow variables of interest are positive and the nonnegativity constraints (1e) are not binding, the nonnegativity in terms of path flows may be omitted in the model formulation without affecting the equilibrium solution. The acquisition of the positive paths is executed by column generation and illustrated in section 4. The path set will only include those positive variables through all the paper.

Consistent with the necessary and sufficient conditions of optimality, the following equilibrium condition for every origin-destination pair is built by Inoue (1986a, b).

$$\left.\begin{array}{l}\sum_a (t_a + d_a)\delta_{ak}^w = u^w \text{ if } f_k^w > 0 \\ \sum_a (t_a + d_a)\delta_{ak}^w \ge u^w \text{ if } f_k^w = 0\end{array}\right\} \forall k,\, w \quad (2)$$

The solution of the capacitated UE assignment model is considered to have the characterization of a Wardrop principle when the travel cost is articulated in terms of running times and waiting delays. This generalized travel cost is, in fact, the ordinary one to be minimized by the individual travelers in a congested network with queuing. The waiting time or queuing delay is equivalent to the Lagrange multipliers associated with capacity constraints on each link (Inoue, 1986a). Then equilibrium flow pattern and generalized cost over network can be obtained once the problem is solved.

By the way, Daganzo (1977) showed a method to incorporate implicitly capacity constrains into conventional UE problem, by the Davidson cost function, in which,

$$\lim_{x_a \to c_a} t_a(x_a, c_a) = \infty \quad (3)$$

The proposed method may overestimate the travel times, and furthermore it cannot foretell the waiting times occurred at the particular congested link.

3. Integrate capacity constraints into the objective

In the case where the gradient projection method to traffic assignment problem, the properties of conventional model are desired to retain, so that some familiar knowledge can be utilized, such as the all-or-nothing load, the shortest route search. For the purpose, in this section the capacity constraints will be translated into the objective, a penalty function. A following function is defined with respect to each link.

$$\psi_a(x_a) = -\log \frac{c_a - x_a}{c_a} \quad (4)$$

let $\{\gamma^m\}$, m=0, 1, 2,...denote penalty parameters, then an augmented objective function is obtained

$$\begin{aligned}\bar{z}(\mathbf{x}, \gamma^m) &= \sum_a \int_0^{x_a} t_a(x)dx + \gamma^m \sum_a \psi_a(x_a) \\ &= \sum_a \int_0^{x_a} \{t_a(x) + \gamma^m \psi_a'(x_a)\}dx \\ &= \sum_a \int_0^{x_a} \left\{t_a(x) + \frac{\gamma^m}{c_a - x_a}\right\}dx \end{aligned} \quad (5)$$

In this way, the capacitated UE model can be transformed into an analogue of the conventional UE, by substituting $\bar{z}(\mathbf{x}, \gamma^m)$ for $z(\mathbf{x})$. The following model will be solved to approach the solution of the capacitated model (1).

minimize $\bar{z}(\mathbf{x}, \gamma^m)$ (6a)

subject to

$\sum_k f_k^w = q^w \quad \forall w$ (6b)

$x_a = \sum_w \sum_k f_k^w \delta_{ak}^w \quad \forall a$ (6c)

When γ^m approaches 0, the series of the optimal solution of these problems, $\{\mathbf{x}^{(m)}\}$, converges to the optimal solution of the original problem. In parallel, the penalty item ($\gamma^m \psi_a(x)$) added in the link cost function is associated with the Lagrange multiplier for the given penalized constraint, and interpreted as the waiting time at the link exit. The relationship between penalty item and delay is explicitly described by the following limit equation:

$$d_a = \lim_{\gamma^m \to 0^+} \frac{\gamma^m}{c_a - x_a} \quad (7)$$

Since the model may be regarded as a conventional UE problem with the augmented cost function $\bar{t}_a$ =

$t_a(x) + \gamma^n \psi'_a(x)$, the existing methods can be used to achieve the equilibrium flow, such as Frank-Wolfe algorithm (Inoue, 1986b; Yang and Yagar, 1994), Bersekas gradient projection method (Prashjer and Toledo, 2000). In this paper, a Rosen gradient projection is adopted, to facilitate convergence and obtain accuracy solution.

4. UE Solution by Rosen GP Algorithm

The well-known Frank-Wolfe method is the one which solution moving is within the feasible region, while Rosen gradient projection solution project the negative gradient in a such a way that make rapid changes to the active set. Here a Rosen gradient projection method (Bazarra, 1979, Inoue, 1981) is adapted to the solution of the capacitated equilibrium model.

Provided that a current feasible solution exists, RGP algorithm for the translated model (6) is summarized:

Initialization: generate an initial solution

step 0 set iteration counter n=0,

select an initial solution, and corresponding path set $K^w(0)$, $\forall w \in W$

Column Generation: generate the shortest paths and augment the path set

step 1 increment iteration counter: $n:=n+1$

step 2 update cost, perform 0-1 assignment and record a set of shortest paths $\bar{k}^w(n)$

step 3 augment the path set

set $K^w(\mathrm{n}) = K^w(n-1) \cup \bar{k}^w(n)$, if $\bar{k}^w(n) \notin K^w(n-1)$; otherwise, set $K^w(\mathrm{n}) = K^w(n-1)$

Equilibration: solve assignment problem over the augmented path set

step 4 calculate search direction of path flows

$$\Delta f_k^w(n) = q^w\left(\frac{1}{r^w}\sum_{k\in K^w}\bar{\tau}_k^w(n) - \bar{\tau}_k^w(n)\right), \quad \forall k, \forall w$$

step 5 if $\Delta f_k^w = 0$ for any path k within any pair w, then repeat ***Column Generation***;

if $f_k^w = 0$, and $\Delta f_k^w < 0$, drop the path k: $K^w(n) = K^w(n) \mid k$

step 6 limit step size of path flows

$$\lambda_{\text{path}} = \min_{k,w}\left\{-\frac{f_k^w(n)}{\Delta f_k^w(n)} \,\middle|\, \Delta f_k^w(n) < 0\right\}$$

step 7 calculate search direction of link flows by assigning flows into links:

$$\Delta x_a(n) = \sum_{w\in W}\sum_{k\in K^w(n)}\Delta f_k^w(n)\delta_{ak}^w, \quad \forall a$$

step 8 limit step size of link flows

$$\lambda_{link} = \min_{a\in A}\left\{-\frac{\sum_a \Delta x_a \bar{t}_a}{\sum_a \Delta x_a^2 \bar{t}_a'}\right\}$$

step 9 limit step size due to link capacities:

$$\lambda_{\text{capacity}} = \min_a\left\{\frac{c_a - x_a(n)}{\Delta x_a(n)} \,\middle|\, \Delta x_a(n) > 0\right\}$$

step 10 optimal step size for path flow and link flow,

$$\lambda^* \cong \min\{\lambda_{link}, \lambda_{\text{path}}, \lambda_{\text{capacity}}\}$$

step 11 update flow pattern

$$f_k^w(n+1) = f_k^w(n) + \lambda^*\Delta f_k^w(n)$$

$$x_a^w(n+1) = x_a^w(n) + \lambda^*\Delta x_a^w(n)$$

Termination: terminate the algorithm if it satisfies the stopping criterion

step 12 if $\max_w \sum \frac{f_k^w(n)}{q^w}\left(\frac{\bar{\tau}_k^w(n) - \bar{\tau}_{\bar{k}}^w(n)}{\bar{\tau}_k^w(n)}\right) \le \varepsilon$,

terminate; otherwise, go to ***Column Generation***

For solving the capacitated UE model, the Rosen gradient projection algorithm must be still integrated into the following penalty function procedure:

step 0 set an initial penalty parameter γ, and penalty counter m=0

step 1 $\gamma := \sigma\gamma$, $m := m+1$

step 2 Rosen gradient projection algorithm (RGA)

step 3 if flows converge, then terminate; otherwise, go to step 1

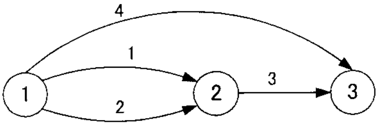

Fig. 1 Network

Table 1 Trip Table

2	3	D / O
600	400	1
	600	2

Table 2: Link Performance

link	t_a^0	c_a
1	10	600
2	17	500
3	9	800
4	60	400

5. A Numerical Example

In this section, we present a numerical example to illustrate capacitated UE assignment problem solved by Rosen gradient projection method. The test network, shown in Fig. 1, consists of 4 links and 3 nodes. The origin destination demand and link performance are listed in Table 1 and Table 2. The standard BPR function is used and parameters γ= 1000, σ=0.1 are adopted. The initial feasible solution is achieved by Daganzo technique (1977). The path flows converge to the optimal solution by four times penalty iterations. At each iteration the 100 times RGP repetition is executed.

Table 3: Estimated Link Solution

	The capacited UE			The conventional UE	
Link	Flow	Time	Delay	Flow	Time
1	600	11.5	5.6	882.1	17
2	200	17.1	0	117.9	17
3	800	10.3	33.2	1000	12.3
4	200	60.6	0	0	60

Table 4: Estimated Path Solution

Pairs	Path Flow	Link Makeup	Time	Delay
(1, 2)	403.6	1	11.5	5.6
	196.4	2	17.1	0
(1, 3)	196.4	1_3	21.8	38.8
	200.0	4	60.6	0
	3.6	2_3	27.4	33.2
(2, 3)	600.0	3	10.3	33.2

The link solution, illustrated in Table 3, shows waiting time occurred at links 1 and 3. The path solution in Table 4 is also provided by RGP method. It can be seen that the Wardrop principle still comprises in terms of the generalized travel costs. The gradient projection method is a path-based solution and a byproduct of the RGP method is the alternative path flow pattern. Although such path flows are not of uniqueness, it is indispensable in some traffic problems, such as sensitivity analysis of capacitated network flow. Table 4 shows that more than two paths are concurrently used in pairs (1, 2) and (1,3). In contrast to the RGP solution, the Frank-Wolfe algorithm cannot offer such multiple path flows for each pair, due to the presumption that only one shortest path is chosen within any centroid pair. If this assumption is not to fit right in with reality, the approach to the solution might be a burdensome computation.

6. Conclusions

A path-based solution algorithm for UE assignment with capacity constraints is presented. The algorithm integrates Rosen gradient projection method into the penalty function procedure, and may provide flow, cost of both link and path. The efficiency of the algorithm will be verified in the future research.

7. References

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