FLOW-DEPENDENT PATH FLOW ESTIMATIOR BY FISK TYPE MODEL

Lin Cheng* Yasunori Iida** Nobuhiro Uno***

1. Introduction

With the rapid development of vehicle information and communication technologies, it is urgently required to collect up-to-date information about the state of road network, such as link flows, locations of congestion, path flows and path travel times. For example, dynamic route guidance requires path travel times, while traffic detectors are set up only at some sections of road network to collect such data as link flows. These observations need to be complemented by some processors, in order to build a complete picture of current network states.

Sherali *et al.* (1994) proposed a linear programming Path Flow Estimator (abbreviated to PFE) for estimating user equilibrium path flows, which may be aggregated to yield an OD matrix. Bell and Iida (1997a,b) proposed non-linear programming PFE to estimate stochastic user equilibrium path flows, which may be used for estimating travel time reliability.

The output data of stochastic PFE include path flows, link flows, together with travel times and the OD trip table, Due to the potential of PFE in evaluating travel time reliability, this study is in the spirit of the latter and makes two improvements in the modularity of stochastic PFE. First, the link travel time is not assumed to be constant, but more realistically a monotone function of link flow, or say, the undelayed travel time is flow-dependent function. Second, overload delay is explicitly explained to be the waiting time of queuing, and is connected to queue by introducing link service rate equal to link capacity. Furthermore, this paper accounts for necessary number of required traffic detectors for estimating path flow, by analyzing the mechanism of mathematical programming model of stochastic PFE.

2. Flow, Service rate and Queue

The capacity of each link is available and regarded as the link service rate (vehs / time unit), which is defined by length, width and signal split *et al.* of the road segment. Dependent on the vehicle positioning in relation to the link, the flow that passes through the link may sequentially enter three states, called inflow, queue and outflow, shown in Fig 1.

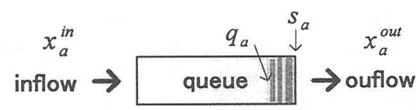

Fig. 1 schematic link flow

The occurrence of queue depends on size relationship of inflow and service rate of the link. Inflow could smoothly pass through the link and equal to outflow if inflow is not in excess of the link service rate, then queue should not exist; once if inflow overreaches service rate, queue will cumulate and increase with the growing inflow and then outflow is tantamount to the service rate. This awareness is applicable to both unobserved links and observed links. The flow mentioned in this paper usually denotes inflow, except for particular designation.

$$\left.\begin{array}{l} x_a^{out} = s_a,\ q_a > 0,\ \text{if } x_a^{in} > s_a \\ x_a^{out} \le x_a^{in},\ q_a = 0,\ \text{if } x_a^{in} \le s_a \end{array}\right\} \quad (1)$$

Additionally, It is because of queuing that the part of inflow will experience delay on the correspondent link. In terms of the noticed time width, the queue equals to the product of overload delay times the service rate.

$$q_a = d_a s_a \quad (2)$$

where q_a denotes the existing queue on the link a, and d_a denotes overload delay.

Equation (2) is useful in the iterative procedure of estimating flows of unobserved links. It provides a feedback mechanism for the balancing of inflow, queue and outflow with respect to unobserved links of congestion. Typically, the flow must equal to the summation of queue and outflow.

3. Representation of Link Cost

Link travel cost of each link is partitioned into two components: flow-dependent travel time, t_a, and overload delay, d_a due to the limited exit capacity. In general, traveling speed is restricted by the number of vehicles on the road section, and the travel time increases with higher traffic density. On the network of urban areas, most vehicle delay occurs at road intersections due to limited exit capacity, and is significant in the travel cost under the condition of congestion. The overload delay is usually determined

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*student member of JSCE, Department of Civil Engineering, School of Engineering, Kyoto University
Sakyoku, Kyoto, Japan, T: 075-753-5126, F: 075-7535907
cheng@urbanfac.kuciv.kyouto-u.ac.jp
**Fellow of JSCE, Ph.D., Professor, Department of Civil Engineering, Kyoto University
***Regular Member of JSCE, Ph.D., Associate Professor, Department of Civil Engineering, Kyoto University

by both network flow pattern and link service rate, i.e.

$$\left.\begin{aligned} d_a &= 0 \quad \text{if } x_a^{out} < s_a \\ d_a &> 0 \quad \text{if } x_a^{out} = s_a \end{aligned}\right\} \tag{3}$$

where s_a is the exit capacity equal to the link service rate, and x_a^{out} denotes link outflow. Equation (3) couples queue and delay on one link. The delay may be regarded as a penalty of bringing overload to the capacity-limited link. Fig.1 shows the link cost-volume function. By assuming that the travel cost on a link is equivalent to the sum of a flow-dependent travel time and overload delay at the link exit, both link flow and delay can be obtained from a convex programming problem (1997b). Accordingly, the links are assumed of two types: uncongested links and congested links. On the other hand, according to the availability of traffic volume, links are also assumed of observed and unobserved links.

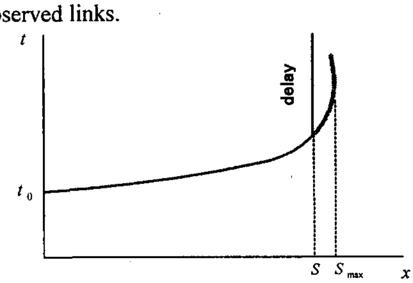

Fig.2 relation of volume−cost

4. Nature of Logit Route Choice

Consider a network of two routes k and k' connecting one origin and one destination. The choice probability is offered by well-known logit model,

$$\Pr[k] = \frac{\exp(-\theta C_k)}{\exp(-\theta C_k) + \exp(-\theta C_{k'})} \tag{4}$$

where C_k is travel cost on route k, θ is dispersion parameter. Since the route flow h_k is proportion to the choice probability, it leads the following equivalent logit model with respect to route flows.

$$h_k / h_{k'} = \exp(-\theta C_k) / \exp(-\theta C_{k'}) \tag{5a}$$

$$\text{or} \quad \ln(h_k/h_{k'}) = -\theta(C_k - C_{k'}) \tag{5b}$$

This is a fundamental equation discussed in the paper, more specifically, stochastic user equilibrium is achieved when and only when the trip allocation between alternative paths confirms to the equation. We rewrite equation as

$$\tfrac{1}{\theta}\ln h_k + C_k(h_k) = \tfrac{1}{\theta}\ln h_{k'} + C_{k'}(h_{k'}) = u^w \tag{6}$$

The equation express the significant feature that each OD-pair w holds a constant value u^w respectively at the equilibrium point. This constant value, called OD-pair equilibrium cost, consists of the inherent travel cost $C_k(h_k)$ of each path, between OD-pair w, and the additional cost, $\frac{1}{\theta}\ln h_k$, produced by correspondent path flows. It signifies that SUE assigns the more OD flow to the lower cost route, the less OD flow to the higher cost route. It evidently differs from the shortest route choice in which the OD flow gathers in the route of minimum cost.

5. Path Flow Estimator

Because of uncertainty of route choice, and imperfect information about network state, one assumption behind UE is evidently violated and hence the stochastic user equilibrium is preferred to simulate drivers' route choice behaviours. The logit-based SUE is formulated as mathematical programming

$$\begin{aligned} &\textit{Minimize } f(\mathbf{h}) = \frac{1}{\theta}\mathbf{h}^{\mathrm{T}}(\ln\mathbf{h} - \mathbf{1}) + \sum_a \int_0^{x_a(\mathbf{h})} c_a(x)dx \\ &\textit{subject to} \quad \mathbf{s} \ge \mathbf{A}\mathbf{h}, \quad \mathbf{v} = \mathbf{\Lambda}\mathbf{h}, \quad \mathbf{h} > \mathbf{0} \end{aligned} \tag{7}$$

where $\mathbf{h}$ denotes column vector of route flows, $\mathbf{A}$ represents incidence matrix of unobserved links and routes, $\mathbf{\Lambda}$ the relationship of observed links and routes, $\mathbf{v}$ is column vector of link flows. Readers can note that this mathematical programming is distinguished from the primitive Fisk's model by explicit inclusion of capacity constraints acting on unobserved links.

The following Lagrange equation can be formed

$$L(\mathbf{h},\boldsymbol{\mu},\boldsymbol{\lambda}) = f(\mathbf{h}) + \boldsymbol{\mu}^{\mathrm{T}}(\mathbf{A}\mathbf{h} - \mathbf{s}) + \boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{v} - \mathbf{\Lambda}\mathbf{h}) \tag{8}$$

At optimality, the first-order conditions

$$\tfrac{1}{\theta}\ln\mathbf{h} + \mathbf{C} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\mu} = \mathbf{\Lambda}^{\mathrm{T}}\boldsymbol{\lambda} \tag{9a}$$

$$\tfrac{1}{\theta}\ln h_k + C_k + \mu_k = \lambda_k \tag{9b}$$

where μ_k denotes the sum of capacity-constraint Lagrange multipliers of unobserved links along route k, and λ_κ is that of observed link Lagrange multipliers also along route k.

There are two points different between equation (6) and (9). First in the left of equation the route travel cost is augmented by one additional cost μ_k, interpreted as path delay consisting of delays of link on the route. The fact that capacity-constrained Lagrange multipliers are interpreted as link delay is explained by its complementary slackness conditions

$$\begin{aligned} \mu_a &= 0 \quad \text{if } \mathrm{s_a} > \sum \delta_{ak} h_k \\ \mu_a &> 0 \quad \text{if } \mathrm{s_a} = \sum \delta_{ak} h_k \end{aligned} \tag{10}$$

The comparison of Eq. (1) and Eq.(10) shows that if let $d_a = \mu_\alpha$, $x_a^{out} = \sum \delta_{ak} h_k$, the two equations are the same. Therefore, the assumption about travel cost is accommodated with mathematical programming (7). Second, in the right side of equation (9), λ_k replaces OD-pair equilibrium cost associated with OD-pair demand in equation (6), and represents the intrinsic attractiveness showed by flows from traffic detectors. Here hence the logit-based flow pattern in PFE is dependent on both augmented link costs and detected flows, while the flow pattern of the primitive logit model depends on OD trip table, although either of them follows stochastic user equilibrium.

Note that at the equilibrium point of PFE the equilibrium cost is not equal for each route of one OD-pair. The reason for this extraordinary phenomenon is that the network flow pattern is also swayed by distribution of detected link flows. In other words, the OD-pair equilibrium cost is also determined by detected data of links along the routes between OD-pair. Furthermore, the property figures out the allocation problem of traffic count stations over one network. See the right side of equation (9b), as described above, λ_k is associated with observed link flows along route k. If let the factor be zero, that is to say, no traffic counts be established on the route, the desired path flow would be not solvable. Accordingly, each considered path must include at least one link with traffic count. In brief, at any rate there exits one cutset separating an origin-destination, and all its elements, the links pertaining to the cutest, must be set up traffic count stations. By rights, the more traffic counts, the more representative the simulated flow.

6. Determination of Parameter θ

The determination of parameter θ is similar to that of stochastic assignment. It may be interpreted as a measure of drivers' sensibility to the path costs; by varying this parameter it is possible to represent different traveller behaviour. The influence of the second part of the objective function (7) grows with the increasing parameter. In the limit as $\theta \to \infty$, the user equilibrium comes out. This corresponds to a situation where travellers are extremely sensitive to an increased path cost, or closely say, the travellers have perfect information about the actual travel cost. In reality, the value of the parameter θ is uniquely determined by the optimal flows, and may therefore be calibrated in a practical application, for instance from observed flows. This paper takes the parameter as 0.1, so as to give a comparison between the result from Bell and Iida (1997a) and that of our study.

7. Solution by Iterative Balancing

Consider the Lagrange equation (8). The saddle point theorem says that at the optimum it is minimized with respect to the primal variables and maximized with respect to the dual variables. In the light of this theorem, the optimal solution may be searched by the sequentially optimizing the Lagrange function in terms of the dual and the primal. In other words, given initial values of $\boldsymbol{\mu}, \boldsymbol{\lambda}$, note so-called Lagrange problem.

$$D(\boldsymbol{\mu}^{(i)}, \boldsymbol{\lambda}^{(i)}) = \min_{\mathbf{h}} L(\mathbf{h}, \boldsymbol{\mu}^{(i)}, \boldsymbol{\lambda}^{(i)}) \qquad (11)$$

The solution of path flow $\mathbf{h}^{(i)}$ only relies on the temporal dual values of i-th iteration. The iterative procedure ended when the value of Lagrange function cannot further be improved. Otherwise, the new dual ($\boldsymbol{\mu}^{(i+1)}, \boldsymbol{\lambda}^{(i+1)}$) can be formulated, which may make Lagrange function ameliorable.

On the basis of this mind, the developed model of PFE is advantageous in its simplicity of calculation of flow and cost on relevant routes. By introducing link performance functions and queues, the iterative balancing procedure proposed by Bell & Iida (1997 a, b) is modified as follows:

Step 1 initialisation
$\mathbf{x} \leftarrow \mathbf{0}$
Step 2 updating cost by performance function
$\mathbf{c} = \mathbf{c}(\mathbf{x})$
go to **Step 4** if cost equilibrium
Step 3 balancing flow
$\boldsymbol{\lambda} \leftarrow \mathbf{0}, \boldsymbol{\mu} \leftarrow \mathbf{0}$
Repeat
For all measure links a
$\ln(\mathbf{h}) \leftarrow \theta(\mathbf{\Lambda}^T\boldsymbol{\lambda} - \mathbf{C} - \mathbf{A}^T\boldsymbol{\mu})$
$\lambda_a \leftarrow \lambda_a + \ln(v_a) - \ln(\mathbf{\Lambda}^T\mathbf{h})_a$
For all unmeasured links a
$\ln(\mathbf{h}) \leftarrow \theta(\mathbf{\Lambda}^T\boldsymbol{\lambda} - \mathbf{C} - \mathbf{A}^T\boldsymbol{\mu})$
if $(\mathbf{\Lambda}^T\mathbf{h})_a \geq s_a + q_a$ then
$\mu_a \leftarrow \mu_a + \ln(\mathbf{\Lambda}^T\mathbf{h})_a - \ln(s_a + q_a)$
queue occurring
$q_a = \max\{0, (\mathbf{\Lambda h})_a - s_a\}$, for measured links
$\left.\begin{matrix}\mu_a = d_a \\ q_a = d_a s_a\end{matrix}\right\}$, for unmeasured links
until *inflow* = *outflow* + *queue*
Step 4 output flow and delay

8. Numerical Example

A same network used in Bell and Iida (1997a) is presented in Fig. 3, to illustrate the effectiveness of modified Iterative Balancing method in estimating path flow. The free travel time and capacity are given in the brackets. There is only one OD pair of the origin 1 and the destination 9. It is assumed that traffic detectors are installed in links 1 and 3, clearly all the paths from origin 1 to destination 9 include either link 1 or link 3. Since the detectors are installed in the upstream of each link, the queuing at the downstream of link is assumed to have no effect to observed data. In other word, traffic volume observed by the detectors can be regarded as the link demand. The observed link flows of link 1, 3 are assumed to equal to their capacity respectively like Bell and Iida. The link cost function used here are

$$t_a(x) = t_{a0} + 0.01x_a^2 \quad (x \le s_a)$$

where, t_{a0} represents the free travel time of the link a, the first items of the brackets illustrated in Fig 3; x_a represents the flow of link a.

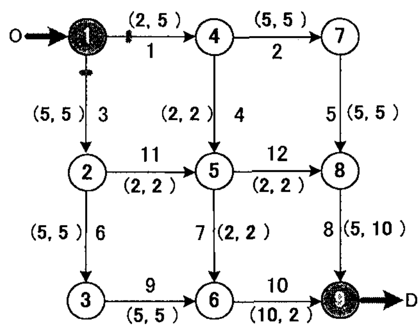

Fig. 3 Network of example

When the link travel time is independent on link flow, the result of this paper is equivalent of that from Bell and Iida (1997a). Under the same condition of input data and parameters, the value of objective function in this study is 7.66, which is superior to 11.22, the result calculated according to Bell and Iida. This is explained by the fact that relation between delay and queue has been explicitly included in the procedure of iterative algorithm. The overload delay on any congested link must be in congruence with the queue, and the queue size give a feedback to the calculated inflow. Because of introducing link cost function, the travel time will rely on network flow pattern, the value of the objective function becomes 8.24 when stochastic user equilibrium is achieved.

Table 2 link flow and cost

Link #	inflow	capacity	queue	cost	time	delay
1	5.00	5	0	2.25	2.25	0
2	1.11	5	0	5.01	5.01	0
3	5.00	5	0	5.25	5.25	0
4	3.89	2	1.89	2.98	2.04	0.94
5	1.11	5	0	5.01	5.01	0
6	1.36	5	0	5.02	5.02	0
7	4.22	2	2.22	3.15	2.04	1.11
8	4.42	10	0	5.20	5.20	0
9	1.36	5	0	5.02	5.02	0
10	5.58	10	0	2.31	2.31	0
11	3.64	2	1.63	2.86	2.04	0.82
12	3.31	2	1.31	2.69	2.04	0.65

The calculated link flow and cost are shown in Table 2. Since the observed flows of link 1,3 equals to their capacities, queuing does not occur on these links. Of course, queue could exist only when the observed link flow is above the link capacity, and equal the difference of inflow minus capacity. Similarly, the delay and queue occur on the unobserved links 4, 7, 11, 12, because they hold limited capacities and lower costs. Although the unobserved link 9 also has lower cost, there is no congestion on it. The reason for this phenomenon is that the link 9 merely lies on the longest path.

The path cost and flow are displayed in Table 3. The column of time shows the path travel time in accordance with cost function under current outflow pattern. The occurrence of delay exists mostly in the lower cost path. In the situation of congestion, pursuing the most benefit of individuals, the drivers prefer the experience of delay in the lower cost path to higher cost path. The delay size of each path is in inverse proportion to the path travel time. The distribution of path flow is mainly characterized by logit model and observed link flows.

Table 3 path cost and flow

path	cost	time	delay	flow
1-4-7-10	10.69	8.64	2.05	2.18
1-4-8-12	13.12	11.53	1.60	1.71
1-2-5-8	17.47	17.47	0.00	1.11
3-7-10-11	13.57	11.64	1.93	2.04
3-8-11-12	16.00	14.53	1.47	1.60
3-6-9-10	17.60	17.60	0.00	1.36

9. Conclusions

This paper describes an improved PFE. Link travel cost consists of flow-dependent travel time and overload delay. Traffic demand is governed by logit model and observed link flow pattern. By introducing the link service rate equal to link exit capacity, queuing is associated with delay and equals to service rate times delay. Queue is used in balancing inflow and outflow to guarantee the convergence of PFE. And then a modified iterative balancing procedure is set out the equivalent convex programming problem. The partitioning of inflow and outflow may be applicable to providing travel time reliability.

The further subjects will apply improved PFE to the estimating of the network performance measure, and confirm whether the different type of the performance function can be used in PFE.

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